Force on a static charge outside a Schwarzschild black hole

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An exact calculation of the gravitationally induced electrostatic self-force on a charged test particle held stationary outside a Schwarzschild black hole is presented. After mass renormalization, there remains a finite repulsive force proportional to the square of the charge of the particle and to the mass of the black hole. In a local, freely falling frame momentarily at rest with respect to the charge, the repulsive force varies as the inverse cube of the Schwarzschild coordinate distance of the charge from the origin.

I. INTRODUCTION

Over the last twenty years a number of authors[1-4] have investigated the influence of a gravitational field on charged test particles. Such work has been motivated both by formal and by practical considerations. On the formal side, the questions involved are interesting because they are simple yet subtle. As DeWitt and DeWitt[1] pointed out, they raise "some of the most delicate issues in classical particle physics." On the practical side, this work may be viewed as a small part of the ongoing astrophysical research program[5] to understand electrodynamical effects in processes such as accretion and x-ray emission around black holes and other strong gravitational sources.

One of the results of these investigations is that a gravitational field modifies the electrostatic self-interaction of a charged particle in such a way that the particle experiences a finite self-force. The origin of this force is the distortion of the charge's long-range Coulomb field by the spacetime curvature associated with the gravitational field. A variety of techniques have been used to demonstrate this. DeWitt and DeWitt[1] employed a formalism of curved-space covariant Green's functions previously developed by DeWitt and Brehme.[6] Berends and Gastmans[2] used covariant perturbation theory to do a quantum-field-theoretic derivation. MacGruder[3] used an approximate Lagrangian for charged-particle motion derived by Bazanski[7] from the Einstein field equations. Most recently, Vilenkin[4] has approached the problem in perhaps the most straightforward way, using the curved-space Maxwell equations. In all of these approaches the authors assumed that the gravitational field was weak; thus they worked to leading order in the small dimensionless quantity GM/c^2r, where M is the mass of the gravitational source and r is the distance from it. They found that the self-force on a charge e is repulsive (i.e., directed away from the gravitational source) and has magnitude GMe^2/c^2r^3.

The purpose of this paper is to present an exact calculation, valid to all orders in GM/c^2r, of the gravitationally induced electrostatic self-force on a charged test particle which is held stationary outside a Schwarzschild black hole. In this admittedly special case an exact calculation is made possible by the fortuitous existence of a previously discovered[8,9] analytic solution to the curved-space Maxwell equations which are the basis of our approach. Our result for the magnitude of the repulsive self-force is

$$F_{\text{self}}=\frac{GMe^2}{c^2r_S{}^3}\ . \tag{1}$$

Here r_S is the Schwarzschild radial coordinate of the stationary test particle. This formula gives the self-force which would be measured by an instantaneously comoving, freely falling observer at the position of the test particle. We note again that Eq. (1) is exact. Its similarity to the weak-field result obtained by previous authors is a coincidence resulting from our particular choice of Schwarzschild coordinates and a freely falling observer. For example, in terms of isotropic coordinates [see Eq. (11)], the self-force takes the form

$$\begin{aligned}F_{\text{self}}&=\frac{GMe^2}{c^2r^3}\,\frac{1}{(1+GM/2c^2r)^6}\\&=\frac{e^2}{r^2}\,\frac{GM}{c^2r}\left(1-\frac{3GM}{c^2r}+\cdots\right)\ .\end{aligned} \tag{2}$$

We arrive at our result by calculating the external force required to hold the test particle static and find that it is given by

$$F_{\text{ext}}=\frac{GMm}{r_S{}^2}\left(1-\frac{2GM}{c^2r_S}\right)^{-1/2}-\frac{GMe^2}{c^2r_S{}^3}\ , \tag{3}$$

where m is the mass of the test particle.

The first term is present for uncharged as well as for charged test particles. In Newtonian language, it is just the negative of the gravitational force that the hole exerts on the test particle.

