Gravitation, geometry, and nonrelativistic quantum theory

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In Cartan's description, classical particles freely falling in a Newtonian gravitational field follow geodesics of a curved spacetime. We cast this geodesic motion into generalized Hamiltonian form and quantize it by Dirac's constraint method in a coordinate-independent way. The Dirac constraint takes a simplified form in special noninertial frames (nonrotating, rigid, Galilean, and Gaussian). Transformation theory of the state function allows us to compare descriptions of a given quantum state by two different observers and to illustrate how the principle of equivalence works for quantum systems. In particular, we show that quantum states of a particle moving in a homogeneous gravitational field and of the gravitational harmonic oscillator can be reduced to the study of plane waves in an appropriate frame.

I. NEWTONIAN GRAVITY AND QUANTUM THEORY

Recent attempts to reconcile quantum field theory with the general theory of relativity uncovered an unsuspected number of conceptual difficulties. While the least understood ones stem from attempts to quantize geometry, others arise at a more elementary level: It is not entirely clear how to interpret relativistic quantum fields propagating on curved classical backgrounds.[1] A little reflection shows that at the heart of this class of difficulties lies a conflict between the local character of the principle of equivalence and the global nature of some basic procedures in quantum theory.

In its simplest application, the principle of equivalence asserts that freely falling particles within a freely falling elevator are all brought to rest or into a uniform rectilinear motion. The situation is often described by saying that gravity is locally indistinguishable from inertia and cannot be logically conceived as a force. Globally, however, the tidal effects prevent the stacking of local freely falling elevators into an all-encompassing inertial frame. Unfortunately, such a frame is a logical prerequisite for many conventional patterns in quantum theory. For example, elementary particles are introduced as irreducible unitary representations of the Poincaré group connecting different inertial frames.[2,3] The construction of these representations relies on the identification of momentum and energy with generators of spacetime translations. In curved spacetimes, translational symmetry is broken and if we try to introduce particles by reference to their momenta and energies measured by a noninertial observer, we are led to the unpalatable conclusion that different observers do not agree on how many particles there are in space at a given instant of time.[4,5]

After we trace such difficulties back to the principle of equivalence, it becomes suddenly strange that ordinary nonrelativistic quantum mechanics can avoid similar dilemmas when Newtonian gravitational fields are present. The principle of equivalence is clearly applicable to Newtonian spacetimes as well as to the general relativistic ones. When Einstein initially calculated light deflection,[6] he used the principle of equivalence in the Newtonian context. A complete geometrical description of Newtonian spacetimes incorporating the principle of equivalence was given by Cartan[7] and developed later by many authors. As in relativistic spacetimes, there is no global inertial frame in a Newtonian spacetime which is curved by Newtonian gravitation. Still, most of us are inclined to evade the problem and quantize the motion of a nonrelativistic particle in a Newtonian gravitational field in the same way we quantize the motion in an ordinary potential field of force, starting from the Schrödinger equation

$$i\hbar\partial_T\psi=[-(\hbar^2/2m)\,\delta^{ab}\partial_a\partial_b+m\phi\,]\psi\,. \tag{1.1}$$

We use the Kronecker delta δ^{ab} and the gravitational potential ϕ without an explanation. By writing the Schrödinger equation in this form we are tacitly assuming that there is an inertial frame and that gravity is a force. Neither of these assumptions being true, we face the problem of whether Eq. (1.1) can be justified within the framework of Cartan's geometrical theory which respects the universal character of Newtonian gravitation, speaking about it in terms of a curvature of spacetime rather than in terms of a force. More generally, we want to discuss what form the Schrödinger equation takes in noninertial frames and how different noninertial observers describe a given quantum state of a nonrelativistic system.

Let us briefly state the main conclusions. The Schrödinger equation (1.1) makes sense when properly interpreted. There is a geometrial procedure allowing us to write the Schrödinger equation in an arbitrary noninertial frame in a spacetime curved by Newtonian gravitation. In special frames, this equation takes on the form (1.1). Moreover, different noninertial observers agree that there is just one particle in space and on its position distribution, though they disagree on its energy and momentum. Nonrelativistic quantum mechanics escapes the difficulties of the relativistic theory because Newtonian spacetimes possess a unique foliation by leaves of absolute time. It is the light-cone structure combined with the principle of equivalence rather than the principle of equivalence by itself which is to be blamed for the conceptual complications of generally relativistic quantum theory.

A different way of posing our problem is to order basic physical theories in a three-dimensional diagram (Fig. 1) indicating the role played by the fundamental constants c, G, and h. What emerges from the diagram is the limiting position which some of the theories occupy with respect to others.

Start from what we shall call "the classical vertex"—the beginning of every physicist's curriculum. There, quantum effects are neglected ($h=0$), the curvature of spacetime caused by gravitation is unimportant ($G=0$), and interactions are instantaneous ($c^{-1}=0$). Global inertial frames, connected by the Galilei group, set the stage on which nonrelativistic particles move, acting on each other at a distance.

Let us next pass along the c^{-1} axis into the special theory of relativity. Interactions start propagating with finite speed and are described by Poincaré-invariant field equations. Quantum effects are still neglected and global inertial frames are at our disposal. Let us continue further into the "classical plane" ($c^{-1}\neq 0$, $G\neq 0$, $h=0$) of the general theory or relativity. The principle of equivalence asserts that the global inertial frame no longer exists but has been broken into local inertial frames by the tidal effects of the gravitational field. Classical fields are described by "generally covariant" equations and the gravitational field possesses its own dynamical degrees of freedom. It is identified with the geometry of spacetime.

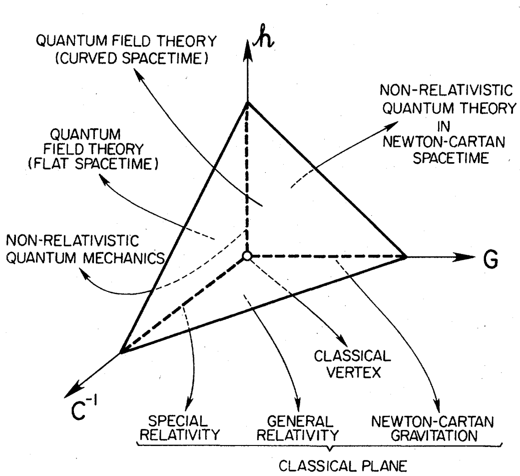

FIG. 1. Dimensional pyramid.

On the other border, the classical plane is bounded by the G axis. On it the dynamical degrees of freedom belonging to the gravitational field become frozen and interaction again becomes instantaneous. However, the principle of equivalence is still at work so that the global inertial frame cannot be recovered from the local inertial pieces. Spacetime continues to be curved by Newtonian gravitation described in geometrical terms by Cartan.

Let us return to the classical vertex and take off again, now in the vertical direction. The h axis is the seat of nonrelativistic quantum mechanics, comparatively well understood in its conceptual structure. From the Planck axis, we move into the plane ($c^{-1}\neq 0$, $G=0$, $h\neq 0$) representing relativistic quantum field theory. Beset by problems of its own, this theory can at least rely on rigid inertial frames for its formulation.

Through the equilateral triangle facing us in the diagram, we must finally gaze into the interior of our schematic pyramid, the *terra incognita* of generally relativistic quantum field theory and quantum gravity. The conceptual difficulties we mentioned all reside in this largely unexplored region. All theories we represented by axes and planes meeting in the classical vertex are but limiting cases of the problematic theory hidden inside the pyramid.

One boundary, curiously enough, has, so far, not been given much attention: It is the plane ($c^{-1}=0$, $G\neq 0$, $h\neq 0$). It represents nonrelativistic quantum theory in the Newton-Cartan spacetime. There is at least one thing we need not worry about on this side of the pyramid: quantum gravity. This is because the dynamical degrees of freedom of the gravitational field remain frozen in the $c^{-1}=0$ plane. Such a simplification unfortunately also means that we cannot learn anything about quantum gravity from this limit. On the other hand, spacetime is curved and global inertial frames do not exist in the $c^{-1}=0$ plane, while spacetime is flat and there are global inertial frames in the $G=0$ plane. We can thus hope to get some insight into quantum theory on curved backgrounds from the $c^{-1}=0$ limit, insight which is not readily available in the usual $G=0$ limit. Our diagram simply illustrates various "correspondence principles" imposed on the yet unknown generally relativistic quantum field theory.

Following this program, we build a geometrical description of the quantum mechanics of a single nonrelativistic particle freely falling in an external Newtonian gravitational field. We leave to a later paper the quantum field theory of an unspecified number of identical particles interacting through their own Newtonian gravitational field.

The classical part of the theory, characterizing the free fall of a point particle as a geodesic motion in an appropriate spacetime, was developed almost concurrently with the formulation of the theory of relativity. The geometry of Galilean spacetimes without gravitation was discussed by Frank[8] and Weyl,[9] of Newtonian spacetimes with gravitation by Cartan[7] and Friedrichs.[10] Cartan's theory was elaborated by Havas,[11] Trautman,[12-14] Misner,[15,16] Dombrowski and Horneffer,[17,18] and Künzle.[19] The correspondence transition from general relativity to the Newton-Cartan theory was spelled out by Dautcourt[20] and Künzle.[21] We shall summarize the main geometrical features of Newtonian spacetimes in Sec. II. The Galilei group connecting inertial frames in spacetimes free of gravitation gets extended into a "quasi-Galilean gauge group" connecting Galilean (rigid, nonrotating) frames in the presence of Newtonian gravitational fields.[14,15] Because the four-velocity of a Galilean observer enters as an auxiliary structure into the equation of a Newtonian geodesic, it is important that both the classical and quantum theory of freely falling particles be invariant with respect to this gauge group.

In Sec. III we derive the equation of a Newtonian geodesic from a homogeneous Lagrangian and cast the action principle into generalized Hamiltonian form. The action principle is invariant under the quasi-Galilean gauge group.

In Sec. IV we write the action as it appears to an arbitrary observer in the general gauge. We also introduce a hierarchy of special observers, including the Galilean observer and the Gaussian (freely falling and nonrotating) observer. We deparametrize the action by labeling the world lines by the absolute time. We let each observer use his comoving coordinates and discuss how to simplify its action by gauge transformations.

By casting the action into generalized Hamiltonian form, we prepared everything for quantization by Dirac's contraint method. The transition to quantum theory is accomplished in Sec. V for an arbitrary observer and an arbitrary gauge in a coordinate-independent manner. We define an inner product between any two state functions which satisfy the Hamiltonian constraint. We prove that this product does not depend on absolute time and that it makes the Hamiltonian Hermitian for an arbitrary observer. We also analyze the gauge behavior of the state function.

The Hamiltonian constraint is deparametrized by using the absolute time and comoving coordinates of an observer (Sec. VI). We show that the resulting equation is nothing else but the Schrödinger equation which could have been obtained directly by quantizing the deparametrized action principle. We notice the simplifications achieved by special observers. In particular, we recover the Schrödinger equation (1.1) for the Galilean observer.

The last two sections are devoted to a detailed discussion of two simple examples. We want to see how the principle of equivalence works when applied to nonrelativistic quantum mechanics. In Sec. VII we observe the state of a free particle in a space without gravitation from two different frames, an inertial frame and uniformly accelerated frame (Einstein's elevator), and we compare the two descriptions. We show how to obtain the stationary states in the accelerated frame (Airy's functions) as appropriate "transforms" of stationary states in the inertial frame (plane waves). In Sec. VIII we apply the same technique to a system moving in a true gravitational field, namely, to the gravitational harmonic oscillator. We compare the descriptions of the oscillator by a Galilean and a Gaussian observer and we again interpret stationary states in the Galilean frame (Hermite functions) as appropriate "transforms" of stationary states in the Gaussian frame (plane waves). We hope that these two examples help to develop an intuitive feeling about the relationship between the principle of equivalence and quantum mechanics.

