Gauge-invariant coupled gravitational, acoustical, and electromagnetic modes on most general spherical space-times

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The coupled Einstein-Maxwell system linearized away from an arbitrarily given spherically symmetric background space-time is reduced from its four-dimensional to a two-dimensional form expressed solely in terms of gauge-invariant geometrical perturbation objects. These objects, which besides the gravitational and electromagnetic, also include mass-energy degrees of freedom, are defined on the two-manifold spanned by the radial and time coordinates. For charged or uncharged arbitrary matter background the odd-parity perturbation equations for example, reduce to three second-order linear scalar equations driven by matter and charge inhomogeneities. These three equations describe the intercoupled gravitational, electromagnetic, and acoustic perturbational degrees of freedom. For a charged black hole in an asymptotically de Sitter space-time the gravitational and electromagnetic equations decouple into two inhomogeneous scalar wave equations.

I. INTRODUCTION AND SUMMARY

This article extends the computational, structural, and conceptual streamlining of the Einstein field equations (linearized around a generic, in general matter-occupied, spherically symmetric space-time) from the pure Einstein system[1,2] to the coupled Einstein-Maxwell system. Coupled electromagnetic and gravitational perturbations are being investigated in an active way.[3-18] So far the interest has centered itself solely on vacuum background space-times, Reissner-Nordström space-time being the favorite. The analysis of this article does not suffer from this restriction. The approach of this article is naturally tailored to any spherical background. This allows one, therefore, to consider (a) the intercoupling between acoustical, electromagnetic, as well as gravitational degrees of freedom, and (b) the effect of the temporal or spatial structure of the background on the evolution (and structure) of any or all of these degrees of freedom. Thus, in regard to item (b), for example, one has at one's disposal the means to discover which type of background best facilitates the generation of gravitational radiation.

The discussion and classification of many physically and astrophysically interesting phenomena (involving the Einstein field equations linearized around spherical symmetry) has involved an inordinate expenditure of effort by a number of workers dealing with just the perturbation-theory formalism itself. The lack of speedy progress towards a detailed understanding of slightly aspherical relativistic configurations has been due to the lack of a sufficiently well-tailored perturbational formalism. This has led to claims that calculations in perturbation theory are typically long, tedious, and filled with long mathematical expressions, and at times are downright messy. Such an assessment does not, however, do justice to perturbation theory itself but rather reflects (i) the proliferation of *ad hoc* coordinate-dependent expressions with its concomitant difficulties on one hand and (ii) the ambiguities associated with infinitesimal coordinate (i.e., "gauge") transformations on the other.

Now gauge ambiguities in perturbations away from certain spherically symmetric space-times have been eliminated by Moncrief's explicit introduction of gauge invariants.[19,20] The equations describing perturbations on a Robertson-Walker space-time have also been given in terms of gauge invariants only.[21] But the perturbation theory reveals its most striking structural beauty, versatility, and ease of applicability when expressed in terms of geometrical objects that are gauge invariants. In fact, it is difficult to find formulations that require less investment to achieve equal or higher levels of applicability.

The formulation in terms of gauge-invariant geometrical objects[1,2] (tensors, vectors, and scalars on a two-dimensional space-time) captures the best of two worlds: namely, geometric formulation (neither a coordinate nor a background geometry commitment is necessary) and gauge invariance (representation of any perturbed quantity is invariant and hence unique with respect to infinitesimal coordinate changes).

The background geometry as well as the coordinate system used to describe it can be that of an arbitrarily selected spherically symmetric space-time. These space-times may be grouped into overlapping classes associated with the names (1) Reissner-Nordström,[22] (2) Bertotti-Robinson,[23] (3) Vaidya,[24] (4) Robertson-Walker,[22,25] (5) Kantowski-Sachs,[26] (6) de Sitter and anti-de Sitter,[22] or (7) some collapsing[27] or radially pulsating star with an asymptotically flat or de Sitter exterior.

The properties of matter, gravitational, and electromagnetic perturbations on several of the above-listed spherically symmetric space-times have been considered by many workers. Their works are best classified as to the manner in which the perturbational degrees of freedom intercouple among each other. The intercoupling is determined by the individual presence (or absence) of (a) background matter stress energy, (b) background electromagnetic field, or (c) background charge current.

Thus, in the absence of all three of these background quantities each perturbational degree of freedom evolves independently. This corresponds, for example, to the propagation of gravitational and electromagnetic waves in a Schwarzschild geometry.[29,30,19,31-34] On the other hand, energy might be fed by some matter sources into the gravitational or electromagnetic degrees of freedom. This corresponds, for example, to (i) radiation from an uncharged[30,35] or charged[34,36] particle falling into or passing by a Schwarzschild black hole, (ii) Machian effects of a slowly rotating uncharged[37,38] or charged[39] star, or (iii) star suffering from odd-parity perturbations such as differential rotations[40] or torsional oscillations.

If only the background matter stress-energy tensor is nonzero then there is only an intercoupling between the acoustical and the gravitational waves. An example of this is a star undergoing even-parity pressure[41,40] oscillations. If only the background electromagnetic tensor is nonzero, then electromagnetic and gravitational waves are intercoupled.[3-18] The associated coupled modes are very directly described in terms of a beating phenomenon, Faraday rotation, or both.[6] The source of such coupled modes may be an uncharged[42] or a charged particle falling into, say, a Reissner-Nordström black hole.

Nothing seems to be known when all three waves (gravitational, acoustical, electromagnetic) are intercoupled by virtue of both the matter as well as the electromagnetic (together with possibly the charge current) background being nonzero.

It is clear that any spherically symmetric space-time together with a gauge-invariant perturbation, projected onto a spacelike hypersurface, ordinarily represents a moving point in superspace.[28] The background space-times under consideration in this article have two Killing vectors. Consequently, if the spacelike hypersurface is compact, then a perturbation must satisfy additional initial-value constraints associated with these Killing vectors. This is due to the linearization instability.[43,44]

The body of the paper is arranged as follows. Various background tensor fields (metric, Maxwell, electromagnetic, and matter stress-energy) as well as the background Einstein and Maxwell field equations for any spherically symmetric space-time are given geometrically in reduced form on a two-dimensional manifold in Sec. II. In Sec. III the linearized Einstein field equations are given both for odd and even parity in reduced form in terms of gauge-invariant geometrical objects on the above-mentioned two-dimensional manifold. In Sec. IV the same is done for the linearized Maxwell equations, in Sec. V for the linearized electromagnetic stress-energy tensor, and in Sec. VI for the odd-parity coupled Einstein-Maxwell system, including the perturbed mass-energy and charge conservation law. In addition, if there is no charged background matter, the linearized field equations are reduced to three coupled scalar equations which are decoupled for a charged black hole in an asymptotically de Sitter space. Whereas Sec. VI considers the equations for $l \geq 2$, Sec. VII considers and decouples them in an uncharged matter background for the case $l = 1$. Section VIII presents the even-parity coupled Einstein-Maxwell system in terms of gauge-invariant geometrical objects only.