The second term is peculiar to charged test particles. Since the hole is uncharged, we must interpret it as arising from the test particle's electrostatic self-interaction. It vanishes as $M \to 0$, indicating that the effect is induced by the hole's spacetime curvature. We conclude that the hole's gravitational field causes a stationary test particle to feel a repulsive self-force as given in (1). As a consequence, the external force needed to support a charged test particle differs by this amount from that for an uncharged one of the same mass.

It may appear strange at first glance that a gravitational field can induce such a self-force, because the equivalence principle assures us that, for a freely falling observer, in the neighborhood of the test particle the same Maxwell equations hold as in flat spacetime, where there is no self-force. But this paradox is resolved when one realizes that the equivalence principle does *not* say that these same differential equations will have the same solution locally as in flat spacetime. Indeed, a solution for the potential which locally looks just like the flat-space one would not have the correct asymptotic behavior at spatial infinity, because spacetime curvature affects the long-range behavior. The solution which *does* behave correctly at infinity is not the same locally as the flat-space one, although it still satisfies the flat-space Maxwell equations locally. The different local behavior gives rise to the self-force. The long-range nature of electromagnetism has forced us to impose boundary conditions far outside the local jurisdiction of the equivalence principle.

II. CALCULATION OF THE SELF-FORCE

A. Metric and freely falling observer

We consider a gravitational field described by the Schwarzschild metric,

$$ds^2 = -(1-2M/r_S)dt^2 + (1-2M/r_S)^{-1}dr_S{}^2 + r_S{}^2(d\theta^2 + \sin^2\theta d\phi^2) , \quad (4)$$

written here in terms of the standard Schwarzschild coordinates t, r_S, θ, and ϕ. (In this section $G=c=1$.) We shall compute the external force required to hold a test particle with mass m and charge e fixed at the point $r_S = b_S$, $\theta = 0$. It is anticipated that this external force will have two contributions, one of which is to be identified as the negative of the gravitational force on the test particle and the other as the negative of the electrostatic self-force.

We expect to encounter the usual divergences associated with a point particle, and indeed the electrostatic self-force will turn out to be infinite. However, the infinite part of the self-force will have precisely the same location dependence as the gravitational force on the "bare mass" of the test particle, and thus it can be absorbed by a renormalization of the test particle's mass. A finite part of the self-force will remain, though.

In order to focus on measurable quantities, we must specify the observer who measures the external force. The most convenient choice is a freely falling observer, located at the test particle, instantaneously at rest with respect to it.

To transform quantities into this observer's frame, we need expressions for his coordinates in terms of the Schwarzschild coordinates. Since he is freely falling, his coordinates $x^{\bar{0}} \equiv \bar{t}$, $x^{\bar{1}} \equiv \bar{x}$, $x^{\bar{2}} \equiv \bar{y}$, and $x^{\bar{3}} \equiv \bar{z}$ can be chosen to be locally flat; this means that, at the spacetime point $\mathcal{P}$ at which the force is measured,

$$(g_{\bar{\alpha}\bar{\beta}})_{\mathcal{P}} = \eta_{\bar{\alpha}\bar{\beta}} \quad (5)$$

and

$$(\partial_{\bar{\gamma}}\, g_{\bar{\alpha}\bar{\beta}})_{\mathcal{P}} = 0 . \quad (6)$$

The construction of such locally flat coordinates is a well-known procedure.[10] Given an arbitrary set of coordinates x^μ in which the metric is $g_{\mu\nu}$, the coordinates

$$x^{\bar{\alpha}} = (\Lambda^{\bar{\alpha}}{}_{\mu})_{\mathcal{P}} [(x-b)^\mu + \tfrac{1}{2}(\Gamma^\mu_{\kappa\lambda})_{\mathcal{P}}(x-b)^\kappa (x-b)^\lambda + \cdots] \quad (7)$$

are locally flat at the point $\mathcal{P}$, the coordinates of which are $x^\mu = b^\mu$ in the arbitrary coordinates and $x^{\bar{\alpha}} = 0$ in the locally flat ones. Here $\Gamma^\mu_{\kappa\lambda}$ is the affine connection

$$\Gamma^\mu_{\kappa\lambda} = \tfrac{1}{2} g^{\mu\nu}(\partial_\kappa g_{\nu\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\nu g_{\kappa\lambda}) \quad (8)$$

and $\Lambda^{\bar{\alpha}}{}_{\mu}$ must satisfy

$$g_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}} \Lambda^{\bar{\alpha}}{}_{\mu} \Lambda^{\bar{\beta}}{}_{\nu} . \quad (9)$$