II. GEOMETRY OF NEWTONIAN SPACETIMES

In this section we summarize the main geometrical features of Newtonian spacetimes. For details, we refer the reader to papers by Trautman,[14] Künzle,[19] and the review paper by the author.[22]

Newtonian spacetimes are endowed with three interrelated structures: the space metric $g^{\alpha\beta}$, the time metric $h_{\alpha\beta}$, and the symmetric affine connection ∇_γ. These structures satisfy the following conditions:

(A). The space metric $g^{\alpha\beta}$ is degenerate, with signature $(0;1,1,1)$. Similarly, the time metric is degenerate, with signature $(1;0,0,0)$. The two metrics are mutually orthogonal,

$$g^{\alpha\beta}h_{\beta\gamma}=0\,. \tag{2.1}$$

(B). The symmetric affine connection ∇_γ is compatible with both metrics,

$$\nabla_\gamma g^{\alpha\beta} = 0 = \nabla_\gamma h_{\alpha\beta}. \tag{2.2}$$

(C). The curvature tensor of the affine connection has the properties[23,24]

$$h_{\alpha[\beta} R^{\gamma}{}_{\delta]\mu\nu} = 0, \tag{2.3}$$

$$R^{[\alpha}{}_{\beta}{}^{\gamma]}{}_{\delta} = 0. \tag{2.4}$$

The coordinates of an event are $x^\alpha = \{t, x^a\}$, with $\alpha = 0, 1, 2, 3$ and $a = 1, 2, 3$. Spacetime covariant differentiation will also be denoted by a semicolon, $\nabla_\beta w^\alpha \equiv w^\alpha{}_{;\beta}$.

Greek indices are raised by the space metric $g^{\alpha\beta}$. The curvature tensor $R^\alpha{}_{\beta\gamma\delta}$ is introduced by the convention

$$w^\alpha{}_{;[\gamma\delta]} = w^\beta R^\alpha{}_{\beta\gamma\delta}. \tag{2.5}$$

The time metric helps us to distinguish timelike vectors from spacelike vectors,

$$h_{\alpha\beta} w^\alpha w^\beta \begin{cases} > 0 \Longleftrightarrow w^\alpha \text{ timelike} \\ = 0 \Longleftrightarrow w^\alpha \text{ spacelike} \end{cases} \tag{2.6}$$

The space metric determines the norms of spacelike vectors: To every spacelike vector w^α we can find a covector w_α which generates w^α by

$$w^\alpha = g^{\alpha\beta} w_\beta. \tag{2.7}$$

The norm of w^α is then defined by

$$\|w\|^2 \equiv g^{\alpha\beta} w_\alpha w_\beta. \tag{2.8}$$

The covectors

$$\tilde{w}_\alpha = w_\alpha + h_{\alpha\beta} v^\beta \tag{2.9}$$

generate the same vector w^α. The norm $\|w\|$, however, does not depend on the auxiliary vector v^α.

The basic structures $h_{\alpha\beta}$, $g^{\alpha\beta}$, and ∇_α acquire an operational meaning through the correlation postulates

(a) timelike intervals can be measured by standard clocks,

(b) spacelike intervals can be measured by standard rods,

(c) neutral test particles move along timelike geodesics.

From the fundamental postulates (A)–(C) there follows the existence of privileged spacetime fields

T ··· absolute time,

X^a ··· Galilean (nonrotating, Cartesian) coordinates,

u^α ··· four-velocity of the Galilean (rigid, nonrotating) observer (frame),

$-\phi^\alpha$ ··· gravitational field strength,

ϕ ··· scalar gravitational potential,

which satisfy the conditions

$$h_{\alpha\beta} = T_\alpha T_\beta, \text{ with } T_\alpha \equiv T_{,\alpha}, \tag{2.10}$$

$$X^{a\alpha}{}_{;\beta} = 0, \text{ with } X^{a\alpha} \equiv g^{\alpha\beta} X^a{}_{,\beta}, \tag{2.11}$$

$$g^{\alpha\beta} = \delta_{ab} X^{a\alpha} X^{b\beta}, \tag{2.12}$$

$$u^\alpha T_\alpha = 1, \tag{2.13}$$

$$u^\alpha X^a{}_{,\alpha} = 0, \tag{2.14}$$

$$u^\alpha{}_{;\beta} = \phi^\alpha T_\beta, \tag{2.15}$$

$$\phi^\alpha = g^{\alpha\beta} \phi_{,\beta}, \tag{2.16}$$

and are connected with the curvature tensor by

$$R^\alpha{}_{\beta\gamma\delta} = T_\beta \phi^\alpha{}_{;[\gamma} T_{\delta]}. \tag{2.17}$$

Equation (2.11) implies that the vectors $X^{a\alpha}$ are parallel propagated, so that the Galilean coordinates are nonrotating. According to Eq. (2.12), they are also Cartesian coordinates. The leaves of a constant absolute time are thus flat. Equation (2.13) means that the world lines of the Galilean observer are parametrized by the absolute time T. Equation (2.14) states that X^a are the comoving coordinates of the Galilean observer. From Eq. (2.15), we learn that the Galilean observer is rigid,[24] $u^{(\alpha;\beta)} = 0$, and nonrotating, $u^{[\alpha;\beta]} = 0$. We also learn that $-\phi^\alpha$ has the meaning of the gravitational field strength,

$$(\nabla_u u)^\alpha \equiv u^\alpha{}_{;\beta} u^\beta = \phi^\alpha. \tag{2.18}$$

According to Eq. (2.16), ϕ^α has a potential ϕ.

Newton's law of gravitation[25]

$$\Delta\phi = 4\pi\mu \tag{2.19}$$

can be then written as a restriction on the Ricci tensor $R_{\alpha\beta}$,

$$R_{\alpha\beta} = 4\pi\mu h_{\alpha\beta}. \tag{2.20}$$

The curvature tensor $R^\alpha{}_{\beta\gamma\delta}$ is uniquely determined by the elements T, X^a, u^α, and ϕ by Eq. (2.17). However, there exist many elements T, X^a, u^α, and ϕ interlocked by the relations (2.10)–(2.16) and generating the same curvature tensor (2.17). Different Galilean observers are thus connected by a gauge group which is a generalization of the Galilei Lie group to spacetimes with gravitational fields,[22]

$$\tilde{X}^a(x) = A^a{}_b X^b(x), \quad \tilde{T}(x) = T(x) - T_0,$$

$$\delta^{mn} A^a{}_m A^b{}_n = \delta^{ab}, \quad A^a{}_b = \text{const}, \quad T_0 = \text{const}, \tag{2.21}$$

$$\tilde{X}^a(x) = X^a(x) - R^a(T), \tag{2.22}$$

$$\tilde{u}^\alpha(x) = u^\alpha(x) + U_a(T) X^{a\alpha}(x), \tag{2.23}$$

$$\tilde{\phi}(x) = \phi(x) + \mathfrak{A}_a(T) \tilde{X}^a(x) + \varphi(T), \tag{2.24}$$

$$U^a \equiv dR^a/dT, \quad \mathfrak{A}^a \equiv dU^a/dT = d^2R^a/dT^2. \tag{2.25}$$

While (2.21) is a Lie group, Eqs. (2.22)–(2.25) contain four arbitrary functions of T, namely, $R^a(T)$ and $\varphi(T)$, and consequently represent a gauge group. Equations (2.22)–(2.25) connect two Galilean frames by an arbitrary accelerated translation. The potential $\tilde{\phi} = \mathcal{a}_a(T)\tilde{X}^a + \varphi(T)$ automatically satisfies the Laplace equation and therefore it cannot be distinguished from the "true" gravitational potential. Equation (2.24) expresses the principle of Einstein's elevator in a Newtonian spacetime. We require that all equations, classical and quantum, be form invariant with respect to the quasi-Galilean gauge group (2.21)–(2.25).

III. NEWTONIAN GEODESICS

Geodesics can be defined as "straightest paths" by the requirement that the tangent vector $w^\alpha \equiv \dot{x}^\alpha$ be parallel propagated along them,

$$(\nabla_w w)^\alpha \sim w^\alpha, \quad \text{or } \ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma \sim \dot{x}^\alpha. \tag{3.1}$$

For nondegenerate metrics, the compatibility condition $g_{\alpha\beta;\gamma} = 0$ fixes $\Gamma^\alpha{}_{\beta\gamma}$ to be the Christoffel symbol constructed from $g_{\alpha\beta}$. In Newtonian spacetimes, the compatibility conditions (2.2) for the degenerate metrics $g^{\alpha\beta}, h_{\alpha\beta}$ are insufficient to fix the affine connection. To express the Newtonian affine connection in terms of the potentials $\{g^{\alpha\beta}, u^\alpha, T, \phi\}$, we must adjoin to Eq. (2.2) the supplementary conditions (2.15), (2.16) characterizing the Galilean observer.

We introduce first the covariant space metric $g_{\alpha\beta}$ by the relations[19]

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta^\beta_\alpha - T_\alpha u^\beta, \quad g_{\alpha\beta} u^\beta = 0. \tag{3.2}$$

This metric is again degenerate, with signature (0;1,1,1). We use it for lowering the Greek indices. However, $g_{\alpha\beta}$ is not gauge independent. If we change u^α according to Eq. (2.23), $g_{\alpha\beta}$ changes into

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - T_{(\alpha} X^a_{\beta)} U_a + U^2 T_\alpha T_\beta. \tag{3.3}$$

From $g^{\alpha\beta}$ and $g_{\alpha\beta,\gamma}$ we construct the Christoffel symbol ${}^0\Gamma^\alpha{}_{\beta\gamma}$ by the standard prescription. Equations (2.2), (2.15), and (2.16) then ensure[22] that the affine connection $\Gamma^\alpha{}_{\beta\gamma}$ has the form

$$\Gamma^\alpha{}_{\beta\gamma} = {}^0\Gamma^\alpha{}_{\beta\gamma} + u^\alpha T_{,\beta\gamma} + \phi^\alpha T_\beta T_\gamma. \tag{3.4}$$

While ${}^0\Gamma^\alpha{}_{\beta\gamma}$ is not gauge invariant, the total connection (3.4) is gauge invariant.

To quantize the geodesic motion, we must find the action

$$S[x] = \int d\tau\, L(x,\dot{x}) \tag{3.5}$$

which yields the geodesics (3.1), (3.4) as extremal paths. The Lagrangian $L(x,\dot{x})$ must be a homogeneous function of the first degree in the four-velocity $\dot{x}^\alpha$ so that the action (3.5) be parametrization independent. To write the Lagrangian, we select a Galilean observer $\{u^\alpha, \phi\}$ in the given Newtonian spacetime $\{g^{\alpha\beta}, h_{\alpha\beta}, \nabla_\alpha\}$ and write[26]

$$L(x,\dot{x}) \equiv \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta (T_\gamma \dot{x}^\gamma)^{-1} - \phi T_\gamma \dot{x}^\gamma. \tag{3.6}$$

The Euler-Lagrange equations of the action (3.5) and (3.6) imply[22] that

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = N^{-1}\dot{N}\dot{x}^\alpha, \tag{3.7}$$

where $\Gamma^\alpha{}_{\beta\gamma}$ is given by Eq. (3.4) and

$$N \equiv T_\alpha \dot{x}^\alpha = dT/d\tau \tag{3.8}$$

is called the lapse function. For $T=\tau$, the right-hand side of Eq. (3.7) vanishes, which shows that absolute time is an affine parameter.