Notation: *Use lower-case letters for the background geometrical objects; indicate their perturbations by the prefix* Δ *and their gauge invariants by using capitals.* We apply this rule consistently to all geometrical objects except the Einstein tensor and the metric tensor $g_{\mu\nu}$, where we attempt to deviate as little as possible from the now well-established notation of Regge and Wheeler.[29]

II. GEOMETRY AND ELECTROMAGNETIC FIELD ON M^2: BACKGROUND

Consider a spherically symmetric space-time with a metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{AB}dx^A dx^B + r^2(x^C)[d\theta^2 + \sin^2\theta\, d\varphi^2], \tag{2.1}$$

a Maxwell field

$$\tfrac{1}{2}f_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2}f_{AB}dx^A \wedge dx^B + \tfrac{1}{2}f_{ab}dx^a \wedge dx^b, \tag{2.2}$$

a charge current

$$j_\mu dx^\mu = j_A dx^A , \tag{2.3}$$

and a stress-energy tensor

$$t_{\mu\nu} dx^\mu dx^\nu = t_{AB} dx^A dx^B + \tfrac{1}{2} t_a{}^a r^2(x^C)[d\theta^2 + \sin^2\theta\, d\varphi^2]. \tag{2.4}$$

Capital latin indices A, B, $C,\ldots$ refer to some as-yet-unspecified radial and time coordinates, while lower-case latin indices a, b, $c,\ldots$ refer to θ and φ. The functions $r(x^C)$, $g_{AB}(x^C)$, $f_{AB}(x^C)$, and $j_A(x^C)$ as well as the partial trace $t_b{}^b = t_2{}^2 + t_3{}^3$ are scalar, vector, and tensor fields on the totally geodesic submanifold M^2 spanned by $x^C (C = 0, 1)$. Covariant derivatives on M^2 and on the unit two-sphere (spanned by x^a) will be indicated by a vertical bar and a colon, respectively. Observe that spherical symmetry implies the vanishing of the angular part of the charge current. The off-diagonal elements (t_{aA} and f_{aA}) of any second-rank tensor vanish for the same reason.

The stress-energy tensor decomposes into a part due to matter and a part due to electromagnetism

$$t_{\mu\nu} = t^{\text{mat}}_{\mu\nu} + t^{\text{em}}_{\mu\nu} ,$$

where

$$t^{\text{em}}_{\mu\nu} = \frac{1}{4\pi}\left(f_{\mu\alpha} f_{\nu\beta} g^{\alpha\beta} - \tfrac{1}{4} f_{\alpha\beta} f^{\alpha\beta} g_{\mu\nu}\right). \tag{2.5}$$

Let

$$\tfrac{1}{2}\epsilon_{AB} dx^A \wedge dx^B = \tfrac{1}{2}(\det g_{CD})^{1/2}[AB] dx^A \wedge dx^B = |g|^{1/2} dx^0 \wedge dx^1$$

be the antisymmetric unit tensor on M^2. Any antisymmetric tensor, including the Maxwell field, on M^2 is a multiple of this unit tensor. Thus,

$$f_{AB} dx^A \wedge dx^B = f^e \epsilon_{AB} dx^A \wedge dx^B , \tag{2.6}$$

where

$$f^e = -\tfrac{1}{2} f_{CD} \epsilon^{CD}$$

is the radial electric field.

Similarly, if ϵ_{ab} is the antisymmetric unit tensor on the sphere of surface area $4\pi r^2$,

$$f_{ab} dx^a \wedge dx^b = f^{\text{mag}} \epsilon_{ab} dx^a \wedge dx^b ,$$

where

$$f^{\text{mag}} = \tfrac{1}{2} f_{cd} \epsilon^{cd}$$

is the radial magnetic field. The Maxwell field in a general spherically symmetric space-time is therefore

$$\tfrac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu = \tfrac{1}{2} f^e \epsilon_{AB} dx^A \wedge dx^B + \tfrac{1}{2} f^{\text{mag}} \epsilon_{ab} dx^a \wedge dx^b . \tag{2.7}$$

It follows that on a spherical space-time the electromagnetic stress-energy tensor, Eq. (2.5), has the form

$$t^{\text{em}}_{\mu\nu} dx^\mu dx^\nu = \frac{-1}{8\pi}[(f^e)^2 + (f^{\text{mag}})^2] g_{AB} dx^A dx^B + \frac{1}{8\pi}[(f^e)^2 + (f^{\text{mag}})^2] g_{ab} dx^a dx^b . \tag{2.8}$$

It turns out that if the radial magnetic field (due to magnetic monopoles, say) is nonvanishing, then the parity of the electromagnetic and gravitational perturbations is not conserved. In other words, there a radial magnetic field brings about a coupling between even- and odd-parity modes, which are described by spherical harmonics of order l, m. Instead of decoupling them after the linearized Maxwell-Einstein system has been written down, it is simpler to first do a duality transformation:

$$\bar{f}^e = f^e \cos\alpha + f^{\text{mag}} \sin\alpha ,$$

$$\bar{f}^{\text{mag}} = -f^e \sin\alpha + f^{\text{mag}} \cos\alpha ,$$

which changes the background to an "extremal" one,[45] i.e., one having no radial magnetic field. Linearize the Maxwell-Einstein system around it, obtain the even and odd normal modes, and then finally perform the inverse duality transformation on them to obtain the coupled even- and odd-parity modes, which solve the original linearized system under consideration. In this paper we shall only consider the Maxwell-Einstein perturbation problem on an extremal background, i.e., one where the background tensors are

$$\tfrac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu = \tfrac{1}{2} f \epsilon_{AB} dx^A \wedge dx^B , \tag{2.9}$$

$$t_{\mu\nu} dx^\mu dx^\nu = \frac{-1}{8\pi} f^2 g_{AB} dx^A dx^B + \frac{1}{8\pi} f^2 g_{ab} dx^a dx^b . \tag{2.10}$$

Let

$$v_A = r_{,A}/r$$

be the vector field on M^2 constructed from the scalar $r(x^C)$, which characterizes the concentric spheres. The Einstein field equations for any spherically symmetric space-time assume the form of one tensor and one scalar equation on M^2,

$$8\pi t_{AB} = -2(v_{A|B} + v_A v_B) + (2v_C{}^{|C} + 3v_C v^C - r^{-2}) g_{AB} \equiv G_{AB} , \tag{2.11}$$

$$8\pi\tfrac{1}{2}t_a{}^a = v_C{}^{|C} + v_C v^C - \mathcal{R} \equiv \tfrac{1}{2}G_a{}^a . \qquad (2.12)$$

The conservation law implied by these equations is

$$r^{-2}(r^2 t_A{}^B)_{|B} - v_A t_a{}^a = 0 .$$

The vertical bars refer to covariant derivatives on M^2, and $\mathcal{R}$ to its Gaussian curvature defined by

$$w_{A|B|C} - w_{A|C|B} = w_D R^D{}_{ABC}$$
$$= w_D \mathcal{R}(\delta^D_B g_{AC} - \delta^D_C g_{AB}) . \qquad (2.13)$$

The stress-energy tensor is expressed in terms of geometrical units (length^{-2}). The Maxwell equations are

$$f^{\mu\nu}{}_{;\nu} = 4\pi j^\mu , \qquad (2.14)$$

$$f_{[\alpha\beta;\gamma]} = 0 , \qquad (2.15)$$

and the Maxwell tensor as well as the current vector are also expressed in term of geometrical units (length^{-1}). In a spherically symmetric space-time they are satisfied by expressions (2.2) and (2.3). The only nontrivial equations are

$$(r^2 f^{AB})_{|B} = 4\pi r^2 j^A , \qquad (2.16a)$$

$$f_{[ab,C]} = 0 . \qquad (2.16b)$$

For the extremal field Eq. (2.9), Eq. (2.16b) is satisfied identically. Using Eq. (2.9) and the fact that $\epsilon^{AB}{}_{|C} = 0$ the equation becomes

$$\epsilon^{AB}(r^2 f)_{,B} = 4\pi r^2 j^A . \qquad (2.17)$$

The charge conservation equation implied by these Maxwell equations is

$$(r^2 j^A)_{|A} = 0 .$$

The field equations and their conservation laws thus formulated, incorporate only the spherical symmetry of space-time. They do not make any coordinate or background geometry commitment as is done explicitly or implicitly in standard formulations. This can clearly be a great advantage if one wishes to let the equations speak for themselves as to what, for example, the most natural coordinate chart should be.