We note that the condition of local flatness determines the coordinates $x^{\bar{\alpha}}$ only through second order in $(x-b)^\mu$, but this will be sufficient. Also, Eq. (9) determines $\Lambda^{\bar{\alpha}}{}_{\mu}$ only up to an arbitrary Lorentz transformation.

In the present case, it proves convenient to take the x^μ to be isotropic coordinates[11]

$$\begin{aligned} x^0 &\equiv t , \\ x^1 &\equiv x = r(r_S)\sin\theta\cos\phi , \\ x^2 &\equiv y = r(r_S)\sin\theta\sin\phi , \\ x^3 &\equiv z = r(r_S)\cos\theta , \end{aligned} \quad (10)$$

rather than the standard Schwarzschild coordinates themselves, where

$$r = \tfrac{1}{2}(r_S - M + [r_S(r_S - 2M)]^{1/2}),$$

or

$$r_S = r(1+M/2r)^2 \ . \tag{11}$$

In these coordinates the metric is given by

$$ds^2 = -\left(\frac{1-M/2r}{1+M/2r}\right)^2 dt^2 + (1+M/2r)^4(dx^2+dy^2+dz^2) \ . \tag{12}$$

The test particle is located at $x=0$, $y=0$, $z=b = \frac{1}{2}\{b_S - M + [b_S(b_S-2M)]^{1/2}\}$ for all t. Choosing the time of the measurement of the external force to be $t=0$, we take $b^\mu = (0,0,0,b)$. We impose the requirement that the observer be instantaneously at rest relative to the test particle at $t=0$ and that his local coordinate axes to be aligned with the isotropic ones at that instant; then $\Lambda^{\bar\alpha}{}_\mu$ is uniquely determined to be

$$\Lambda^{\bar\alpha}{}_\mu = \begin{pmatrix} \sqrt{-g_{00}} & & & \\ & \sqrt{g_{11}} & & \\ & & \sqrt{g_{22}} & \\ & & & \sqrt{g_{33}} \end{pmatrix} . \tag{13}$$

Using (7), we then find the following transformation between the two frames:

$$\bar t = \frac{1-M/2b}{1+M/2b}\, t + \frac{M}{b^2}\, \frac{1}{(1+M/2b)^2}\, t(z-b) + O((x^\mu - b^\mu)^3), \tag{14a}$$

$$x^{\bar i} = (1+M/2b)^2(x^i - b^i) + \frac{M}{2b^2}\, \frac{1-M/2b}{(1+M/2b)^5}\, \delta^{\bar i \bar z} t^2 - \frac{M}{2b^2}(1+M/2b)[2(x^i-b^i)(z-b) - \delta^{\bar i \bar z}|x^k - b^k|^2] + O((x^\mu - b^\mu)^3) \ . \tag{14b}$$

In the locally flat frame the metric takes the form $g_{\bar\alpha\bar\beta} = \eta_{\bar\alpha\bar\beta} + O((x^{\bar\gamma})^2)$; i.e., it is Minkowskian up to second-order corrections in the coordinates.

With these preliminaries out of the way, we may proceed to calculate the self-force, which we do in the next two subsections by two different methods. In subsection B we use the fact that the external force can be obtained by integrating the covariant divergence of the energy-momentum tensor over the spatial extent of the test particle. This is a local approach to the problem. In subsection C we employ a global energy-conservation argument to obtain the external force.