From the homogeneous action (3.5) and (3.6), we can pass to the generalized Hamiltonian formalism.[27] The four-momentum conjugate to x^α is

$$\begin{aligned} p_\alpha &\equiv L_{,\dot{\alpha}} = N^{-1} g_{\alpha\beta}\dot{x}^\beta - E T_\alpha, \\ E &\equiv \tfrac{1}{2} N^{-2} g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \phi. \end{aligned} \tag{3.9}$$

The canonical variables x^α and p_α are not independent, being subject to the constraint

$$\mathcal{H} \equiv \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + u^\alpha p_\alpha + \phi = 0 \tag{3.10}$$

which follows from Eqs. (3.9), (2.13), and (3.2). The original action principle is thus transformed into the form

$$\delta S = 0, \quad S[x,p] = \int d\tau\, p_\alpha \dot{x}^\alpha, \tag{3.11}$$

where the variations of x^α and p^α are restricted by the auxiliary condition (3.10). If we adjoin (3.10) to the action (3.11) by a Lagrange multiplier N, we get the variational principle

$$\delta S = 0, \quad S[x,p;N] = \int d\tau (p_\alpha \dot{x}^\alpha - N\mathcal{H}) \tag{3.12}$$

in which all the variables x^α, p_α, and N can be freely varied. The Hamilton equations imply that N is the lapse function, Eq. (3.8). The function $\mathcal{H}(x,p)$ is often called the super-Hamiltonian.

Under the gauge transformation (2.22)–(2.25), the Lagrangian (3.6) changes by a total time derivative

$$\tilde{L}(x,\dot{x}) = L(x,\dot{x}) + \dot{\Lambda}(x), \tag{3.13}$$

$$\Lambda(x) = U_a(T)[R^a(T) - X^a(x)] - \int^T dT\, \tfrac{1}{2} U^2(T) + \int^T dT\, \varphi(T). \tag{3.14}$$

The action thus changes only by a boundary term and the equations of motion remain the same. The transformation (3.13) of the Lagrangian induces the change

$$\tilde{p}_\alpha = p_\alpha + \Lambda_{,\alpha} \tag{3.15}$$

of the canonical momentum. This change is exactly compensated by the gauge transformation (2.23) and (2.24) of the potentials u^α and ϕ, so that the super-Hamiltonian stays invariant:

$$\tilde{\mathcal{H}} \equiv \mathcal{H}(x^\alpha, \tilde{p}_\alpha; g^{\alpha\beta}, \tilde{u}^\alpha, \tilde{\phi}) = \mathcal{H}(x^\alpha, p_\alpha; g^{\alpha\beta}, u^\alpha, \phi) \equiv \mathcal{H}. \tag{3.16}$$

Equation (3.15) implies that the canonical action (3.12) again changes only by a boundary term.

IV. GENERAL AND SPECIAL OBSERVERS: DEPARAMETRIZATION

The action (3.12), (3.10) describes the motion from the standpoint of a Galilean observer. We now want to find the action as it appears to an arbitrary observer. We characterize him by the normalized four-velocity vector v^α,

$$v^\alpha T_\alpha = 1, \tag{4.1}$$

and introduce the vector

$$A^\alpha \equiv v^\alpha - u^\alpha \tag{4.2}$$

which tells us how his motion deviates from that of a Galilean observer. If we introduce v^α into the super-Hamiltonian (3.10),

$$\mathcal{H} = v^\alpha p_\alpha + \tfrac{1}{2} g^{\alpha\beta}(p_\alpha - A_\alpha)(p_\beta - A_\beta) + \phi_v, \tag{4.3}$$

we can interpret A^α as the vector potential and

$$\phi_v \equiv \phi - \tfrac{1}{2} g^{\alpha\beta} A_\alpha A_\beta \tag{4.4}$$

as the scalar potential of the gravitational field for the observer v^α. Once we admit A^α in the super-Hamiltonian, we can also modify p_α by an arbitrary gauge transformation

$$p_\alpha \to \tilde{p}_\alpha = p_\alpha + \lambda_{,\alpha}(x). \tag{4.5}$$

This brings the action into the most general form

$$S[x^\alpha, \tilde{p}_\alpha; N] = \int d\tau (\tilde{p}_\alpha \dot{x}^\alpha - N\mathcal{H}), \quad \mathcal{H} = v^\alpha \tilde{p}_\alpha + \tfrac{1}{2} g^{\alpha\beta}(\tilde{p}_\alpha - \tilde{A}_\alpha)(\tilde{p}_\beta - \tilde{A}_\beta) + \tilde{\phi}_v, \tag{4.6}$$

where

$$\tilde{A}^\alpha = A^\alpha + \lambda^{,\alpha}, \quad \tilde{\phi}_v = \phi_v - v^\alpha \lambda_{,\alpha} \tag{4.7}$$

is the general gauge transformation of the gravitational potentials.

Galilean observer is *rigid* and *nonrotating*. Another important observer is a *freely falling* observer who moves along geodesics. Combining these properties, we get the whole hierarchy of special observers,

$$\left.\begin{array}{l} v^{(\alpha;\beta)} = 0 \text{ rigid} \\ v^{[\alpha;\beta]} = 0 \text{ nonrotating} \\ v^\alpha{}_{;\beta} v^\beta = 0 \text{ freely falling} \end{array}\right\} \begin{array}{l}\text{Galilean} \\ \text{Gaussian}\end{array} \Big\} \text{ inertial} \tag{4.8}$$

Gaussian observers are widely used in Riemannian spacetimes. To pass from a Galilean observer to a Gaussian observer in the Newtonian spacetime, we write the Hamilton-Jacobi equation

$$\mathcal{H}(x^\alpha, p_\alpha = A_{,\alpha}) \equiv u^\alpha A_{,\alpha} + \tfrac{1}{2} g^{\alpha\beta} A_{,\alpha} A_{,\beta} + \phi = 0 \tag{4.9}$$

for the principal function $-A$. Any particular solution $A(x)$ of that equation defines a Gaussian observer

$$v^\alpha \equiv u^\alpha + g^{\alpha\beta} A_{,\beta} \tag{4.10}$$

and, conversely, any Gaussian observer can be generated in this way.[22]

Inertial observers exist only in spacetimes which are free of gravitation; all other observers in the table (4.8) exist in arbitrary Newtonian spacetimes.

For a given observer, we can simplify the action (4.7) by (i) the choice of coordinates and (ii) the choice (4.7) of gauge.

(i) Any three independent scalar functions $Y^a(x^\alpha)$ which are constant along the observer's world lines are called his "comoving coordinates",

$$v^\alpha Y^a_\alpha = 0 \quad (Y^a_\alpha \equiv Y^a{}_{,\alpha}). \tag{4.11}$$

We insert the privileged coordinates $\{T, Y^a\}$ into the action (4.6). This yields

$$\tilde{p}_\alpha \dot{x}^\alpha = \tilde{P}_T \dot{T} + \tilde{P}_a \dot{Y}^a, \tag{4.12}$$

with

$$\tilde{P}_T = p_\alpha v^\alpha, \quad \tilde{P}_a = \tilde{p}_\alpha Y_a^\alpha, \quad Y_a^\alpha \equiv g^{\alpha\beta} Y^b_\beta g_{ba}. \tag{4.13}$$

The super-Hamiltonian reduces to

$$\mathcal{H} = \tilde{P}_T + \tilde{H}, \tag{4.14}$$

where

$$\tilde{H} = \tfrac{1}{2} g^{ab}(\tilde{P}_a - \tilde{A}_a)(\tilde{P}_b - \tilde{A}_b) + \tilde{\phi}_v = \tfrac{1}{2} g^{ab} \tilde{P}_a \tilde{P}_b - \tilde{A}^a \tilde{P}_a + \phi. \tag{4.15}$$

Because A^α is a spacelike vector, it is completely characterized by the three projections $A^a \equiv A^\alpha Y^a_\alpha$.

If we solve the constraint $\mathcal{H} = 0$ with respect to $\tilde{P}_T$ and parametrize the world lines by the absolute time $\tau = T$, the action reduces to the "deparametrized form"

$$S[Y^a, \tilde{P}_a] = \int dT \left(\tilde{P}_a \frac{dY^a}{dT} - \tilde{H}\right). \tag{4.16}$$

We see that $\tilde{H} = -\tilde{P}_T$ is the energy of the particle measured by the general observer v^α and expressed in an arbitrary gauge.

(ii) For a given observer, we can simplify the gravitational potentials ϕ_v and $\tilde{A}^a$ by the gauging (4.7). We shall summarize the final results[22] in a table:

Observer	Potentials
General	$\tilde{A}^a{}_{\|a}=0$ or $\tilde{\phi}_v=0$
Nonrotating	$\tilde{A}^a=0$
Rigid	$\tilde{A}_a=\Omega_{ab}(T)Y^b,\ \ \Omega_{ab}(T)=-\Omega_{ba}(T)$, $\phi_v=\phi-\frac{1}{2}\delta^{ab}\Omega_{am}\Omega_{bn}Y^mY^n - \Omega_{ab}V^a(T)Y^b-[dV_b(T)/dT]Y^b$, $g^{ab}=\delta^{ab}$
Galilean	$A^a=0,\ \ \phi_u=\phi,\ \ g^{ab}=\delta^{ab}$
Gaussian	$\tilde{A}^a=0,\ \ \tilde{\phi}_v=0,\ \ g^{ab}=g^{ab}(T,Y^c)$
Inertial	$A^a=0,\ \ \phi=0,\ \ g^{ab}=\delta^{ab}$

(4.17)

In particular, for a Galilean observer the gravitational field influences the motion only through the scalar potential, while for a Gaussian observer only through the space metric $g^{ab}(T,Y^c)$.

V. CONSTRAINT QUANTIZATION

We shall quantize the motion of a freely falling particle by Dirac's method.[28] We start from the generalized Hamiltonian form of the action written for an arbitrary observer and in an arbitrary gauge. First, we turn the coordinates x^α and the momenta $\tilde{p}_\alpha$ into operators

$$\underline{x}^\alpha=x^\alpha\cdot\,,\quad \underline{\tilde{p}}_\alpha=-i\nabla_\alpha\,. \tag{5.1}$$

Next, we interpret the super-Hamiltonian (4.6) as an operator

$$\begin{aligned}\underline{\tilde{\mathcal{H}}}&=-\tfrac{1}{2}i[v^\alpha(x)\nabla_\alpha+\nabla_\alpha v^\alpha(x)]+\underline{\tilde{H}}_v\\ &=-i(v^\alpha\nabla_\alpha+\tfrac{1}{2}v^\alpha{}_{;\alpha})+\underline{\tilde{H}}_v\,.\end{aligned} \tag{5.2}$$

Here,

$$\begin{aligned}\underline{\tilde{H}}_v&=\tfrac{1}{2}g^{\alpha\beta}(-i\nabla_\alpha-\tilde{A}_\alpha)(-i\nabla_\beta-\tilde{A}_\beta)+\tilde{\phi}_v\\ &=-\tfrac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta+i(\tilde{A}^\alpha\nabla_\alpha+\tfrac{1}{2}\tilde{A}^\alpha{}_{;\alpha})+\phi\end{aligned} \tag{5.3}$$

is the Hamiltonian operator of the particle observed by v^α. Finally, we impose the super-Hamiltonian constraint as a restriction on the state function $\tilde{\psi}(x)$,

$$\underline{\tilde{\mathcal{H}}}\tilde{\psi}(x)=0\,. \tag{5.4}$$

The factor ordering in the super-Hamiltonian (5.2) is such that the terms quadratic in the momenta are replaced by the covariant Laplacian with respect to the (degenerate) space metric $g^{\alpha\beta}$, and the terms linear in the momenta are symmetrically ordered through anticommutators.