III. THE PERTURBED EINSTEIN EQUATIONS

Perturbations away from a given background geometry are determined by

$$(h_{\mu\nu;\alpha}{}^{;\alpha} - h_{\mu\alpha;\nu}{}^{;\alpha} - h_{\nu\alpha;\mu}{}^{;\alpha} + h_\alpha{}^\alpha{}_{;\mu;\nu}) + g_{\mu\nu}(h_{\alpha\beta}{}^{;\alpha;\beta} - h_\alpha{}^\alpha{}_{;\beta}{}^{;\beta} - h_{\alpha\beta}R^{\alpha\beta}) + h_{\mu\nu}R = -16\pi\Delta t_{\mu\nu} . \qquad (3.1)$$

For a general spherically symmetric background metric (2.1) and stress-energy tensor (2.4) these perturbations can be expanded in terms of spherical harmonics characterized by the integers l, m.

Suppressing these angular integers, one has for odd-parity perturbations

$$h_{\mu\nu}dx^\mu dx^\nu = h_A(x^C)S_a(\theta,\varphi)(dx^A dx^a + dx^a dx^A) + h(x^C)(S_{a:b} + S_{b:a})dx^a dx^b \quad \text{(metric)} \qquad (3.2)$$

and

$$\Delta t_{\mu\nu}dx^\mu dx^\nu = \Delta t_A(x^C)S_a(\theta,\varphi)(dx^A dx^a + dx^a dx^A) + \Delta t(x^C)(S_{a:b} + S_{b:a})dx^a dx^b \quad \text{(stress-energy)} , \qquad (3.3)$$

where the expansion coefficients h_A, h, Δt_A, and Δt have the usual geometrical significance on M^2, the totally goedesic submanifold spanned by $x^C (C = 0, 1)$. The odd-parity geometrical gauge-invariant perturbation objects are[1]

$$k_A = h_A - r^2(h/r^2)_{,A} \quad \text{(metric)} , \qquad (3.4)$$

$$T_A = \Delta t_A - (t_a{}^a/2)h_A \quad \text{(stress-energy)} , \qquad (3.5a)$$

$$T = \Delta t - (t_a{}^a/2)h \quad \text{(stress-energy)} . \qquad (3.5b)$$

The odd-parity linearized Einstein field equations are

$$(S_{a:b} + S_{b:a}):\ k_A{}^{|A} = 16\pi T \quad (l \geq 2) , \qquad (3.6a)$$

$$S_a:\ -[r^4(r^{-2}k^A)^{|C} - r^4(r^{-2}k^C)^{|A}]_{|C} + (l-1)(l+2)k^A = 16\pi r^2 T^A \quad (l \geq 1) . \qquad (3.6b)$$

Similarly, for even-parity perturbations one has

$$h_{\mu\nu}dx^\mu dx^\nu = h_{AB}Y dx^A dx^B + h_A Y_{,a}(dx^A dx^a + dx^a dx^A) + (r^2 K Y \gamma_{ab} + r^2 G Y_{,a:b})dx^a dx^b \quad \text{(metric)} \qquad (3.7)$$

and[46]

$$\Delta t_{\mu\nu}dx^\mu dx^\nu = \Delta t_{AB}Y dx^A dx^B + \Delta t_A Y_{,a}(dx^A dx^a + dx^a dx^A) + (r^2\Delta t^1 Y \gamma_{ab} + \Delta t^2 Y_{,a:b})dx^a dx^b \quad \text{(stress-energy)} . \qquad (3.8)$$

The even geometric perturbation objects are

$$\left.\begin{aligned} k_{AB}&=h_{AB}-p_{A|B}-p_{B|A},\\ k&=K-2v^C p_C \end{aligned}\right\}\ \text{(metric)},\tag{3.9}$$

$$T_{AB}=\Delta t_{AB}-t_{AB|C}p^C-t_A{}^C p_{C|B}-t^C{}_B p_{C|A},\tag{3.10a}$$

$$T_A=\Delta t_A-t_A{}^C p_C-r^2(t_a{}^b/4)G_{,A},\tag{3.10b}$$

$$T^1=\Delta t^1-(p^C/r^2)(r^2 t_a{}^a/2)_{,C},\quad\text{(stress-energy)},\tag{3.10c}$$

$$T^2=\Delta t^2-(r^2 t_a{}^a/2)G\tag{3.10d}$$

where

$$p_C=h_C-\tfrac{1}{2}r^2 G_{,C}.$$

The reduced even-parity linearized Einstein field equations are obtained by first substituting Eqs. (3.7) and (3.8) into Eq. (3.1) and then by equating the coefficients of the linearly independent harmonics

$$\{Y,\ Y_{,a},\ Y\gamma_{ab},\ Y_{,a:b}\}\tag{3.11}$$

on the left-hand side to the corresponding coefficients on the right-hand side. This has been done for $l\geq 2$ in Ref. 1, where the corresponding equations on M^2 are given. However, for $l=0,1$ the set of harmonics is not linearly independent. In fact,

$$Y_{,a:b}+\tfrac{1}{2}l(l+1)Y\gamma_{ab}=0\quad\text{for } l=0,1\,,$$

as one can easily verify. This is remedied by a slight change in basis functions in Eq. (3.8),

$$\Delta t_{\mu\nu}dx^\mu dx^\nu=\Delta t_{AB}Y\,dx^A dx^B+\Delta t_A Y_{,a}(dx^A dx^a+dx^a dx^A)+r^2\Delta t^3 Y\gamma_{ab}dx^a dx^b+\Delta t^2[Y_{,a:b}+\tfrac{1}{2}l(l+1)Y\gamma_{ab}]dx^a dx^b.$$

This is the orthogonal basis used by Zerilli.[30] The stress-energy gauge-invariant perturbation objects with respect to this basis are the same as those given by Eqs. (3.10a)–(3.10d) except for the first scalar (3.10c):

$$\left.\begin{aligned} T_{AB}&=\Delta t_{AB}-t_{AB|C}p^C-t_A{}^C p_{C|B}-t_B{}^C p_{C|A},\\ T_A&=\Delta t_A-t_A{}^C p_C-r^2(t_a{}^a/4)G_{,A},\\ T^3&=\Delta t^3-(p^C/r^2)(r^2 t_a{}^a/2)_{,C}+l(l+1)(t_a{}^a/4)G\\ &=T^1-\frac{1}{2}\frac{l(l+1)}{r^2}T^2,\\ T^2&=\Delta t^2-(r^2 t_a{}^a/2)G \end{aligned}\right\}\ \text{(stress-energy)}.\tag{3.12}$$