B. Local method

The test particle is assumed to be held fixed by some external force. The density $G^{\bar\alpha}_{\rm ext}$ of the external force measured by the freely falling observer is given by[11]

$$G^{\bar\alpha}_{\rm ext} = T^{\bar\alpha\bar\beta}{}_{;\bar\beta} \ . \tag{15}$$

The energy-momentum tensor $T^{\bar\alpha\bar\beta}$ of the system has two contributions:

$$T^{\bar\alpha\bar\beta} = T^{\bar\alpha\bar\beta}_{\rm mech} + T^{\bar\alpha\bar\beta}_{\rm em} \ . \tag{16}$$

The first term,

$$T^{\bar\alpha\bar\beta}_{\rm mech} = \frac{m_0}{(-\bar g)^{1/2}} \int \frac{d\xi^{\bar\alpha}}{d\tau}\, \frac{d\xi^{\bar\beta}}{d\tau}\, \delta^4(x^{\bar\gamma} - \xi^{\bar\gamma}(\tau))\, d\tau \ , \tag{17}$$

is the "mechanical" energy-momentum tensor for a particle of mechanical mass m_0. Here $\bar g \equiv \det(g_{\bar\alpha\bar\beta})$; τ is the proper time along the world line $x^{\bar\alpha} = \xi^{\bar\alpha}(\tau)$ of the particle and is given by

$$d\tau^2 = -g_{\bar\alpha\bar\beta}\, d\xi^{\bar\alpha} d\xi^{\bar\beta} = -g_{\mu\nu} d\xi^\mu d\xi^\nu \ . \tag{18}$$

The second term,

$$T^{\bar\alpha\bar\beta}_{\rm em} = \frac{1}{4\pi}\,(F^{\bar\alpha\bar\gamma} F^{\bar\beta}{}_{\bar\gamma} - \tfrac{1}{4} g^{\bar\alpha\bar\beta} F^{\bar\gamma\bar\delta} F_{\bar\gamma\bar\delta}) \ , \tag{19}$$

is the energy-momentum tensor for the particle's electromagnetic field. The field $F^{\bar\alpha\bar\beta}$ and the corresponding potential $A^{\bar\alpha}$ are related to the current density

$$J^{\bar\alpha} = \frac{e}{(-\bar g)^{1/2}} \int \frac{d\xi^{\bar\alpha}}{d\tau}\, \delta^4(x^{\bar\gamma} - \xi^{\bar\gamma}(\tau))\, d\tau \tag{20}$$

by the Maxwell equations

$$F_{\bar\alpha\bar\beta} = A_{\bar\beta;\bar\alpha} - A_{\bar\alpha;\bar\beta} \tag{21}$$

and

$$F^{\bar\alpha\bar\beta}{}_{;\bar\beta} = 4\pi J^{\bar\alpha} \ . \tag{22}$$

The external force $F^{\bar i}_{\rm ext}$ on the test particle is obtained by integrating the space components $G^{\bar i}_{\rm ext}$ of the force density at $\bar t = 0$ over the extent of the test particle. For a point test particle, we cannot do this; instead we shall integrate over a small sphere of radius $\bar\epsilon$ centered on the test particle and take the limit $\bar\epsilon \to 0$ afterward. In freely falling coordinates the sphere $\bar\epsilon = $ const is a true metric sphere. By symmetry, it is clear that the external force can only have a $\bar z$ component, so we need only calculate $F^{\bar z}_{\rm ext}$. We have

$$F^{\bar z}_{\rm ext} = \lim_{\bar\epsilon\to 0} \int_{\bar r \le \bar\epsilon} [G^{\bar z}_{\rm ext}]_{\bar t = 0}\, d^3\bar x$$

$$= \lim_{\bar\epsilon\to 0} \int_{\bar r \le \bar\epsilon} [T^{\bar z\bar\beta}{}_{;\bar\beta}]_{\bar t=0}\, d^3\bar x \ , \tag{23}$$