We now turn the space of state functions $\tilde{\psi}(x)$ into a Hilbert space. The inner product between two functions which satisfy the constraint (5.4) is defined by picking out an arbitrary leaf of absolute time T and by putting

$$\langle\tilde{\psi}_1|\tilde{\psi}_2\rangle\equiv\int_T dV_T\,\tilde{\psi}_1^*(x)\tilde{\psi}_2(x)\,. \tag{5.5}$$

Here, dV_T is the invariant volume element on the leaf T. To obtain an explicit expression for dV_T, we introduce the contravariant Levi-Civita pseudotensor $\epsilon^{\alpha\beta\gamma\delta}$. We can characterize it completely by its antisymmetry and by the property that in oriented Galilean coordinates $\{T,X^a\}$

$$\epsilon^{\alpha\beta\gamma\delta}\,T_\alpha X^1{}_{,\beta}X^2{}_{,\gamma}X^3{}_{,\delta}=1\,. \tag{5.6}$$

We then form the parallelepiped with the edges d_ax^α, $a=1,2,3$, lying in the leaf T,

$$T_\alpha d_ax^\alpha=0\,. \tag{5.7}$$

To each vector d_ax^α, there corresponds a covector d_ax_α such that

$$d_ax^\alpha=g^{\alpha\beta}d_ax_\alpha\,. \tag{5.8}$$

The volume element of the parallelepiped is

$$dV_T=T_\alpha\,\epsilon^{\alpha\beta\gamma\delta}d_1x_\beta d_2x_\gamma d_3x_\delta\,. \tag{5.9}$$

The covectors d_ax_α are arbitrary up to terms proportional to T_α; such terms, however, do not contribute to the volume element (5.9).

The Hamiltonian (5.3) is Hermitian under the inner product (5.5). The proof is based on Gauss's theorem which takes the following form in Newtonian spacetimes:

Let w^α be a spacelike vector, $w^\alpha T_\alpha=0$, and n^α a unit spacelike vector normal to the boundary ∂V_T of a volume V_T on the leaf T; put $n^\alpha=g^{\alpha\beta}n_\beta$. Further, let d_1x^α, d_2x^α be the edges of a two-surface element on the boundary and $d(\partial V_T)=\epsilon^{\alpha\beta\gamma\delta}T_\alpha n_\beta\,d_1x_\gamma d_2x_\delta$ its surface area. Then

$$\int_{V_T}dV_T\nabla_\alpha w^\alpha=\int_{\partial V_T}d(\partial V_T)w^\alpha n_\alpha\,. \tag{5.10}$$

Note that Eq. (5.10) deals only with space vectors on leaves of absolute time; the spacetime volume element is not defined in Newtonian spacetimes. One can easily check Eq. (5.10) by passing to the Galilean system of coordinates.

Helped by the Gauss theorem, we are able to evaluate the difference

$$\begin{aligned}\langle\tilde{\psi}_1|\underline{\tilde{H}}_v\tilde{\psi}_2\rangle-\langle\underline{\tilde{H}}_v\tilde{\psi}_1|\tilde{\psi}_2\rangle&=\int_T dV_T\nabla_\alpha\tilde{j}^\alpha_{12}\\ &=\int_{\partial T}d(\partial T)\tilde{j}^\alpha_{12}n_\alpha\,.\end{aligned} \tag{5.11}$$

The application of the theorem is justified, because

$$\tilde{j}^\alpha_{12}\equiv\tfrac{1}{2}[\tilde{\psi}_1^*(\nabla^\alpha\tilde{\psi}_2)-(\nabla^\alpha\tilde{\psi}_1^*)\tilde{\psi}_2]+i\tilde{A}^\alpha\tilde{\psi}_1^*\tilde{\psi}_2 \tag{5.12}$$

is a spacelike vector. For state functions $\tilde{\psi}_1$, $\tilde{\psi}_2$ vanishing sufficiently fast at spatial infinity ∂T, the last integral in Eq. (5.11) goes to zero. This means that $\underline{\tilde{H}}_v$ is a Hermitian operator.

It is fairly straightforward to show that the inner product (5.5) does not depend on the choice of the leaf T. To do that, follow how the expression (5.5) changes along the world lines v^α from one leaf to another:

$$\mathcal{L}_v \int_T dV_T\, \tilde{\psi}_1^* \tilde{\psi}_2 = \int_T dV_T [\tilde{\psi}_1^*(v^\alpha \nabla_\alpha \tilde{\psi}_2) + (v^\alpha \nabla_\alpha \tilde{\psi}_1^*)\tilde{\psi}_2 + v^\alpha{}_{;\alpha} \tilde{\psi}_1^* \tilde{\psi}_2]. \quad (5.13)$$

The last term in Eq. (5.13) comes from the change in the volume element,

$$\mathcal{L}_v \int_{V_T} dV_T = \int_{V_T} dV_T\, v^\alpha{}_{;\alpha}. \quad (5.14)$$

Equation (5.14) can be proved by going to the comoving coordinates $\{T, Y^a\}$ of the observer v^α; the necessary details are found in Sec. VI.

Because $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are subject to the constraint (5.4), we can rearrange the expression (5.13) into

$$\mathcal{L}_v \int_T dV_T\, \tilde{\psi}_1^* \tilde{\psi}_2 = i(\langle \tilde{\psi}_1 | \underline{\tilde{H}}_v \tilde{\psi}_2 \rangle - \langle \underline{\tilde{H}}_v \tilde{\psi}_1 | \tilde{\psi}_2 \rangle) = i \int_T dV_T \nabla_\alpha \tilde{j}_{12}^\alpha \quad (5.15)$$

and conclude that it vanishes by the argument used in Eq. (5.11). The inner product is thus time independent.

Putting $\tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\psi}$ in Eq. (5.15), we identify

$$\tilde{w} \equiv \tilde{\psi}^* \tilde{\psi} \quad (5.16)$$

as the probability density and

$$\tilde{j}^\alpha = \tfrac{1}{2} i \tilde{\psi}^* \overleftrightarrow{\nabla}^\alpha \tilde{\psi} - \tilde{A}^\alpha \tilde{\psi}^* \tilde{\psi} \quad (5.17)$$

as the probability current. Both expressions are referred to unit *proper* volume.

The quantum mechanics of a freely falling particle is based on the constraint (5.2)–(5.4) and the inner product (5.5). The theory is expressed in the language which is manifestly covariant with respect to the changes of (i) spacetime coordinates, (ii) the observer, and (iii) the gauge. We shall discuss these three changes in succession.

(i) The super-Hamiltonian (5.2) and (5.3) is a scalar operator and the state function $\tilde{\psi}(x)$ is a scalar function. The Hamiltonian constraint (5.4) is thus manifestly covariant with respect to the change of coordinates. The probability density (5.16) is a scalar and the inner product (5.5) is an invariant.

(ii) When we pass from one observer to another,

$$v^\alpha \to \bar{v}^\alpha = v^\alpha + \delta v^\alpha, \quad T_\alpha \delta v^\alpha = 0, \quad (5.18)$$

the vector and scalar potentials change,

$$\bar{\tilde{A}}^\alpha = \tilde{A}^\alpha + \delta v^\alpha, \qquad \bar{\tilde{\phi}}_{\bar{v}} = \tilde{\phi}_v - \tilde{A}_\alpha \delta v^\alpha - \tfrac{1}{2} g^{\alpha\beta} \delta v_\alpha \delta v_\beta, \quad (5.19)$$

and the super-Hamiltonian is split differently into a $-i(v^\alpha \nabla_\alpha + \frac{1}{2} v^\alpha{}_{;\alpha})$ part and the $\underline{\tilde{H}}_v$ part. The total super-Hamiltonian, however, stays the same and so does state function $\psi(x)$ satisfying the constraint (5.4). The transition amplitude (5.5) and the probability density (5.16) are observer independent.

(iii) Let us discuss finally how the constraint quantization in the new gauge is related to the constraint quantization in the old gauge. The gauge transformation

$$\tilde{x}^\alpha = x^\alpha, \quad \tilde{p}_\alpha = p_\alpha + \lambda_{,\alpha} \quad (5.20)$$

can be considered as a canonical transformation of the classical variables x^α, p_α. When we turn these variables into operators (5.1) the transformation (5.20) is induced by the unitary operator

$$\underline{U}_\lambda = e^{i\lambda(x)} \quad (5.21)$$

through the formulas

$$\underline{\tilde{x}}^\alpha = \underline{U}_\lambda^{-1} \underline{x}^\alpha \underline{U}_\lambda, \quad \underline{\tilde{p}}_\alpha = \underline{U}_\lambda^{-1} \underline{p}_\alpha \underline{U}_\lambda. \quad (5.22)$$

We introduce the new state function

$$\tilde{\psi} = \underline{U}_\lambda \psi. \quad (5.23)$$

From Eq. (5.22), it follows that

$$\langle \psi | \underline{\tilde{x}}^\alpha \psi \rangle = \langle \tilde{\psi} | \underline{x}^\alpha \tilde{\psi} \rangle, \quad \langle \psi | \underline{\tilde{p}}_\alpha \psi \rangle = \langle \tilde{\psi} | \underline{p}_\alpha \tilde{\psi} \rangle. \quad (5.24)$$

More generally, let $\underline{\tilde{F}}(\underline{\tilde{x}}, \underline{\tilde{p}})$ be a polynomial operator function of $\underline{p}$. Then

$$\underline{\tilde{F}}(\underline{\tilde{x}}, \underline{\tilde{p}}) = \tilde{F}(x, \underline{U}_\lambda^{-1} \underline{p} \underline{U}_\lambda) = \underline{U}_\lambda^{-1} \underline{\tilde{F}}(\underline{x}, \underline{p}) \underline{U}_\lambda \quad (5.25)$$

and

$$\langle \psi | \underline{\tilde{F}}(\underline{\tilde{x}}, \underline{\tilde{p}}) \psi \rangle = \langle \tilde{\psi} | \underline{\tilde{F}}(\underline{x}, \underline{p}) \tilde{\psi} \rangle. \quad (5.26)$$

In particular, let $\underline{\tilde{F}}(\underline{\tilde{x}}, \underline{\tilde{p}})$ be the super-Hamiltonian (4.6) of the particle in the new gauge. This super-Hamiltonian was constructed so that

$$\underline{\tilde{\mathcal{H}}}(\underline{\tilde{x}}, \underline{\tilde{p}}) = \underline{\mathcal{H}}(\underline{x}, \underline{p}). \quad (5.27)$$

Putting Eqs. (5.25) and (5.27) together,

$$\underline{\mathcal{H}}(\underline{x}, \underline{p}) = \underline{U}_\lambda^{-1} \underline{\tilde{\mathcal{H}}}(\underline{x}, \underline{p}) \underline{U}_\lambda, \quad (5.28)$$

we see that the constraint (5.4) on the old state function ψ is equivalent to the constraint

$$\underline{\tilde{\mathcal{H}}}(x, -i\nabla)\, \tilde{\psi} = 0 \quad (5.29)$$

on the new state function $\tilde{\psi}$.