The corresponding perturbed Einstein field equations are[47,48,49]

$$\begin{aligned} Y:\ &2v^C(k_{AB|C}-k_{CA|B}-k_{CB|A})-[l(l+1)/r^2+G_C{}^C+G_a{}^a+2\mathcal{R}]k_{AB}-2g_{AB}v^C(k_{ED|C}-k_{CE|D}-k_{CD|E})g^{ED}\\ &+g_{AB}(2v^{C|D}+4v^C v^D-G^{CD})k_{CD}+g_{AB}[l(l+1)/r^2+\tfrac{1}{2}(G_C{}^C+G_a{}^a)+\mathcal{R}]k_D{}^D+2(v_A k_{,B}+v_B k_{,A}+k_{,A|B})\\ &-g_{AB}\left[2k_{,C}{}^{|C}+6v^C k_{,C}-\frac{(l-1)(l+2)}{r^2}k\right]=-16\pi T_{AB}, \end{aligned}\tag{3.13a}$$

$$Y_{,a}:\ k_{,A}-k_{AC}{}^{|C}+k_C{}^C{}_{|A}-v_A k_C{}^C=-16\pi T_A,\tag{3.13b}$$

$$\begin{aligned} Y\gamma_{ab}:\ &-(k_{,C}{}^{|C}+2v^C k_{,C}+G_a{}^a k)+[k_{CD}{}^{|C|D}+2v^C k_{CD}{}^{|D}+2(v^{C|D}+v^C v^D)k_{CD}]\\ &-\left[k_C{}^C{}_{|D}{}^{|D}+v^C k_D{}^D{}_{|C}+\mathcal{R}k_C{}^C-\frac{l(l+1)}{2r^2}k_C{}^C\right]=-16\pi T^3, \end{aligned}\tag{3.13c}$$

$$Y_{,a:b}+\tfrac{1}{2}l(l+1)Y\gamma_{ab}:\ k_C{}^C=-16\pi T^2.\tag{3.13d}$$

Consider the linearized conservation equation $\Delta(t_{\mu\nu}{}^{;\nu})=0$. It is in fact implied by the linearized field equations. It consists of a scalar and a vector equation, namely,[50]

$$Y_{,a}:\ r^{-2}(r^2 T^A)_{|A}+T^3-\frac{(l-1)(l+2)}{2r^2}T^2=\tfrac{1}{2}t_a{}^a(k-\tfrac{1}{2}k_G{}^G)+\tfrac{1}{2}t^{AB}k_{AB},\tag{3.14a}$$

$$Y:\; r^{-2}(r^2 T_{AB})^{|B} - T_A l(l+1)/r^2 - 2v_A T^3 = \tfrac{1}{2}k_{BC|A}t^{BC} + k_{CB}{}^{|B}t^C{}_A - \tfrac{1}{2}k_C{}^C{}_{|B}t^B{}_A - k_{,C}t^C{}_A + \tfrac{1}{2}(k_{,A} - kv_A)t_a{}^a + 2v^B k_{BC}t^C{}_A + k^B{}_c t^c{}_{A|B}. \tag{3.14b}$$

These equations are identities if one uses the background field Eqs. (2.11), (2.12), the linearized field Eqs. (3.13), and the identity

$$(k_{AB|C} - k_{CA|B} - k_{CB|A})^{|C} + k_C{}^C{}_{|A|B} + g_{AB}(k_{CD}{}^{|C|D} - k_C{}^C{}_{|D}{}^{|D}) = R^C{}_{ADB}(2k_C{}^D - k_E{}^E\delta_C{}^D) = \mathcal{R}(k_E{}^E g_{AB} - k_{AB}).$$

IV. PERTURBED MAXWELL FIELD EQUATIONS

The linearized Maxwell equations together with the linearized charge conservation equation are

$$\Delta f_{\alpha\nu}{}^{;\nu} = 4\pi\Delta j_\alpha + f_{\alpha\mu;\nu}h^{\mu\nu} + f^{\mu\nu}h_{\alpha\mu;\nu} + f_\alpha{}^\mu(h_\mu{}^\nu{}_{;\nu} - \tfrac{1}{2}h^\nu{}_{\nu;\mu}), \tag{4.1}$$

$$\Delta f_{[\mu\nu;\sigma]} = 0, \tag{4.2}$$

$$\Delta j_\mu{}^{;\mu} = j^{\mu;\nu}h_{\mu\nu} + j^\nu h_{\nu\mu}{}^{;\mu} - \tfrac{1}{2}j^\nu h^\mu{}_{\mu;\nu}. \tag{4.3}$$

They govern perturbations in the electromagnetic (em) field and in the charge current. For a general spherically symmetric background metric (2.1), em field (2.2), and charge current (2.3), these perturbations can be expanded in terms of spherical harmonics characterized by the integers l, m.

A. Odd parity

Suppressing these integers, one has for odd-parity perturbations

$$\tfrac{1}{2}\Delta f_{\mu\nu}dx^\mu \wedge dx^\nu = \Delta f_A(x^C)S_a\,dx^A \wedge dx^a + \Delta f(S_{b:a} - S_{a:b})dx^a \wedge dx^b, \tag{4.4}$$

$$\Delta j_\mu dx^\mu = \Delta j(x^C)S_a\,dx^a. \tag{4.5}$$

The corresponding gauge invariants are very easy to obtain because Δf_A, Δf, and Δj are unaltered by an infinitesimal odd-parity background coordinate transformation. The odd-parity gauge-invariant geometrical perturbation objects are simply

$$\left.\begin{array}{l} F_A = \Delta f_A, \\ F = \Delta f \end{array}\right\}\ \text{(Maxwell)}, \tag{4.6}$$

$$J = \Delta j \quad \text{(current)}, \tag{4.7}$$

and they satisfy the linearized Eqs. (4.1)–(4.3),

$$-F_A{}^{|A} + \frac{l(l+1)}{r^2}F - r^2 f^{AB}(r^{-2}k_A)_{|B} = 4\pi J, \quad l \geq 1 \tag{4.8a}$$

$$F_{A|B} - F_{B|A} = 0, \quad l \geq 1 \tag{4.8b}$$

$$F_A = F_{,A}, \quad l \geq 1. \tag{4.8c}$$

The perturbed charge conservation Eq. (4.3) is trivially satisfied.

B. Even parity

Even-parity Maxwell and current perturbations have the form (suppress the angular integers l and m)

$$\tfrac{1}{2}\Delta f_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2}\Delta f_{AB}(x^C)Y dx^A \wedge dx^B + \Delta f_A(x^C)Y_{,a}dx^A \wedge dx^a + \tfrac{1}{2}\Delta f Y\epsilon_{ab}dx^a \wedge dx, \tag{4.9}$$

$$\Delta j_\mu dx^\mu = \Delta j_A(x^C)Y dx^A + \Delta j(x^C)Y_{,a}dx^a. \tag{4.10}$$

The corresponding gauge-invariant geometrical perturbation objects are therefore (see Appendix for details)

$$\left.\begin{array}{l} F_{AB} = \Delta f_{AB} - f_{AB}{}^{|C}p_C - f^C{}_B p_{C|A} - f_A{}^C p_{C|B}, \\ F_A = \Delta f_A - f_A{}^C p_C, \\ F = \Delta f \end{array}\right\}\ \text{(Maxwell)}, \tag{4.11}$$

$$\left.\begin{array}{l} J_A = \Delta j_A - j_{A|C}p^C - j^C p_{C|A}, \\ J = \Delta j - j^A p_A \end{array}\right\}\ \text{(current)}. \tag{4.12}$$

The gauge invariants for the metric perturbations are

$$\left.\begin{array}{l} k_{AB} = h_{AB} - p_{A|B} - p_{B|A}, \\ k = K - 2v^C p_C \end{array}\right\}\ \text{(metric)}. \tag{4.13}$$

In terms of these objects the linearized Maxwell equations (4.1) and (4.2) are

$$Y:\; (r^2F_{AB})^{|B} - l(l+1)F_A = 4\pi r^2 J_A + (r^2 f_{AB})_{|C}k^{BC} + \tfrac{1}{2}r^2 f^{BC}(k_{BA|C} + k_{BC|A} - k_{AC|B}) + \tfrac{1}{2}r^2 f_A{}^C g^{BD}(k_{CB|D} + k_{CD|B} - k_{BD|C}) - r^2 f_A{}^C k_{,C}, \quad l \geq 0 \tag{4.14a}$$

$$Y_{,a}:\; -F_B{}^{|B} = 4\pi J, \quad l \geq 1 \tag{4.14b}$$

$$Y_{,a}:\; F_{AB} = F_{A|B} - F_{B|A}, \quad l \geq 1 \tag{4.14c}$$

$$Y\epsilon_{ab}:\; F_{|A} = 0, \quad l \geq 0. \tag{4.14d}$$

The linearized conservation equation (4.3) is

$$[r^2(J_A - j^C k_{CA})]^{|A} - l(l+1)J + j^A[k + \tfrac{1}{2}k_C{}^C]_{,A} = 0\,, \quad l \geq 0\,. \tag{4.15}$$

It is implied by Eqs. (4.14).