where $\bar r^2 = \bar x^2 + \bar y^2 + \bar z^2$. Now the observer's metric is Minkowskian up to $O((x^{\bar\alpha})^2)$ corrections, so the covariant derivative in (23) is $T^{\bar z\bar\beta}{}_{;\bar\beta} = \partial_{\bar 0} T^{\bar z\bar 0} + \partial_{\bar j} T^{\bar z\bar j} + O(x^{\bar\alpha} T^{\bar\beta\bar\gamma})$. Therefore

$$F^{\bar{z}}_{\text{ext}} = \lim_{\bar{\epsilon}\to 0}\left\{ \int_{\bar{r}\le\bar{\epsilon}} \left[\frac{\partial}{\partial \bar{t}} T^{\bar{z}\bar{0}}_{\text{mech}}\right]_{\bar{t}=0} d^3\bar{x} \right. \tag{24a}$$

$$+\int_{\bar{r}=\bar{\epsilon}} [T^{\bar{z}\bar{j}}_{\text{mech}}]_{\bar{t}=0}\, d^2S_{\bar{j}} \tag{24b}$$

$$+\int_{\bar{r}\le\bar{\epsilon}} [0(x^{\bar{\alpha}} T^{\bar{\beta}\bar{\gamma}}_{\text{mech}})]_{\bar{t}=0}\, d^3\bar{x} \tag{24c}$$

$$+\int_{\bar{r}\le\bar{\epsilon}} \left[\frac{\partial}{\partial \bar{t}} T^{\bar{z}\bar{0}}_{\text{em}}\right]_{\bar{t}=0} d^3\bar{x} \tag{24d}$$

$$+\int_{\bar{r}=\bar{\epsilon}} [T^{\bar{z}\bar{j}}_{\text{em}}]_{\bar{t}=0}\, d^2S_{\bar{j}} \tag{24e}$$

$$\left. +\int_{\bar{r}\le\bar{\epsilon}} [0(x^{\bar{\alpha}} T^{\bar{\beta}\bar{\gamma}}_{\text{em}})]_{\bar{t}=0}\, d^3\bar{x} \right\}, \tag{24f}$$

where the divergence theorem has been employed to obtain the surface integrals in (24b) and (24e). Here $d^2S_{\bar{i}} \equiv \delta_{\bar{i}\bar{j}} n^{\bar{j}} \bar{r}^2 d\bar{\Omega}$, with $n^{\bar{i}} \equiv x^{\bar{i}}/\bar{r}$ being a radial unit vector and $d\bar{\Omega}$ an element of solid angle in the locally flat frame. We proceed to evaluate each term in order.

Our first task is to obtain $\xi^{\bar{\alpha}}(\tau)$, the world line of the test particle in the observer's frame, since this enters into $T^{\bar{\alpha}\bar{\beta}}_{\text{mech}}$. In the isotropic coordinate basis, the test particle is at rest at $x^i = b^i = b\delta^{iz}$ for all t, so the world line in this frame is $\xi^{\mu}(\tau) = (t(\tau), 0, 0, b)$. Using the metric in (12), one finds from (18) that

$$d\tau = \frac{1-M/2b}{1+M/2b}\, dt \ . \tag{25}$$

Choosing $\tau = 0$ when $t = 0$, we have

$$\xi^{\mu}(\tau) = \left(\frac{1+M/2b}{1-M/2b}\,\tau, 0, 0, b\right) . \tag{26}$$

Transforming this into the observer's frame by means of (14), one finds

$$\xi^{\bar{\alpha}}(\tau) = \left(\tau, O, O, \frac{M}{2b^2}\frac{1}{(1+M/2b)^3}\frac{1}{1-M/2b}\,\tau^2\right) + O(\tau^3) \ . \tag{27}$$