Equation (5.29) tells us that the constraint quantization in the new gauge is achieved by substituting the old operator $-i\nabla_\alpha$ for the new momentum $\tilde{p}_\alpha$ in the classical super-Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}, \tilde{p})$, and by imposing the super-Hamiltonian constraint on the new state function $\tilde{\psi}$. Equation (5.26) implies that the mean value of an observable $\tilde{F}(\tilde{x},\tilde{p})$ constructed from the new variables $\tilde{x},\tilde{p}$ can be calculated by substituting the old operators x^α and $-i\nabla_\alpha$ for $\tilde{x}^\alpha$ and $\tilde{p}^\alpha$ into $\tilde{F}(\tilde{x},\tilde{p})$ and sandwiching $\tilde{F}(x,-i\nabla)$ between the new state functions $\tilde{\psi}$. Finally, Eqs. (5.21) and (5.23) give the new state function in terms of the old state function. We shall see in Secs. VII and VIII how these prescriptions work for specific examples.

Let us finally note that the probability density and the probability current are gauge independent,

$$\tilde{w} \equiv \tilde{\psi}^*\tilde{\psi} = \psi^*\psi \equiv w\,, \tag{5.30}$$

$$\tilde{j}^\alpha \equiv \tfrac{1}{2}i\tilde{\psi}^*\overleftrightarrow{\nabla}^\alpha\tilde{\psi} - \tilde{A}^\alpha\tilde{\psi}^*\tilde{\psi} = \tfrac{1}{2}i\psi^*\overleftrightarrow{\nabla}^\alpha\psi - A^\alpha\psi^*\psi \equiv j^\alpha\,. \tag{5.31}$$

VI. DEPARAMETRIZATION: SCHRÖDINGER'S EQUATION FOR SPECIAL OBSERVERS

The Hamiltonian constraint (5.4) is nothing more than the Schrödinger equation of the particle. This fact clearly emerges when we write the constraint in the comoving coordinates $\{T, Y^a\}$.

To do that, we first express in these coordinates the two divergences, $v^\alpha{}_{;\alpha}$ and $\tilde{A}^\alpha{}_{;\alpha}$. For an arbitrary vector field w^α,

$$\begin{aligned} w^\alpha{}_{;\alpha} &= w^\alpha{}_{;\beta}(T_\alpha v^\beta + Y^a_\alpha Y^\beta_a) \\ &= (w^\alpha T_\alpha)_{,\beta}v^\beta - (w^\gamma T_\gamma)v^\alpha Y^a_{\alpha;\beta}Y^\beta_a \\ &\quad + w^a{}_{,a} - w^b Y^\alpha_b Y^a_{\alpha;\beta}Y^\beta_a\,. \end{aligned} \tag{6.1}$$

Because

$$Y^\alpha_b Y^a_{\alpha;\beta}Y^\beta_c = \Gamma^a{}_{bc} \tag{6.2}$$

is the Christoffel symbol of the metric g_{ab} and

$$\begin{aligned} g_{,T}/g &= -g_{ab}g^{ab}{}_{,T} = -g_{ab}(g^{\alpha\beta}Y^a_\alpha Y^b_\beta)_{;\gamma}v^\gamma \\ &= -2v^\alpha Y^a_{\alpha;\beta}Y^\beta_a \end{aligned} \tag{6.3}$$

is the logarithmic time derivative of its determinant, Eq. (6.1) can be written as

$$w^\alpha{}_{;\alpha} = (w^\alpha T_\alpha)_{,\beta}v^\beta + \tfrac{1}{2}(g_{,T}/g)w^\alpha T_\alpha + w^a{}_{|a}\,. \tag{6.4}$$

Specializing now to $w^\alpha = v^\alpha$ and $w^\alpha = \tilde{A}^\alpha$, we get

$$v^\alpha{}_{;\alpha} = \tfrac{1}{2}(g_{,T}/g),\quad \tilde{A}^\alpha{}_{;\alpha} = \tilde{A}^a{}_{|a}\,. \tag{6.5}$$

From here, we read out the action of the linear differential operators contained in $\underline{\mathcal{H}}$,

$$\begin{aligned} (v^\alpha\nabla_\alpha + \tfrac{1}{2}v^\alpha{}_{;\alpha})\tilde{\psi} &= \tilde{\psi}_{,T} + \tfrac{1}{4}(g_{,T}/g)\tilde{\psi} \\ &= g^{-1/4}(g^{1/4}\tilde{\psi})_{,T}\,, \\ (\tilde{A}^\alpha\nabla_\alpha + \tfrac{1}{2}\tilde{A}^\alpha{}_{;\alpha})\tilde{\psi} &= \tilde{A}^a\tilde{\psi}_{,a} + \tfrac{1}{2}\tilde{A}^a{}_{|a}\tilde{\psi} \\ &= g^{-1/4}(\tilde{A}^a\partial_a + \tfrac{1}{2}\tilde{A}^a{}_{,a})g^{1/4}\tilde{\psi}\,. \end{aligned} \tag{6.6}$$

The quadratic operator $g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ reduces to the Laplacian Δ in space with the instantaneous metric g_{ab},

$$\begin{aligned} g^{\alpha\beta}\nabla_\alpha\nabla_\beta\tilde{\psi} &= g^{\alpha\beta}(\tilde{\psi}_{,TT}T_\alpha T_\beta + \tilde{\psi}_{,Tb}Y^b{}_{(\alpha}T_{\beta)} \\ &\quad + \tilde{\psi}_{,ab}Y^a_\alpha Y^b_\beta + \tilde{\psi}_{,a}Y^a_{\alpha;\beta}) \\ &= g^{ab}\tilde{\psi}_{|ab} \equiv \Delta\tilde{\psi}\,. \end{aligned} \tag{6.7}$$

The Hamiltonian constraint (6.4) thus assumes the familiar form

$$i(g^{1/4}\tilde{\psi})_{,T} = [-\tfrac{1}{2}\Delta + i(\tilde{A}^a\nabla_a + \tfrac{1}{2}\tilde{A}^a{}_{|a}) + \phi]g^{1/4}\tilde{\psi}\,. \tag{6.8}$$

Equation (6.8) is the ordinary Schrödinger equation for a nonrelativistic particle moving in a space with a time-dependent metric $g_{ab}(T, Y^c)$ under the influence of a vector potential $\tilde{A}^a(T, Y^c)$ and a scalar potential $\phi(T, Y^c)$. The operator acting on the state function $g^{1/4}\tilde{\psi}$ is the Hamiltonian operator obtained from the classical Hamiltonian (4.6) by standard Dirac rules: The momentum $\tilde{P}_a$ is replaced by the operator $-i\nabla_a$ and linear terms are symmetrically ordered by anticommutators. In a nonrigid system, it is important to use the state function $g^{1/4}\tilde{\psi}$ which is space density of weight $\frac{1}{2}$ rather than a scalar. The conservation of probability follows then by standard arguments, the inner product (5.5) assuming the form

$$\langle\tilde{\psi}_1|\tilde{\psi}_2\rangle = \int_T d^3Y(g^{1/4}\tilde{\psi})^*(g^{1/4}\tilde{\psi})\,. \tag{6.9}$$

Also, applied to the density $g^{1/4}\tilde{\psi}$, the linear differential operator

$$\begin{aligned} (\tilde{A}^a\nabla_a + \tfrac{1}{2}\tilde{A}^a{}_{|a})g^{1/4}\tilde{\psi} &= (\tilde{A}^a\partial_a + \tfrac{1}{2}\tilde{A}^a{}_{,a})g^{1/4}\tilde{\psi} \\ &= \mathfrak{L}_{\tilde{A}^a}g^{1/4}\tilde{\psi} \end{aligned} \tag{6.10}$$

is simply the Lie derivative along the space vector $\tilde{A}^a$ on the leaf of absolute time.

The expression (6.5) for $v^\alpha{}_{;\alpha}$ can be used to prove Eq. (5.14) by going to comoving coordinates $\{T, Y^c\}$.

For the general observer v^α, we can simplify the Schrödinger equation by the choice of gauge. We have seen in Sec. IV that we can either make $\tilde{A}^a$ divergence-free, or remove the scalar potential $\tilde{\phi}_v$. For special observers, the equation simplifies even further. We summarize the canonical forms of the Schrödinger equation in table (6.11) which reflects the canonical forms [table (4.17)] of gravitational potentials:

Observer	Schrödinger's equation
General	$i(g^{1/4}\tilde{\psi})_{,T} = [-\frac{1}{2}\Delta + i\tilde{A}^a\nabla_a + \phi]g^{1/4}\tilde{\psi}$
	or
	$i(g^{1/4}\tilde{\psi})_{,T} = [\frac{1}{2}g^{ab}(-i\nabla_a - \tilde{A}_a) \times(-i\nabla_b - \tilde{A}_b)]g^{1/4}\tilde{\psi}$

Nonrotating	$i(g^{1/4}\tilde{\psi})_{,T} = [-\frac{1}{2}\Delta + \tilde{\phi}_v]\, g^{1/4}\tilde{\psi}$
Rigid	$i\tilde{\psi}_{,T} = [-\frac{1}{2}\delta^{ab}\partial_a\partial_b - i\Omega^{ab}(T)Y_a\partial_b + \tilde{\phi}_v]\tilde{\psi}$ [see table (4.17) for $\tilde{\phi}_v$]
Galilean	$i\psi_{,T} = [-\frac{1}{2}\delta^{ab}\partial_a\partial_b + \phi]\psi$
Gaussian	$i(g^{1/4}\tilde{\psi})_{,T} = -\frac{1}{2}\Delta(g^{1/4}\tilde{\psi})$
Inertial	$i\tilde{\psi}_{,T} = -\frac{1}{2}\delta^{ab}\partial_a\partial_b\psi$

(6.11)

In particular, we see that the geometrical approach justifies the Schrödinger equation (1.1) in Galilean frames, which was obtained by treating gravity as an ordinary force and disregarding the difficulties arising from the nonexistence of inertial frames.

VII. PROBLEM 1: EINSTEIN'S ELEVATOR IN NONRELATIVISTIC QUANTUM MECHANICS

Because the Schrödinger equation has its familiar form in Galilean frames, there are no surprising quantum effects, caused by Newtonian gravity, which we could not foresee without the geometrical formulation. However, the transformation theory of the state function allows us to compare the descriptions of a given quantum state by two different observers and to obtain in this way interesting insights into how the "principle of equivalence" works for quantum systems. We are ready to illustrate this point on two typical examples:

(i) In this section, we shall look at the inertial motion of a particle in a space free of gravitation, first from an inertial frame of reference and then from a uniformly accelerated frame (Einstein's elevator) which simulates a homogeneous gravitational field.

(ii) In the next section we shall observe a simple system subject to a true gravitational field, namely, the gravitational harmonic oscillator, first from a Galilean and then from a Gaussian frame of reference.