V. THE PERTURBED ELECTROMAGNETIC STRESS-ENERGY TENSOR

The linearized Einstein equations (3.1) are coupled to the linearized Maxwell equations (4.1) and (4.2) by virtue of the fact that, in the presence of a nonzero background electromagnetic field $f_{\mu\nu}$, first-order deviations $\Delta f_{\mu\nu}$ away from $f_{\mu\nu}$ result in first-order deviations $\Delta t_{\mu\nu}$ away from the em stress-energy tensor

$$t_{\mu\nu} = \frac{1}{4\pi}(f_{\mu\alpha} f_\nu{}^\alpha - \tfrac{1}{4} g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta})\,. \tag{5.1}$$

In fact, one has

$$\Delta t_{\mu\nu} = \frac{1}{4\pi}[\Delta f_{\mu\alpha} f_\nu{}^\alpha + \Delta f_{\nu\alpha} f_\mu{}^\alpha - f_\mu{}^\alpha f_\nu{}^\beta h_{\alpha\beta} - \tfrac{1}{4} h_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta} - \tfrac{1}{2} g_{\mu\nu} \Delta(f_{\alpha\beta} f^{\alpha\beta})]\,. \tag{5.2}$$

Focus, as usual, on a particular harmonic mode characterized by the integers l and m, which are, as usual, suppressed. The to-be-determined odd- and even-parity gauge-invariant stress-energy objects, Eqs. (3.5) and (3.10), must now be expressed in terms of the odd- and even-parity gauge-invariant electromagnetic objects, Eqs. (4.6) and (4.11), as well as in terms of the corresponding gauge-invariant metric objects, Eqs. (3.4) and (3.9).

The procedure is particularly easily accomplished for odd-parity modes. The expansion coefficients of Eq. (3.3) are obtained from (5.2) with the help of (4.4) and the fact that

$$f_{\mu\nu} dx^\mu dx^\nu = f_{AB} dx^A \wedge dx^B\,.$$

The result is

$$4\pi \Delta t_A = -f_A{}^C \Delta f_C\,,$$

$$4\pi \Delta t = \tfrac{1}{4} h f_{\alpha\beta} f^{\alpha\beta} = -\tfrac{1}{2} h t_a{}^a\,.$$

Substitute these expressions into Eqs. (3.5), take note of Eqs. (4.6), and obtain

$$\left.\begin{aligned} T_A &= \frac{-1}{4\pi} f_A{}^C F_C\,, \\ T &= 0 \end{aligned}\right\}\text{(odd-parity electromagnetic stress-energy)}\,. \tag{5.3}$$

This is the odd-parity Maxwell source that drives the Einstein perturbation. The odd-parity linearized Maxwell-Einstein system is therefore given by Eqs. (6.1) below.

For even parity the construction is analogous. The result is that the even-parity stress-energy gauge invariants, Eqs. (3.12), expressed in terms of the electromagnetic ones, Eqs. (4.11), are

$$\left.\begin{aligned} &4\pi T_{AB} = F_{AC} f_B{}^C + F_{BC} f_A{}^C - k_{CD} f^C{}_A f^D{}_B - \tfrac{1}{4} k_{AB} f_{CD} f^{CD} - \tfrac{1}{4} H g_{AB}\,, \quad l \geq 0 \\ &4\pi T_A = -f_A{}^C F_C\,, \quad l \geq 1 \\ &4\pi T^2 = 0\,, \quad l \geq 2 \\ &4\pi T^3 = 4\pi T^1 = -\tfrac{1}{4} k f_{CD} f^{CD} - \tfrac{1}{4} H\,, \quad l \geq 0 \end{aligned}\right\}\text{(even-parity electromagnetic stress-energy)}\,, \tag{5.4}$$

where

$$H = 2(F_{CD} f^{CD} - k_{DE} f^{DC} f^E{}_C)$$

is the gauge-invariant perturbation scalar of $f_{\alpha\beta} f^{\alpha\beta}$.

On M^2 the antisymmetric background Maxwell tensor f_{AB}, Eq. (2.6), and the gauge-invariant perturbation tensor F_{AB}, Eq. (4.14c), can be expressed in terms of scalars as follows:

$$f_{AB} = f\epsilon_{AB}\,,$$

$$F_{AB} = -F^{C|D}\epsilon_{CD}\epsilon_{AB}\,.$$

Making use on M^2 of the identity

$$\epsilon_{AC}\epsilon^{CB} = \delta_A{}^B\,,$$

the electromagnetic stress-energy gauge invariants Eqs. (5.4) reduce to

$$\left.\begin{aligned}&4\pi T_{AB}=F^{C|D}\epsilon_{CD}fg_{AB}+\tfrac{1}{2}f^2(k_{AB}+k_C{}^Cg_{AB}-2k_{CD}\epsilon^C{}_A\epsilon^D{}_B)\,,\quad l\geq 0\\&4\pi T_A=-F^C\epsilon_{CA}f\,,\quad l\geq 1\\&4\pi T^2=0\,,\quad l\geq 2\\&4\pi T^3=F^{C|D}\epsilon_{CD}f+\tfrac{1}{2}f^2(k-k_C{}^C)\,,\quad l\geq 0\end{aligned}\right\}\text{(even-parity electromagnetic stress-energy)}\,. \quad (5.5)$$

VI. THE COUPLED LINEARIZED EINSTEIN-MAXWELL SYSTEM

To obtain the equations governing a given odd-parity perturbation mode of the metric and the electromagnetic field, insert (5.3) into (3.6), and (4.8c) into (4.8a). The result is evidently

$$k_C^{|C}=16\pi T^{\rm mat}\,,\quad l\geq 2 \quad (6.1a)$$

$$-[r^4(r^{-2}k_A)_{|C}-r^4(r^{-2}k_C)_{|A}]^{|C}+(l-1)(l+2)k_A+4r^2f_A{}^CF_{,C}=16\pi r^2T_A^{\rm mat}\,,\quad l\geq 1 \quad (6.1b)$$

$$-F_{,A}{}^{|A}+\frac{l(l+1)}{r^2}F-r^2f^{AB}(r^{-2}k_A)_{|B}=4\pi J\,,\quad l\geq 1\,. \quad (6.2)$$

Here $T^{\rm mat}$ and $T_A^{\rm mat}$ include all those stress-energy gauge-invariant perturbations which are not electromagnetic perturbations. The perturbed conservation equation

$$\Delta(t^{\mu\nu}_{\rm mat}+t^{\mu\nu}_{\rm em})_{;\nu}=0$$

for odd parity is ($l\geq 1$)

$$(r^2T_A^{\rm mat})^{|A}-(l-1)(l+2)T^{\rm mat}=r^2j^AF_{,A}\,. \quad (6.3)$$

These are, in fact, implied by Eqs. (6.1) and Maxwell's equation (2.16a). Equation (6.3) is the ("Lorentz") equation of motion for odd-parity matter perturbations.