Substituting this and $\bar{g} = -1 + O((x^{\bar{\alpha}})^2)$ into (17) and performing the integration over τ, one finds that

$$T^{\bar{z}\bar{0}}_{\text{mech}} = \frac{m_0}{1+O((x^{\bar{\alpha}})^2)} \times\left[\frac{M}{b^2}\frac{1}{(1+M/2b)^3}\frac{1}{1-M/2b}\,\bar{t} + O(\bar{t}^2)\right] \times \delta^3(x^{\bar{k}} - X^{\bar{k}}(\bar{t})), \tag{28}$$

where

$$X^{\bar{i}}(\bar{t}) = \tfrac{1}{2} a^{\bar{i}} \bar{t}^2 + O(\bar{t}^3) \tag{29}$$

is the spatial position of the test particle in the observer's frame at time $\bar{t}$, with

$$a^{\bar{i}} = \frac{M}{b^2}\frac{1}{(1+M/2b)^3}\frac{1}{1-M/2b}\,\delta^{\bar{i}\bar{z}} \tag{30}$$

being the test particle's acceleration in this frame at $\bar{t} = 0$. Using (28), one finds that term (24a) is

$$\int_{\bar{r}\le\bar{\epsilon}} \left[\frac{\partial}{\partial \bar{t}} T^{\bar{z}\bar{0}}_{\text{mech}}\right]_{\bar{t}=0} d^3\bar{x} = \frac{Mm_0}{b^2}\frac{1}{(1+M/2b)^3}\frac{1}{1-M/2b} . \tag{31}$$

The surface term (24b) is zero because the integrand contains $\delta^3(x^{\bar{i}})$, which vanishes on the surface $\bar{r} = \bar{\epsilon}$. Term (24c), due to the $O((x^{\bar{\alpha}})^2)$ corrections to $g_{\bar{\alpha}\bar{\beta}}$, is also zero because it entails integrating $O(x^{\bar{i}})\delta^3(x^{\bar{i}})$.

Now we face the more difficult task of evaluating the electromagnetic terms (24d)–(24f). We need to know $A^{\bar{\alpha}}$ in the vicinity of the test particle so that we can then calculate $F^{\bar{\alpha}\bar{\beta}}$ and $T^{\bar{\alpha}\bar{\beta}}_{\text{em}}$. We cannot obtain $A^{\bar{\alpha}}$ by solving (21) and (22), because the locally flat coordinates (7) do not provide a global coordinate system, which is required in order to impose boundary conditions on $A^{\bar{\alpha}}$ at spatial infinity. Even if we were to define $x^{\bar{\alpha}}$ everywhere, rather than just in the neighborhood of $\mathcal{P}$, $g_{\bar{\alpha}\bar{\beta}}$ would be so complicated except near $\mathcal{P}$ that we would be unable to find an exact solution to (21) and (22).

We therefore consider the Maxwell equations in the global isotropic coordinate system. We may write them in the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{32}$$

and

$$\partial_\nu(\sqrt{-g}\,F^{\mu\nu}) = 4\pi\sqrt{-g}J^\mu \ . \tag{33}$$

Now in this basis the situation is static with $J^\mu = (J^0(x,y,z), 0, 0, 0)$ and $A_\mu = (A_0(x,y,z), 0, 0, 0)$. The only nontrivial component of (33) is the $\mu = 0$ one,

$$-\partial_i(\sqrt{-g}\,g^{00} g^{ij}\partial_j A_0) = 4\pi\sqrt{-g}\,J^0 \ . \tag{34}$$

Using the metric in (12), this becomes

$$\left[\nabla^2 - \frac{M}{r^2}\frac{2-M/2r}{(1+M/2r)(1-M/2r)}\frac{\partial}{\partial r}\right]A_0 = 4\pi(1+M/2r)^2(1-M/2r)^2 J^0 \ . \tag{35}$$

Since the test particle is stationary at $x^i = b^i = b\delta^{iz}$, its charge density is

$$J^0 = e\,\frac{1}{(1+M/2r)^5(1-M/2r)}\,\delta^3(x^k - b^k) \ . \tag{36}$$

Thus A_0 satisfies

$$\left[\nabla^2 - \frac{M}{r^2}\frac{2-M/2r}{(1+M/2r)(1-M/2r)}\frac{\partial}{\partial r}\right]A_0 = 4\pi e\,\frac{1-M/2b}{(1+M/2b)^3}\,\delta^3(x^k - b^k)\,. \tag{37}$$

The boundary conditions to be imposed on A_0 are that

$$A_0 \sim -e/r \text{ as } r\to\infty \tag{38}$$

so that, by Gauss's law, the net charge is just e.