In our first example, the particle moving in an inertial frame $\{T, X^a\}$ is described by the state function $\psi(T, X^a)$ satisfying the Schrödinger equation

$$i\psi_{,T}(T, X^a) = -\frac{1}{2}\Delta\psi(T, X^a)\,, \tag{7.1}$$

where Δ is the Laplacian in Cartesian coordinates X^a. We pass to the frame $\{T, \tilde{X}^a\}$ of an Einstein's elevator by the transformation (2.22)–(2.25) with

$$R^a(T) = \{\tfrac{1}{2}\mathfrak{a}T^2, 0, 0\}. \tag{7.2}$$

This creates the homogeneous gravitational field

$$\tilde{\phi}(\tilde{X}^a) = \mathfrak{a}\tilde{X}^1\,, \tag{7.3}$$

of intensity $\mathfrak{a}$, along the negative $\tilde{X}^1$ axis. The Einstein elevator is a Galilean frame and in it the Schrödinger equation assumes the form given in table (6.11),

$$i\tilde{\psi}_{,T}(T, \tilde{X}^a) = (-\tfrac{1}{2}\tilde{\Delta} + \mathfrak{a}\tilde{X}^1)\tilde{\psi}(T, \tilde{X}^a)\,, \tag{7.4}$$

where $\tilde{\Delta}$ is the Laplacian in Cartesian coordinates $\tilde{X}^a$.

The motion in the $X^2 = \tilde{X}^2$ and $X^3 = \tilde{X}^3$ directions can be separated off and, since it is unaffected by the "ficticious" field (7.3), it does not interest us. We can thus focus on the one-dimensional problem in the remaining direction $X^1 \equiv X$, $\tilde{X}^1 \equiv \tilde{X}$.

Stationary solutions of Eq. (7.1)

$$\psi_E(T, X) = u_E(X)e^{-iET} \tag{7.5}$$

are, of course, plane waves:

$$u_{P_0}(X) \equiv u_E^{\pm}(X) = (2\pi)^{-1/2}e^{iP_0X}, \quad P_0 = \pm\sqrt{2E}\ ; \tag{7.6}$$

E must be non-negative, $E \geq 0$, in order that $u_E(X)$ stay finite. In the momentum representation,

$$u_{P_0}(P) = \delta(P - P_0)\,. \tag{7.7}$$

The general solution of Eq. (7.1) is a linear superposition of the plane waves (7.5) and (7.6).

Stationary solutions

$$\tilde{\psi}(T, \tilde{X}) = \tilde{u}_{\tilde{E}}(\tilde{X})e^{-i\tilde{E}T} \tag{7.8}$$

of Eq. (7.4) for a particle moving in a homogeneous field are equally well known (see, e.g., Landau and Lifshitz[29]). In the momentum representation, Eq. (7.4) becomes

$$i\tilde{\psi}_{,T}(T, \tilde{P}) = (-\tfrac{1}{2}\tilde{P}^2 - i\mathfrak{a}\,\partial/\partial\tilde{P})\,\psi(T, \tilde{P})\,, \tag{7.9}$$

and its stationary solutions are

$$\tilde{u}_{\tilde{E}}(\tilde{P}) = (1/\sqrt{2\pi\mathfrak{a}})\,e^{-i[\tilde{E}\tilde{P} - (1/6)\tilde{P}^3]/\mathfrak{a}}. \tag{7.10}$$

They are normalized to the δ function

$$\int_{-\infty}^{\infty} d\tilde{P}\,\tilde{u}_{\tilde{E}_1}{}^*(\tilde{P})\,\tilde{u}_{\tilde{E}_2}(\tilde{P}) = \delta(\tilde{E}_1 - \tilde{E}_2) \tag{7.11}$$

and satisfy the completeness relation

$$\int_{-\infty}^{\infty} d\tilde{E}\,\tilde{u}_{\tilde{E}}{}^*(P_1)\,\tilde{u}_{\tilde{E}}(P_2) = \delta(P_1 - P_2)\,. \tag{7.12}$$

The general solution of Eq. (7.4) is a linear superposition of the stationary solutions (7.8) and (7.10).

In the $\tilde{X}$ representation, the functions (7.8) are mapped into the Airy functions which remain finite for all values of $\tilde{E} \in (-\infty, \infty)$.[29] The energy $\tilde{E}$ can thus take all real values. Our further calculations are simplified if we stay in the $\tilde{P}$ representation (7.10).

In an inertial frame, the stationary solutions (7.6) are simultaneously the eigenfunctions of the momentum operator $\underline{P} = -i\,\partial/\partial X$ corresponding to the eigenvalues $P_0 = \pm\sqrt{2E}$. This is possible because the $\underline{H}$ and $\underline{P}$ operators commute, $[\underline{H},\underline{P}]=0$. In the elevator frame, the stationary solutions (7.10) are not eigenfunctions of the $\underline{P} = -i\,\partial/\partial X$ operator; in fact, the simultaneous eigenfunctions of $\underline{H}$ and $\underline{P}$ do not exist, since

$$[\underline{H},\underline{P}] = i\mathfrak{a} \neq 0\,. \qquad (7.13)$$

Our aim is to relate the $\psi(T,X)$ and $\tilde{\psi}(T,\tilde{X})$ descriptions by our transformation theory. The transformation (7.2) leads by Eq. (3.14) to the gauge function

$$\Lambda(T,\tilde{X}) = -\mathfrak{a}\tilde{X}T - \tfrac{1}{6}\mathfrak{a}^2T^3\,. \qquad (7.14)$$

The particle described by the state function $\psi(T,X)$ in an inertial frame appears to an accelerated observer to be in the state

$$\tilde{\psi}(T,\tilde{X}) = e^{i\Lambda(T,\tilde{X})}\psi(T,\tilde{X}+\tfrac{1}{2}\mathfrak{a}T^2)\,. \qquad (7.15)$$

In particular, the plane wave (7.5), (7.6) appears as the state

$$\tilde{\psi}_{P_0}(T,\tilde{X}) = (2\pi)^{-1/2}e^{i\tilde{P}_0(T)\tilde{X}}e^{-i\int_0^T dT\,\tilde{E}_0(T)}\,, \qquad (7.16)$$

$$\tilde{P}_0(T) = P_0 - \mathfrak{a}T\,, \quad \tilde{E}_0(T) = \tfrac{1}{2}P_0^2(T) \qquad (7.17)$$

to an observer in the elevator. One can check directly that the state function (7.16) solves the Schrödinger equation (7.4). However, this solution is obviously not a stationary solution of Eq. (7.4), in spite of the fact that a classical particle keeps its energy $\tilde{E}$ while falling in the elevator.

To interpret the solution (7.16) we identify $\tilde{P}_0(T)$ and $\tilde{E}_0(T)$ as the momentum and the *kinetic* energy which a classical particle would have at an instant T if it starts falling at $T=0$ with the momentum P_0 in the elevator. Correspondingly, the state function (7.16) is the simultaneous eigenfunction of the operators

$$\underline{\tilde{P}} = -i\,\partial/\partial\tilde{X} \text{ and } \underline{\tilde{H}}_0 = -\tfrac{1}{2}\tilde{\Delta} \qquad (7.18)$$

belonging to the eigenvalue $\tilde{P}_0(T)$ of the momentum operator $\underline{\tilde{P}}$, and to the eigenvalue $\tilde{E}_0(T)$ of the kinetic energy operator $\underline{\tilde{H}}_0$. In other words, it is always an eigenfunction of these operators, but at each instant it belongs to different eigenvalues, the eigenvalues changing with time according to equations expected for a classical particle. We label the function $\tilde{\psi}_{P_0}(T,\tilde{X})$ by the momentum $\tilde{P}_0(0) = \tilde{P}_0 = P_0$ which the particle has at the time $T=0$. In the $\tilde{P}$ representation, $\tilde{\psi}_{P_0}$ becomes

$$\tilde{\psi}_{P_0}(T,\tilde{P}) = \delta(\tilde{P}-\tilde{P}_0(T)) \times \exp\left[-i\int_0^T dT\,\tfrac{1}{2}\tilde{P}_0^2(T)\right]; \qquad (7.19)$$

at $T=0$, the state function (7.19) for the particle in the elevator coincides with the state function (7.5), (7.7) for the particle in the inertial frame:

$$\tilde{\psi}_{P_0}(0,\tilde{P}) = \delta(\tilde{P}-P_0)\,. \qquad (7.20)$$

The solution of the Schrödinger equation matching the initial condition (7.20) is, of course, the propagator $\tilde{K}(\tilde{P},T;\tilde{P}_0,0)$ of the particle. Given the state function $\tilde{\psi}(0,\tilde{P}_0)$ at $T=0$, we obtain the state function $\tilde{\psi}(T,\tilde{P})$ at T from the formula

$$\tilde{\psi}(T,\tilde{P}) = \int d^3\tilde{P}_0\,\tilde{K}(T,\tilde{P};0,\tilde{P}_0)\,\tilde{\psi}(0,\tilde{P}_0)\,. \qquad (7.21)$$

Similarly, the plane wave

$$K(T,P;0,P_0) \equiv \delta(P-P_0)e^{-P_0^2T/2} \qquad (7.22)$$

is the propagator of the particle in the inertial frame. We have thus obtained the propagator for the particle in the elevator by transforming the propagator of the particle from an inertial frame. Due to the commutation relations (7.13), $\tilde{\psi}_{P_0}(T,\tilde{X})$ cannot be an eigenfunction of $\underline{\tilde{H}}$ since it is an eigenfunction of $\underline{\tilde{P}}$. In fact, Eq. (7.13) implies the uncertainty relation

$$\Delta\tilde{P}\,\Delta\tilde{E} \geq \tfrac{1}{2}\mathfrak{a} \qquad (7.23)$$

between the momentum and the total energy of the particle in the elevator. If $\tilde{P}$ has a well-defined value, as it does in the state $\tilde{\psi}_{P_0}(T,\tilde{X})$, the energy $\tilde{E}$ must be completely undetermined. We can explain this further by saying that a particle having a definite momentum $\tilde{P}$ is entirely smeared out in its position $\tilde{X}$, so that it can be found anywhere in the field of the gravitational potential (7.3). While the kinetic energy $\tilde{E}_0$ of such a particle has a definite value, the potential energy (7.3), which is proportional to $\tilde{X}$, can attain any value with equal probability; the same conclusion follows for the total energy $\tilde{E}$.

These remarks help us to understand why the transformed plane wave (7.16) is not a stationary state for an observer in the elevator. We would like, however, to analyze the situation further and find an explicit connection between the stationary states for an inertial observer and the stationary states for a uniformly accelerated observer.

To do that, we take the plane wave (7.5) and (7.7) which is a stationary state in the inertial frame, observe it from the elevator, where it appears as a time-dependent state (7.19), and expand this state in the stationary states (7.8) and (7.9) in the elevator:

$$\tilde{\psi}_{P_0}(T,\tilde{P}) = \int_{-\infty}^{\infty} d\tilde{E}\,\tilde{\psi}_{P_0}(\tilde{E})\,\tilde{u}_{\tilde{E}}(\tilde{P})\,e^{-i\tilde{E}T}\,. \qquad (7.24)$$

We want to use this equation for the construction of stationary states $\tilde{u}_{\tilde{E}}(\tilde{P})$ in the elevator from the stationary states $\tilde{u}_E(P)$ in the inertial frame. We thus forget the actual form (7.10) of the stationary states $\tilde{u}_{\tilde{E}}(\tilde{P})$ and assume only that they form a complete orthonormal set, Eqs. (7.12) and (7.11).