The odd-parity perturbation equations (6.1)–(6.3) can be given a simplified and more transparent form by expressing them in terms of Cartan's calculus of the differential forms.

A. Odd-parity coupled scalar modes

In order to solve the coupled system of Eqs. (6.1)–(6.3), express all relevant dynamical wave fields in terms of dimensionless scalar functions. The gravitational degree of freedom is expressed in terms of the covector field

$$k_Adx^A$$

on the manifold M^2. There are two scalars that can be constructed from k_Adx^A and which each give rise directly to its scalar master equations. They are first of all[1]

$$\Pi=*d(r^{-2}k_Adx^A)=-\epsilon^{AC}(r^{-2}k_A)_{|C}\,, \quad (6.4)$$

the Hodge dual of $d(r^{-2}k_Adx^A)$ with respect to $g_{AB}dx^Adx^B$ and a specified orientation of M^2. A second scalar is Φ with the property

$$d\Phi=*k_Adx^A+\text{other (matter-associated) terms}\,.$$

In terms of the Hodge duals Eq. (6.4) and

$$*k_Adx^A=k^A\epsilon_{AC}dx^C\,,$$

and with the help of Eq. (2.6), the coupled equations (6.1)–(6.3) assume the form[51]

$$-*d*k_Adx^A=16\pi T^{\rm mat}\,,\quad l\geq 2\ \text{(Einstein)} \quad (6.5a)$$

$$*d[r^4*d(r^{-2}k_Adx^A)]+(l-1)(l+2)k_Adx^A-4r^2f*dF=16\pi r^2T_A^{\rm mat}dx^A\,,\quad l\geq 1\ \text{(Einstein)} \quad (6.5b)$$

$$*d*dF+\frac{l(l+1)}{r^2}F+r^2f*d(r^{-2}k_Adx^A)=4\pi J\,,\quad l\geq 1\ \text{(Maxwell)} \quad (6.6)$$

$$*d(r^2*T_A^{\rm mat}dx^A)+(l-1)(l+2)T^{\rm mat}=r^2j_Adx^A\wedge *dF\,,\quad l\geq 1\ \text{(Lorentz)}\,. \quad (6.7)$$

Alternatively, the (co)vectorial linearized Einstein equation (6.5b) could be converted into a scalar equation by operating on it with the operator $*dr^{-2}$ and then using Eq. (6.4). The result is

$$*d*r^{-2}dr^4\Pi+(l-1)(l+2)\Pi-4*d(f*dF)=16\pi*d(T_A^{\rm mat}dx^A)\,. \quad (6.5b')$$

Equations (6.5b′) and (6.7) describe the coupled acousto-electro-gravitational modes on a generic spherically symmetric space-time.

B. Coupled modes in space permeated by uncharged matter: $l\geq 2$

Let us now assume that the spherically symmetric background is permeated by a radial electric field, but that the matter permeating the space is uncharged

$$j_Cdx^C=0\,.$$

Consequently, the background Maxwell field equation (2.17) yields

$$(r^2f)_{,A}=0\,.$$

Thus, the radial background electric field has the well-known form

$$f = e/r^2 \,. \tag{6.8}$$

In the presence of such a special background field the Einstein Eq. (6.5b) assumes the form

$$d[r^4 {*d}(r^{-2} k_A dx^A) - 4eF] = -(l-1)(l+2) {*k_A} dx^A + 16\pi r^2 {*T_A^{\text{mat}}} dx^A$$

and dictates therefore that the right-hand side is the exterior derivative of a scalar, say $(l-1) \times(l+2)\Phi$. Thus,

$$d\Phi = {*k_A} dx^A - 16\pi[(l-1)(l+2)]^{-1} r^2 {*T_A^{\text{mat}}} dx^A \,, \tag{6.9}$$

where

$$(l-1)(l+2)\Phi = r^4 {*d}(r^{-2} k_A dx^A) - 4eF \,. \tag{6.10}$$

The idea now is to consider the scalars Φ and F as the new descriptors of the gravitational and electromagnetic degrees of freedom, instead of $k_A dx^A$ and F. Eliminating $k_A dx^A$ between Eqs. (6.9) and (6.10) yields therefore

$$-r^4 {*d}(r^{-2} {*d\Phi}) + (l-1)(l+2)\Phi = -4eF + 16\pi[(l-1)(l+2)]^{-1} r^4 {*dT_A}^{\text{mat}} dx^A \,. \tag{6.11}$$

It is advantageous to cast this Einstein equation into a form that has its second-order differential operator the same as that of the Maxwell field Eq. (6.6). This is achieved by letting

$$\Phi = r\Psi \,. \tag{6.12}$$

Thus

$$\begin{aligned} {*d}(r^{-2} {*d\Phi}) &= {*d}(r^{-2} {*d}(r\Psi)) \\ &= -[r^{-2}(r\psi)_{,C}]^{|C} \\ &= -r^{-1}\Psi_{,C}{}^{|C} - r^{-2}(v_C{}^{|C} - v_C v^C)\psi \,, \end{aligned}$$

where

$$v_C = r_{,C}/r \,.$$

Thus, the Einstein Eq. (6.11) becomes

$${*d*d\psi} + r^{-2}[v_C v^C - v_C{}^{|C} + l(l+1) - 2]\psi = -\frac{4e}{r^3}F + 16\pi[(l-1)(l+2)]^{-1} r {*dT_A^{\text{mat}}} dx^A \quad \text{(Einstein)} \,. \tag{6.13}$$

The Maxwell Eq. (6.6), using Eqs. (6.8), (6.10), and (6.12) is

$${*d*dF} + r^{-2}\left[l(l+1) + \frac{4e^2}{r^2}\right]F = -\frac{e}{r^3}(l-1)(l+2)\psi + 4\pi J \quad \text{(Maxwell)} \,. \tag{6.14}$$

The conservation Eq. (6.7) for the type of background under consideration is

$${*d}(r^2 {*T_A^{\text{mat}}} dx^A) + (l-1)(l+2)T^{\text{mat}} = 0 \quad \text{(matter)} \,. \tag{6.15}$$

Equations (6.13)–(6.15) are the odd-parity equations that govern coupled gravitational, electromagnetic, as well as matter perturbations ($l \geq 2$) away from any spherically symmetric space-time background occupied by uncharged matter.

C. Coupled modes in charged asymptotically de Sitter space

Let, for example, the background be a Reissner-Nordström black hole in an asymptotically de Sitter space. Then the metric on M^2 is

$$g_{AB} dx^A dx^B = -g(r) dt^2 + \frac{dr^2}{g(r)} \,, \tag{6.16}$$

with

$$g(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2 \,.$$

The equations for the decoupled normal modes can be obtained easily. Indeed for the metric (6.16),

$$v_C v^C - v_C{}^{|C} = \frac{2}{r^2} - \frac{6m}{r^3} + \frac{4e^2}{r^4} \,.$$

Thus, Eqs. (6.13) and (6.14) become

$${*d*\Psi} + \frac{1}{r^2}\left[l(l+1) - \frac{3m}{r} + \frac{4e^2}{r^2}\right]\Psi = \frac{1}{r^3}[3m\Psi - 4eF] + 16\pi[(l-1)(l+2)]^{-1} r {*d}(T_A^{\text{mat}} dx^A) \,,$$

$${*d*dF} + \frac{1}{r^2}\left[l(l+1) - \frac{3m}{r} + \frac{4e^2}{r^2}\right]F = \frac{1}{r^3}[-e(l-1)(l+2)\Psi - 3mF] + 4\pi J \,.$$

By setting

$$\hat{\Psi} = \lambda \Psi ,$$

where

$$\lambda = \tfrac{1}{2}[(l-1)(l+2)]^{1/2} ,$$

these equations have the form

$$*d*d \begin{pmatrix} \hat{\Psi} \\ F \end{pmatrix} + \frac{1}{r^2}\left[l(l+1) - \frac{3m}{r} + \frac{4e^2}{r^2}\right]\begin{pmatrix} \hat{\Psi} \\ F \end{pmatrix} - \frac{1}{r^3}\begin{bmatrix} 3m & -4e\lambda \\ -4e\lambda & -3m \end{bmatrix}\begin{pmatrix} \hat{\Psi} \\ F \end{pmatrix} = \begin{pmatrix} 8\pi\lambda^{-2} r *dT_A^{\text{mat}} dx^A \\ 4\pi J \end{pmatrix} , \tag{6.17}$$

which can be easily separated into the normal modes by means of an orthogonal transformation. Note that the de Sitter aspect of the geometry makes itself felt on the perturbations not through the potential but rather through the nature of the second-order operator $*d*d$. It reduces essentially to Moncrief's result[52] if $\Lambda = 0$.