In a 1926 paper on electrostatics in curved spacetimes, Copson[7] found a particular analytic solution to Eq. (37). It did not satisfy the boundary condition (38), but Linet[9] made the necessary modification. The solution he found is

$$A_0 = -\frac{e}{b(1+M/2b)^2 r(1+M/2r)^2}\times\left[M + b\left(\frac{r^2-2\bar b z+\bar b^2}{r^2-2bz+b^2}\right)^{1/2} + \bar b\left(\frac{r^2-2bz+b^2}{r^2-2\bar b z+\bar b^2}\right)^{1/2}\right], \tag{39}$$

where $\bar b = M^2/4b$. This is the exact electrostatic potential of a point test charge in the Schwarzschild metric.

Now we must transform (39) into the freely falling frame. We have

$$A_{\bar\alpha} = g_{\bar\alpha\bar\beta}A^{\bar\beta} = g_{\bar\alpha\bar\beta}\frac{\partial x^{\bar\beta}}{\partial x^\mu}A^\mu = g_{\bar\alpha\bar\beta}\frac{\partial x^{\bar\beta}}{\partial x^\mu}g^{\mu\nu}A_\nu\,. \tag{40}$$

Using the fact that $A_i = 0$, that $g_{\mu\nu}$ is diagonal, and that $g_{\bar\alpha\bar\beta}$ is Minkowskian plus $O((x^{\bar\alpha})^2)$ corrections, this becomes

$$A_{\bar 0} = \left[-\frac{\partial \bar t}{\partial t} + O((x^{\bar\alpha})^2)\right]g^{00}A_0\,, \tag{41a}$$

$$A_{\bar i} = \left[\frac{\partial x^{\bar i}}{\partial t} + O((x^{\bar\alpha})^2)\right]g^{00}A_0\,. \tag{41b}$$

Because of the limit $\bar\epsilon \to 0$ in (23), all we really need is the first few terms of an expansion of $A_{\bar\alpha}$ about the point $x^{\bar\alpha} = 0$. Towards this end we first expand each factor in (41), for which we have expressions in isotropic coordinates, about the corresponding point $x^\mu = b^\mu$. From (14), we get

$$\frac{\partial \bar t}{\partial t} = \frac{1-M/2b}{1+M/2b}\left[1 + \frac{M}{b^2}\frac{1}{1+M/2b}\frac{1}{1-M/2b}(z-b) + O((x^\mu - b^\mu)^2)\right] \tag{42a}$$

and

$$\frac{\partial x^{\bar i}}{\partial t} = \frac{M}{b^2}\frac{1-M/2b}{(1+M/2b)^5}\,\delta^{\bar i \bar z}t + O((x^\mu - b^\mu)^2)\,; \tag{42b}$$

the expansion of g^{00} is

$$g^{00} = -\frac{(1+M/2b)^2}{(1-M/2b)^2}\left[1 - \frac{2M}{b^2}\frac{1}{1+M/2b}\frac{1}{1-M/2b}(z-b) + O((x^k - b^k)^2)\right], \tag{43}$$

and that of A_0 is

$$A_0 = -\frac{e}{|x^k - b^k|}\frac{1-M/2b}{(1+M/2b)^3}\left[1 + \frac{M}{b^2}\frac{1-M/4b}{(1+M/2b)(1-M/2b)}(z-b) + O((x^k-b^k)^2)\right] - e\frac{M}{b^2}\frac{1}{(1+M/2b)^4}\left[1 - \frac{1}{b}\frac{1-M/2b}{1+M/2b}(z-b) + O((x^k-b^k)^2)\right]. \tag{44}$$