The expansion coefficient $\tilde{\psi}_{P_0}(\tilde{E})$ is the plane-wave state in the $\tilde{E}$ representation. We can see, however, that it must be at the same time the conjugate form of the stationary state $\tilde{u}_{\tilde{E}}(P_0)$, i.e., of the $\underline{\tilde{H}}$ eigenstate in the $\tilde{P}$ representation:

$$\tilde{\psi}_{P_0}(\tilde{E}) = \tilde{u}_{\tilde{E}}^*(P_0)\,. \tag{7.25}$$

This follows by looking at Eq. (7.24) at $T=0$ and by comparing it with the completeness relation (7.12), keeping in mind the initial condition (7.20). We can then invert Eq. (7.23) by taking its Fourier transform in time, obtaining

$$\tilde{u}_{\tilde{E}}(\tilde{P})\tilde{u}_{\tilde{E}}^*(\tilde{P}_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dT\,\tilde{\psi}_{\tilde{P}_0}(T,\tilde{P})e^{i\tilde{E}T}\,. \tag{7.26}$$

The expression on the left-hand side of Eq. (7.26) is the density matrix $\tilde{\rho}_{\tilde{E}}(\tilde{P},\tilde{P}_0)$ of the $\underline{\tilde{H}}$ eigenstate $|\tilde{E}\rangle$ in the $\tilde{P}$ representation:

$$\tilde{\rho}_{\tilde{E}}(\tilde{P},\tilde{P}_0) = \langle\tilde{P}|\underline{\tilde{\rho}}_{\tilde{E}}|\tilde{P}_0\rangle = \tilde{u}_{\tilde{E}}(\tilde{P})u_{\tilde{E}}^*(\tilde{P}_0)\,,\qquad \underline{\tilde{\rho}}_{\tilde{E}} \equiv |\tilde{E}\rangle\langle\tilde{E}|\,. \tag{7.27}$$

Equation (7.26) tells us that this density matrix is the temporal Fourier transform of the plane wave (in the $\tilde{P}$ representation) observed in the elevator.

Equation (7.26) enables us to recover the state function $\tilde{u}_{\tilde{E}}(\tilde{P})$. Substituting the explicit form (7.18) and (7.17) of the state function $\tilde{\psi}_{P_0}(T,\tilde{P})$ into Eq. (7.26), we arrive at the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dT\,\delta(\tilde{P}-P_0+\mathfrak{A}T)\times\exp[-i(\tfrac{1}{2}P_0{}^2T-\tfrac{1}{2}P_0\mathfrak{A}T^2+\tfrac{1}{6}\mathfrak{A}^2T^3)]e^{i\tilde{E}T}\,. \tag{7.28}$$

Because

$$\delta(\tilde{P}-P_0+\mathfrak{A}T) = \mathfrak{A}^{-1}\delta(T-\mathfrak{A}^{-1}(P_0-\tilde{P}))\,, \tag{7.29}$$

everything reduces to the simple substitution $T=(P_0-\tilde{P})/A$ and we get the expected result

$$\tilde{u}_{\tilde{E}}(\tilde{P})\tilde{u}_{\tilde{E}}^*(P_0) = (1/\sqrt{2\pi\mathfrak{A}})e^{-i[\tilde{E}\tilde{P}-(1/6)\tilde{P}^3]/\mathfrak{A}}\times(1/\sqrt{2\pi\mathfrak{A}})e^{-i[\tilde{E}P_0-(1/6)P_0{}^3]/\mathfrak{A}}\,. \tag{7.30}$$

We have thereby reconstructed the stationary state $\tilde{u}_{\tilde{E}}(\tilde{P})$ as the appropriate "transform" of the plane wave $u_E(P)$.

VIII. PROBLEM 2: GRAVITATIONAL HARMONIC OSCILLATOR

The gravitational harmonic oscillator is a neutral particle which, in a Galilean frame, freely falls in a harmonic gravitational potential

$$\phi(X^a) = \tfrac{1}{2}\omega^2R^2,\quad R^2 = \delta_{ab}X^aX^b\,. \tag{8.1}$$

Such a potential is produced, according to the Newton law of gravitation (2.19), within a spherically symmetric body of a uniform density μ, with

$$\omega^2 = \tfrac{4}{3}\pi\mu\,. \tag{8.2}$$

The particle is influenced by the gravitational field of the body, but it is not supposed to interact with its material in any other way.

In the Galilean frame, the Schrödinger equation is [Table (6.11)]

$$i\psi_{,T}(T,X^a) = (-\tfrac{1}{2}\Delta + \tfrac{1}{2}\omega^2R^2)\psi(T,X^a)\,. \tag{8.3}$$

We can separate the Cartesian coordinates X^a and thereby decompose the isotropic harmonic oscillator (8.3) into three independent linear oscillators

$$i\psi_{,T}(T,X) = [-\tfrac{1}{2}(\partial^2/\partial X^2) + \tfrac{1}{2}\omega^2X^2]\psi(T,X)\,. \tag{8.4}$$

For simplicity, X denotes any one of the coordinates X^a, and we have retained the old symbol ψ for the single separated component of the original state function. In the following, we shall focus on the one-dimensional problem (8.4).

Stationary solutions of Eq. (8.4),

$$\psi_n(T,X) = u_n(X)\,e^{-iE_nT}\,, \tag{8.5}$$

are the standard Hermite functions

$$u_n(X) = 2^{-(1/2)n}\pi^{-1/4}(n!)^{-1/2}\times e^{-(1/2)\omega X^2}H_n(\sqrt{\omega}X)\,, \tag{8.6}$$

$$E_n = (n+\tfrac{1}{2})\omega,\quad n=0,1,2,\ldots\,. \tag{8.7}$$

As in our previous example with the Einstein elevator, it is best to carry out all calculations in the P representation. The stationary solutions are again the Hermite functions

$$u_n(P) = 2^{-(1/2)n}\pi^{-1/4}(n!)^{-1/2}\times e^{-(P^2/2\omega)}H_n(P/\sqrt{\omega})\,. \tag{8.8}$$

We note that they are orthogonal and satisfy the completeness relation

$$\sum_{n=0}^{\infty} u_n(P_1)\,u_n^*(P_2) = \delta(P_1-P_2)\,. \tag{8.9}$$

The general solution of Eq. (8.4) is a linear superposition of stationary solutions (8.5)–(8.7).

We now want to look at our harmonic oscillator from a Gaussian frame of reference in which the gravitational potentials $\tilde{\phi}_v$ and $\tilde{A}^a$ are gauged away. Recall that Gaussian frames are freely falling and nonrotating. One such frame is obviously realized by test particles which are all released from rest at a single moment of absolute time, say, at $T=0$, and which then oscillate freely, all with the same frequency ω, in the interior field of the central body. The transformation from the Galilean frame $\{T,X^a\}$ to this Gaussian frame $\{T,Y^a\}$ is thus given by the formula

$$X^a = Y^a \cos\omega T. \tag{8.10}$$

We know from the general theory of Sec. IV that we can generate the transformation (8.10) from a particular solution $A(T, X^a)$ of the Hamilton-Jacobi equation (4.9) which for the harmonic oscillator takes the form

$$-A_{,T}(T, X^a) = \tfrac{1}{2}(\delta^{ab}A_{,a}A_{,b} + \omega^2\delta_{ab}X^aX^b). \tag{8.11}$$

In fact, we need to find this solution, because $\lambda = -A$ is the gauge function which removes the vector potential.

Since the motion (8.9) is spherically symmetric, we seek a spherically symmetric solution of Eq. (8.11),

$$-A_{,T}(T, R) = \tfrac{1}{2}(A_{,R}{}^2 + \omega^2 R^2). \tag{8.12}$$

One such solution is

$$-A = \tfrac{1}{2}\omega R^2 \tan\omega T = \tfrac{1}{4}\omega \sin 2\omega T\, \delta_{ab}Y^aY^b. \tag{8.13}$$

It generates the Gaussian observer (4.10),

$$v^\alpha = \{1, -\omega X^a \tan\omega T\}. \tag{8.14}$$

It is easy to check that the coordinates Y^a given by Eq. (8.10) are comoving coordinates of this observer, as

$$(\partial Y^a/\partial X^\alpha)v^\alpha = 0, \quad X^\alpha = \{T, X^a\}. \tag{8.15}$$

They are uniquely specified by the further condition that Y^a and X^a coincide at $T = 0$.

The metric tensor $g^{ab} \equiv g^{\alpha\beta}Y^a_\alpha Y^b_\beta$ takes the form

$$g^{ab} = \cos^{-2}\omega T\, \delta^{ab} \tag{8.16}$$

in the Gaussian coordinates $\{T, Y^a\}$. We see that the Gaussian system is not rigid and the metric becomes singular every half period, $T = \frac{1}{2}(\pi/\omega)$, $\frac{3}{2}(\pi/\omega)$, $\frac{5}{2}(\pi/\omega), \ldots$. This singularity is a coordinate singularity caused by the isochronous character of the harmonic motion, which forces all the reference points of the Gaussian frame, whatever place they have been released from, to meet simultaneously at the center.

In the Gaussian coordinates (8.9) and the gauge (8.13), the Schrödinger equation for the $\frac{1}{2}$-density state function

$$\begin{aligned}\tilde{\Psi}(T, Y^a) &\equiv g^{1/4}\tilde{\psi} = g^{1/4}e^{i\lambda}\psi \\ &= \cos^{3/2}\omega T \\ &\quad\times \exp(\tfrac{1}{4}i\omega \sin 2\omega T\, \delta_{ab}Y^aY^b) \\ &\quad\times \psi(T, Y^a\cos\omega T)\end{aligned} \tag{8.17}$$

takes the simple form [Table (6.11)]

$$\begin{aligned}&i\tilde{\Psi}_{,T} = -\tfrac{1}{2}\cos^{-2}\omega T\, \Delta_0\tilde{\Psi}, \\ &\Delta_0 \equiv \delta^{ab}(\partial/\partial Y^a)(\partial/\partial Y^b).\end{aligned} \tag{8.18}$$

The gravitational field influences the motion of the particle only through the conformal dependence of the metric g^{ab} on time. We simplify the problem even further if we label the leaves of absolute time by a new parameter

$$\mathcal{T} = \int_0^T dT \cos^{-2}\omega T = \omega^{-1}\tan\omega T. \tag{8.19}$$

With this parameter, the Schrödinger equation formally reduces to that of a free particle

$$i\tilde{\Psi}_{,\mathcal{T}}(\mathcal{T}, Y^a) = -\tfrac{1}{2}\Delta_0\tilde{\Psi}. \tag{8.20}$$

Equations (8.18) or (8.20) can again be separated in coordinates Y^a. For each single component function, we get

$$i\tilde{\Psi}_{,T} = -\tfrac{1}{2}\cos^{-2}\omega T\, \tilde{\Psi}_{,YY} \tag{8.21}$$

or

$$i\tilde{\Psi}_{,\mathcal{T}} = -\tfrac{1}{2}\tilde{\Psi}_{,YY}. \tag{8.22}$$

Any separated component of $\tilde{\Psi}$ is connected with the corresponding separated component of ψ by an appropriate piece of Eq. (8.17):

$$\begin{aligned}\tilde{\Psi}(T, Y) = &\cos^{1/2}\omega T\, e^{(1/4)i\omega Y^2 \sin 2\omega T} \\ &\times \psi(T, Y\cos\omega T).\end{aligned} \tag{8.23}$$

We need to concentrate only on a single component equation (8.22).