VII. THE ODD-PARITY CASE: $l = 1$

For such a mode the three independent equations are given by Eqs. (6.5b), (6.6), and (6.7). If one specializes now to the case where the background charge-current vector $j_C dx^C$ vanishes, then Eq. (6.7) implies that $r^2 * T_A^{\text{mat}} dx^A$ is closed. Consequently, there exists a scalar S such that

$$dS = r^2 * T_A^{\text{mat}} dx^A . \tag{7.1}$$

In addition, Eq. (6.8) holds. It follows from Eqs. (7.1) and (6.8) that the Einstein Eq. (6.5b) can be integrated to give

$$*d(r^{-2} k_A dx^A) = \frac{4e}{r^4} F + 16\pi S , \quad l = 1 \text{ (Einstein)} . \tag{7.2}$$

Introduce this equation into the Maxwell equation (6.6) to eliminate reference to the gravitational perturbation $k_A dx^A$, and obtain

$$*d*F + \left[\frac{2}{r^2} + \frac{4e^2}{r^4}\right] F = 4\pi[J - 4eS] , \quad l = 1 \text{ (Maxwell)} . \tag{7.3}$$

This equation determines the electromagnetic dipole wave field.

Objections against the gauge invariance of the Einstein equation (7.2) can be raised. Indeed the gauge-invariant Eq. (3.4)

$$r^{-2} k_A = r^{-2} h_A - (h/r^2)_{,A} \tag{7.4}$$

is not a gauge invariant at all when $l = 1$. In fact, for that angular mode, $S_{a:b} + S_{b:a} = 0$, and the metric perturbation Eq. (3.2) has only the form

$$h_{\mu\nu} dx^\mu dx^\nu = h_A S_a (dx^A dx^a + dx^a dx^A) .$$

Under an infinitesimal coordinate transformation induced by

$$\xi_\mu dx^\mu = \xi S_a dx^a ,$$

the perturbation (multiplied by r^{-2}) becomes

$$r^{-2} \bar{h}_A = r^{-2} h_A - (\xi/r^2)_{,A} , \tag{7.5}$$

and no gauge invariant such as Eq. (7.4) can therefore be constructed from h_A. The h, necessary for this construction,[1] is indeterminate. Nevertheless, even though Eq. (7.5) is gauge dependent, its curl

$$[(r^{-2} h_A)_{|B} - (r^2 h_B)_{|A}] dx^B \wedge dx^A = d(r^{-2} h_A dx^A) \equiv d(r^{-2} k_A dx^A)$$

in Eq. (7.2) is independent of any gauge change.

It is now clear how to solve the coupled Einstein-Maxwell system Eqs. (7.1)–(7.3) in an arbitrary uncharged spherically symmetric background:

(i) For any $l = 1$ matter perturbation, which must satisfy the equation of motion

$$d(r^2 * T_A^{\text{mat}} dx^A) = 0 ,$$

solve Eq. (7.1) for S as a line integral on M^2,

$$S = \int r^2 * T_A^{\text{mat}} dx^A .$$

(ii) Solve the inhomogeneous Maxwell wave Eq. (7.3) to determine the perturbed Maxwell field F.

(iii) To find the perturbation in the gravitational field $k_A dx^A$, solve the perturbed Einstein Eq. (7.2). This is accomplished by expressing $k_A dx^A$ in terms of a to-be-determined scalar ψ,

$$r^{-2} k_A dx^A = *d\Psi . \tag{7.6}$$

(On M^2, which has an indefinite metric, this decomposition is not unique.) Thus, Eq. (7.2) becomes

$$*d*\psi = \frac{4e}{r^4} F + 16\pi S . \tag{7.7}$$

One sees therefore that, for $l = 1$, odd-parity gravitational perturbation (7.4) and hence $h_A dx^A$ is determined modulo the gradient of some indeterminate scalar.

VIII. COUPLED LINEARIZED EINSTEIN-MAXWELL SYSTEM: EVEN PARITY

To obtain the equations governing a given even-parity perturbation mode of the metric and the electromagnetic field, one must combine the linearized electromagnetic stress-energy gauge-invariant Eq. (5.5) with the linearized Einstein field Eqs. (3.13). Also, using the background field Eqs. (2.11) and (2.12), one has

$$
\begin{aligned}
&2v^C(k_{AB|C}-k_{cA|B}-k_{cB|A})-[(l-1)(l+2)/r^2+4v_C{}^{|C}+6v_Cv^C]k_{AB}-2g_{AB}v^C(k_{ED|c}-k_{cE|D}-k_{cD|E})g^{ED}\\
&\quad+g_{AB}(2v^{c\,|D}+4v^Cv^D-G^{CD})k_{CD}+g_{AB}\left[\frac{l(l+1)-1}{r^2}+2v^{|C}_C+3v_Cv^C\right]k_D{}^D\\
&\quad+2(v_Ak_{,B}+v_Bk_{,A}+k_{,A|B})-g_{AB}\left[2k_{,c}{}^{|C}+6v^Ck_{,C}-\frac{(l-1)(l+2)}{r^2}k\right]\\
&\qquad=-4g_{AB}f\epsilon_{CD}F^{C|D}+2f^2\left[k_{AB}+g_{AB}k_C{}^C-2k_{CD}\epsilon^C{}_A\epsilon^D{}_B\right]-16\pi T^{\text{mat}}_{AB}\,,\quad l\geq 0
\end{aligned}
\tag{8.1a}
$$

$$
k_{,A}-k_{AC}{}^{|C}+k_C{}^C{}_{|A}-v_Ak_C{}^C=-4f\epsilon_A{}^CF_C-16\pi T^{\text{mat}}_A\,,\quad l\geq 1 \tag{8.1b}
$$

$$
\begin{aligned}
&(-k_{,c}{}^{|c}-2v^Ck_{,C}-G_a{}^ak)+[k_{cD}{}^{|c|D}+2v^Ck_{CD}{}^{|D}+2(v^{C|D}+v^Cv^D)k_{CD}]-\left[k_c{}^C{}_{|D}{}^{|D}+v^Ck_D{}^D{}_{|C}+\mathcal{R}k_C{}^C-\frac{l(l+1)}{2r^2}k_C{}^C\right]\\
&\qquad=-4f\epsilon_{cD}F^{c|D}+2f^2(k-k_C{}^C)-16\pi T^{3\text{mat}}\,,\quad l\geq 0\cdots
\end{aligned}
\tag{8.1c}
$$

$$
k_C{}^C=-16\pi T^{2\text{mat}}\,,\quad l\geq 2\,. \tag{8.1d}
$$

These linearized Einstein field equations together with the linearized Maxwell field Eqs. (4.14) govern the even-parity interacting gravitational and electromagnetic degrees of freedom coupled to the uncharged and/or charged matter degrees of freedom. No further dynamical equations are necessary. In particular, the continuum analog of the (perturbed) Lorentz equation of motion and of the (perturbed) charge conservation equation are directly implied by the linearized field Eqs. (8.1) and (4.14), respectively. The perturbed Lorentz equation in terms of gauge-invariant matter and em field objects is Eqs. (3.14a) and (3.14b). There the objects T_{AB}, T_A, T^3, and T^2 must be replaced by the right-hand sides of Eqs. (8.1a)–(8.1d), respectively. The perturbed charge conservation equation is simply Eq. (4.15).

SUMMARY AND CONCLUSION

Suppose one wishes to make some precise statements about first-order perturbations away from some spherically symmetric space-time of one's choice. It is immaterial what the background geometry or the relevant coordinate system might be. If the perturbations are of odd $[(-1)^{l+1}]$ parity, Eqs. (6.5) and (6.6) [or Eqs. (6.1) and (6.2)] give a minimal as well as complete description of the linearized Einstein-Maxwell system. This description includes via Eq. (6.7) [or Eq. (6.3)] the Lorentz equation of motion for the charged matter distribution. The (co)vectorial Einstein Eq. (6.5b) may be replaced by the scalar Eq. (6.5b′). This scalar equation together with the two scalar Eqs. (6.6) and (6.7) gives a necessary and sufficient description of the physically very rich[53] coupling between gravitational, electromagnetic, and acoustic perturbational degrees of freedom. If the matter of the background is uncharged then these equations can be replaced by Eqs. (6.13)–(6.15) for $l\geq 2$, and by Eqs. (7.7) and (7.3) for $l=1$. If the background contains no matter, then the relevant equations are Eqs. (6.17).

In all of these equations the nature of the perturbed matter as expressed by the respective covector and scalar, $T^{\text{mat}}_A dx^A$ and T^{mat}, is as yet unspecified. They must be constructed in accordance with the prescription given by Eqs. (3.5).

If one wishes to make precise statements about even-parity $[(-1)^l]$ perturbations, then, regardless of the background geometry or coordinate system in question, the relevant Einstein-Maxwell set of coupled equations is Eqs. (8.1) and (4.14). As discussed in Sec. VIII, they imply Eqs. (3.14) and (4.15), the continuum perturbational analog of the Lorentz equations of motion and of charge conservation. Thus, once one has constructed the even-parity matter perturbation invariants in accordance with Eqs. (3.12), one can investigate coupled acousto-electro-gravitational modes that might be present in a star perturbed away from equilibrium or in a star undergoing highly nonlinear radial pulsations or general spherical collapse. The junction conditions across the star's surface are already known.[2] Nevertheless, the even-parity equation as exhibited here suffers from a drawback that the odd-parity equations do not have: The former have not been decoupled to exhibit a single master scalar equation for the gravitational perturbation degrees of freedom.

The outstanding problem for perturbation theory away from generic spherically symmetric space-times is this: Find a single scalar master equation for the even-parity gravitational degrees of freedom. One's belief that this can be done is supported not only by the existence of an even-parity master equation for a vacuum background,[30] but also by the fact that at least for a vacuum background even- and odd-parity perturbations are obtainable from the same master equation.[31]

A warning is, however, in order. One should not count on the Zerilli even-parity master equation to be the prototype equation to be generalized to generic spherically symmetric space-times. This warning is motivated by the observation that the odd-parity master scalar equation does not reduce in a vacuum to the one given by Regge and Wheeler.[29] Instead, their equation is in essence the time component of the covectorial equation in the gauge invariant $k_A dx^A$. The decoupling they achieve is made possible by the happy coincidence that time is a cyclic coordinate of the Schwarzschild background. In view of the fact that Zerilli also makes repeated use of the cyclic nature of the time coordinate, one cannot exclude the possibility that his master equation is some coordinate component of a second-rank tensor equation.

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APPENDIX: EVEN-PARITY MAXWELL AND CHARGE GAUGE INVARIANTS

The gauge invariants for the perturbations (4.9) and (4.10) are constructed in the standard way by considering linear combinations of the Maxwell, charge, and metric perturbation that stay unchanged when the background is subjected to an infinitesimal coordinate transformation. Such a transformation is expressed in terms of the Lie derivative with respect to (suppress angular integers l and m)

$$\xi_\mu dx^\mu = \xi_A(x^C) Y\, dx^A + \xi(x^C) Y_{,a} dx^a .$$

The change due to an infinitesimal coordinate transformation is

$$\Delta \bar{f}_{\mu\nu} = \Delta f_{\mu\nu} - (f_{\mu\nu;\sigma}\xi^\sigma + f_{\sigma\nu}\xi^\sigma{}_{;\mu} + f_{\mu\sigma}\xi^\sigma{}_{;\nu}) ,$$

$$\Delta \bar{j}_\mu = \Delta j_\mu - (j_{\mu;J\mu;\sigma}\xi^\sigma + j_\sigma \xi^\sigma{}_{;\mu}) ,$$

and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu\nu} + \xi_{\nu;\mu}) .$$

These expressions, when rewritten in terms of geometrical objects on M^2, have the form

$$\Delta \bar{f}_{AB} = \Delta f_{AB} - (f_{AB}{}^{|C}\xi_C + f^C{}_B \xi_{C|A} + f_A{}^C \xi_{C;B}) ,$$

$$\Delta \bar{f}_A = \Delta f_A - f_A{}^C \xi_C ,$$

$$\Delta \bar{f} = \Delta f ,$$

$$\Delta \bar{j}_A = \Delta j_A - (j_{A|C}\xi^C + j^C \xi_{C|A}) ,$$

$$\Delta \bar{j} = \Delta j - j_C \xi^C ,$$

and

$$\bar{h}_{AB} = h_{AB} - (\xi_{A|B} + \xi_{B|A}) ,$$

$$\bar{h}_A = h_A - [\xi_A + r^2(\xi/r^2)_{,A}] ,$$

$$\bar{K} = K - 2v^C \xi_C ,$$

$$r^2 \bar{G} = r^2 G - 2\xi .$$

Also, if

$$p_A = h_A - \tfrac{1}{2} r^2 G_{,A} ,$$

then

$$\bar{p}_A = p_A - \xi_A .$$

In terms of the vector p_A the set of gauge-invariant geometrical perturbation objects are given by expressions (4.11)–(4.13).

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[46] This corrects a misplaced parenthesis in Eq. (4b) of Ref. 1.

[47] Equation (3.13a) below corrects a typographical error in Eq. (12) in Ref. 1.

[48] Contrary to the assertion made in Ref. 1, Eq. (10b) there holds only for $l \geq 1$, Eq. (10d) holds only for $l \geq 2$; and Eq. (10c), which is correct for $l \geq 2$, should be replaced by (3.13c) in this present paper if one wishes to use it for all $l \geq 0$.

[49] Contrary to the statement made in footnote 30 of Ref. 1, Zerilli's Eq. (c7f) [Phys. Rev. D 2, 2141 (1970)] does in fact agree with our equations.

[50] Equation (15a) in Ref. 1 is correct only for $l \geq 1$.

[51] The omission of a minus sign in each of the Eqs. (16b), (16c), and (16a) of Ref. 1 is hereby corrected by Eqs. (6.5a), (6.5b), and (6.7) below.

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[53] For a classification of all possible (high-frequency) gravitational and electromagnetic coupled modes such as (a) Faraday rotations, (b) beat frequency oscillations, and (c) hybrids between these two, see Fig. 2 in Ref. 6.