Stationary solutions of Eq. (8.22)—stationary in the new time $\mathcal{T}$—are, of course, the plane waves

$$\begin{aligned}&\tilde{\Psi}_{\tilde{P}_0}(\mathcal{T}, Y) = \tilde{u}_{\tilde{P}_0}(Y)e^{-i\tilde{E}\mathcal{T}}, \\ &\tilde{u}_{\tilde{P}_0}(Y) = \frac{1}{\sqrt{2\pi}}\, e^{i\tilde{P}_0 Y}, \quad \tilde{E} = \tfrac{1}{2}\tilde{P}_0{}^2,\end{aligned} \tag{8.24}$$

and the general solution of Eq. (8.22) is their linear superposition.

We have thus transformed the gravitational harmonic oscillator in a Galilean frame into a system which formally looks like a free particle in the Gaussian frame. We want now to know how the plane wave (8.24) looks in the original Galilean frame and what its relationship is to the Hermite functions (8.6).

Equation (8.23) can be inverted, yielding

$$\begin{aligned}\psi(T, X) = &\cos^{-1/2}\omega T\, e^{-(1/2)i\omega X^2 \tan\omega T} \\ &\times \tilde{\Psi}(T, X\cos^{-1}\omega T).\end{aligned} \tag{8.25}$$

In particular, the plane wave (8.24) transforms into the function

$$\begin{aligned}\psi_{\tilde{P}_0}(T, X) = &\frac{1}{\sqrt{2\pi}}\cos^{-1/2}\omega T \\ &\times \exp\{i[\tilde{P}_0 X\cos^{-1}\omega T - \tfrac{1}{2}\omega X^2\tan\omega T \\ &\quad -(1/2\omega)\tilde{P}_0{}^2\tan\omega T]\}.\end{aligned} \tag{8.26}$$

Of course, one can check directly that this rather complicated function of X and T satisfies the Schrödinger equation (8.4). Obviously, the state (8.26) is neither a stationary solution of Eq. (8.4) nor an eigenfunction of the momentum operator $\underline{P}$. We get an insight into the physical meaning of this state by noticing that it reduces to a plane wave in the Galilean frame at the initial time $T=0$,

$$\psi_{\tilde{P}_0}(0,X)=\frac{1}{\sqrt{2\pi}}\,e^{i\tilde{P}_0X}. \tag{8.27}$$

Therefore, the time-dependent state function (8.26) describes the process of the scattering of the plane wave (8.27) by the harmonic potential. The state (8.27) is an eigenstate of the momentum operator $\underline{P}=-i\,\partial/\partial X$ for the eigenvalue $P_0=\tilde{P}_0$. As time goes on, the incident wave breaks into components with other values of P and the state ceases to be an eigenstate of $\underline{P}$. The probability amplitude for the momentum P at the time T is obtained by casting the state function (8.26) into the P representation:

$$\begin{aligned}\psi_{\tilde{P}_0}(T,P)&\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dX\,\psi_{\tilde{P}_0}(T,X)e^{-iPX}\\&=\frac{1}{2\pi}\cos^{-1/2}\omega T\exp\left\{(i/2\omega)\left[(P^2+\tilde{P}_0{}^2)\cot\omega T-\frac{2P\tilde{P}_0}{\sin\omega T}\right]\right\}\\&\quad\times\int_{-\infty}^{\infty}dX\exp\{-\tfrac{1}{2}i\omega\tan\omega T[X+\omega^{-1}(P\cot\omega T-\tilde{P}_0\sin^{-1}\omega T)]^2\}.\end{aligned} \tag{8.28}$$

The Gaussian integral in Eq. (8.28) has the value $(-2\pi i\omega^{-1}\tan^{-1}\omega T)^{1/2}$, so that

$$\begin{aligned}\psi_{P_0}(T,P)&=(1/\sqrt{2\pi i\omega})\sin^{-1/2}\omega T\\&\quad\times\exp\{(i/2\omega)[\cot\omega T(P^2+P_0{}^2)\\&\quad-2\sin^{-1}\omega T\,P\tilde{P}_0]\}.\end{aligned} \tag{8.29}$$

This is the familiar form of the propagator $K(P,T;P_0,0)$ for the harmonic oscillator in the momentum representation. We obtained this propagator by transforming the "free particle" propagator from the Gaussian frame into the Galilean frame. The initial condition (8.27) takes the form

$$\psi_{P_0}(0,P)=\delta(P-P_0) \tag{8.30}$$

in the momentum representation.

An alternative description of the state (8.26) is this: It is an eigenstate of an appropriate component of a conserved time-dependent operator $\underline{P}_a(T)$ which is constructed from the basic operators $\underline{X}^a$ and $\underline{P}_a$ in the Galilean frame in such a way that it reduces to the momentum operator $\underline{P}_a$ at $T=0$. Let us show how to obtain this well-known operator from our transformation theory. We take the momentum operator $\underline{P}_a$ in the Galilean frame and cast it into the momentum operator $\tilde{\underline{P}}_a$ in the Gaussian frame by the coordinate transformation (8.10) accompanied by the gauging (8.13),

$$\begin{aligned}\tilde{\underline{P}}_a&=\underline{P}_b\frac{\partial X^b}{\partial Y^a}+\frac{\partial\lambda(T,Y^c)}{\partial Y^a}\\&=\underline{P}_a\cos\omega T+\omega\underline{X}_a\sin\omega T.\end{aligned} \tag{8.31}$$

The operator $\tilde{\underline{P}}_a$, expressed in this manner as a function of T, $\underline{X}^a$, and $\underline{P}_a$, is the desired operator $\tilde{\underline{P}}_a(T)$. It is conserved because

$$\frac{d\underline{P}_a(T)}{dT}=\frac{\partial\underline{P}_a(T)}{dT}+\frac{1}{i}\,[\underline{P}_a(T),\underline{H}]=0. \tag{8.32}$$

The state function (8.26) is an eigenfunction of the operator (8.31) belonging to the eigenvalue $\tilde{P}_0$ at any time T.

Our last step is to connect "the plane wave in the Gaussian frame observed from the Galilean frame," i.e., the state (8.26) or (8.29), with stationary states in the Galilean frame, i.e., with the Hermite functions (8.5)–(8.8). The procedure is the same as for the Einstein elevator. We decompose the function (8.29) into Hermite functions (8.8),

$$\psi_{P_0}(T,P)=\sum_{n=0}^{\infty}\psi_{P_0}(n)\,u_n(P)\,e^{-i(n+1/2)\omega T}. \tag{8.33}$$

The coefficients $\psi_{P_0}(n)$ express the state (8.29) in the energy representation. For $T=0$, we can take into account the initial condition (8.30) and compare Eq. (8.33) with the completeness relation (8.9), concluding that

$$\psi_{P_0}(n)=u_n^*(P_0). \tag{8.34}$$

The temporal Fourier transform of Eq. (8.33) gives us then the projection operator into the stationary state $|E_n\rangle$ in terms of the plane wave (8.29),

$$u_n(P)\,u_n^*(P_0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dT\,\psi_{P_0}(T,P)\,e^{i(n+1/2)\omega T}. \tag{8.35}$$

From here, we can recover the Hermite functions starting from the plane waves in the Gaussian frame.

The problems we have studied in Secs. VII and VIII illustrate how the solution of the Schrödinger equation in one frame of reference generates the solution of the Schrödinger equation in another frame of reference. In this way, we have been able to reduce the motion of a particle in a homogeneous field and the motion of a harmonic oscillator to the motion of a free particle. Such a simple reduction works only for these elementary systems. In general, the metric tensor in the Gaussian frame depends on T and Y^a in a complicated manner and the rescaling of time does not bring it back to the Kronecker δ. Stationary states do not then exist in the Gaussian frame. Still, we can transform any solution of the Schrödinger equation in one frame into a solution of the corresponding equation in any other frame by a combination of coordinate and gauge transformations.

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[1]For a review of quantum field theory in curved spacetime, see B. S. DeWitt, Phys. Rep. 19C, 297 (1975); or P. C. W. Davies, in *GRG Einstein Centenary Volume*, edited by P. Bergman, J. Boldberg, and A. Held (to be published).

[2]E. P. Wigner, Ann. Math. 40, 149 (1935); see also E. P. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by E. Gürsey (Gordon and Breach, New York, 1964).

[3]V. Bargmann, Ann. Math. 59, 1 (1952).

[4]A. Ashtekar and A. Magnon, Proc. R. Soc. London A346, 375 (1975).

[5]P. Hájíček, Phys. Rev. D 15, 2757 (1977).

[6]A. Einstein, Jahrb. Radioakt. Elektron. 4, 411 (1907).

[7]E. Cartan, Ann. Ec. Norm. 40, 325 (1923); 41, 1 (1924); reprinted in *Oeuvres Complètes* (Gauthier-Villars, Paris, 1955), Vol. III/1, pp. 659 and 799.

[8]P. Frank, Sitzungsber. Akad. Wiss. Wien, Math.-Naturwiss. Kl. 118, Abt. IIa, 373 (1909).

[9]H. Weyl, *Raum-Zeit-Materie* (Springer, Berlin, 1921).

[10]K. Friedrichs, Math. Ann. 98, 566 (1927).

[11]P. Havas, Rev. Mod. Phys. 36, 938 (1964).

[12]A. Trautman, C. R. Acad. Sci. 257, 617 (1963).

[13]A. Trautman, in *Perspectives in Geometry and Relativity*, edited by B. Hoffmann (Indiana Univ. Press, Bloomington, 1966).

[14]A. Trautman, in *Lectures on General Relativity*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, 1965).

[15]C. W. Misner, in *Brandeis Summer Institute 1968, Astrophysics and General Relativity*, edited by M. Chrétien, S. Deser, and J. Goldstein (Gordon and Breach, New York, 1969).

[16]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 12.

[17]H. D. Dombrowski and K. Horneffer, Nachr. Akad. Wiss. Göettingen, Math.-Phys. Kl. 233 (1964).

[18]H. D. Dombrowski and K. Horneffer, Math. Z. 86, 291 (1964).

[19]H. P. Künzle, Ann. Inst. Henri Poincaré 42, 337 (1972).

[20]G. Dautcourt, Acta Phys. Pol. 65, 637 (1964).

[21]H. P. Künzle, Gen. Relativ. Gravit. 7, 445 (1976).

[22]K. Kuchař, University of Utah report, 1980 (unpublished).

[23]For an alternative set of conditions on the curvature tensor, see Künzle, Ref. 19.

[24]Square brackets mean antisymmetrization of the enclosed pair of indices: $w_{[\alpha\beta]} \equiv w_{\alpha\beta} - w_{\beta\alpha}$. Similarly, round brackets mean symmetrization: $w_{(\alpha\beta)} = w_{\alpha\beta} + w_{\beta\alpha}$. Partial differentiation is denoted by a comma.

[25]From this point on, we use the natural units $G=1$, $c=1$.

[26]For simplicity, we are taking the mass of the particle equal to unity, $m=1$. A homogeneous Lagrangian of the type (5.3) was used by Havas (Ref. 11, Sec. IV.4) to yield the motion of particles in a potential field of force in a Galilean spacetime without gravitation. The possibility of a transition to generalized Hamiltonian formalism is mentioned by Havas but not worked out in detail.

[27]See, e.g., C. Lanczos, *The Variational Principles of Mechanics*, 4th ed. (Univ. of Toronto Press, Toronto, 1970), Sec. VI.10.

[28]P. A. M. Dirac, Can. J. Math. 3, 1 (1951); *Lectures on Quantum Mechanics* (Academic, New York, 1965).

[29]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, 3rd ed. (Pergamon, Oxford, 1977), Sec. 24.