Now we use (14) to express these series in terms of the locally flat coordinates. This is straightforward except for the transformation of the factor $1/|x^k - b^k|$, which turns out to be

$$\frac{1}{|x^k - b^k|} = \frac{1}{|x^{\bar k} - X^{\bar k}(\bar t)|}(1+M/2b)^2\left[1 - \frac{M}{2b^2}\frac{1}{(1+M/2b)^3}\bar z + O((x^{\bar\alpha})^2)\right]. \tag{45}$$

The final result is

$$A_{\bar 0} = -e\left[\frac{1+\alpha\bar z + O((x^{\bar\alpha})^2)}{|x^{\bar k} - X^{\bar k}(\bar t)|} + \beta + \gamma\bar z + O((x^{\bar\alpha})^2)\right] \tag{46a}$$

and

$$A_{\bar i} = e\left[\frac{\lambda\delta_{\bar i\bar z}\bar t + O((x^{\bar\alpha})^2)}{|x^{\bar k} - X^{\bar k}(\bar t)|} + \mu\,\delta_{\bar i\bar z}\bar t + O((x^{\bar\alpha})^2)\right], \tag{46b}$$

where

$$-2\alpha=\beta=\lambda=\sqrt{\mu}=\frac{M}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}\,,\quad \gamma=-\frac{M}{b^3}\,\frac{1+M^2/4b^2}{(1+M/2b)^6(1-M/2b)^2}\,. \tag{47}$$

By differentiating (46) one finds the following expressions for the electromagnetic field components $E_{\bar i}\equiv F_{\bar i\bar 0}=\partial_{\bar i}A_{\bar 0}-\partial_{\bar 0}A_{\bar i}$ and $B_{\bar i}\equiv\frac{1}{2}\epsilon_{\bar i\bar j\bar k}F_{\bar j\bar k}=\epsilon_{\bar i\bar j\bar k}\partial_{\bar j}A_{\bar k}$ and their time derivatives at $\bar t=0$:

$$[E_{\bar i}]_{\bar t=0}=e\left[\frac{n^{\bar i}}{\bar r^2}+\frac{\alpha n^{\bar i}n^{\bar z}-(\alpha+\lambda)\delta^{\bar i\bar z}}{\bar r}-(\gamma+\mu)\delta^{\bar i\bar z}\right]+[O(\bar r^0)\text{ terms with an odd number of }n^{\bar k}], \tag{48a}$$

$$[B_{\bar i}]_{\bar t=0}=O(\bar r^0)\,, \tag{48b}$$

$$\left[\frac{\partial E_{\bar i}}{\partial\bar t}\right]_{\bar t=0}=O(1/\bar r)\,, \tag{48c}$$

$$\left[\frac{\partial B_{\bar i}}{\partial\bar t}\right]_{\bar t=0}=-e\epsilon_{\bar i\bar j\bar z}\lambda\frac{n^{\bar j}}{\bar r^2}+[O(1/\bar r)\text{ terms with an even number of }n^{\bar k}]. \tag{48d}$$

We proceed to use these expressions to evaluate the electromagnetic contributions (24d)–(24f) to the external force.

From (19) one obtains

$$T^{\bar z\bar 0}_{\text{em}}=\frac{1}{4\pi}\epsilon^{\bar z\bar j\bar k}E_{\bar j}B_{\bar k}+O((x^{\bar\alpha})^2(F^{\bar\beta\bar\gamma})^2)\,, \tag{49}$$

so term (24d) becomes

$$\int_{\bar r\le\bar\epsilon}\left[\frac{\partial}{\partial\bar t}T^{\bar z\bar 0}_{\text{em}}\right]_{\bar t=0}d^3\bar x=\frac{1}{4\pi}\epsilon^{\bar z\bar j\bar k}\int_0^{\bar\epsilon}\bar r^2d\bar r\int d\bar\Omega\left[\frac{\partial E_{\bar j}}{\partial\bar t}B_{\bar k}+E_{\bar j}\frac{\partial B_{\bar k}}{\partial\bar t}\right]_{\bar t=0}+\int_0^{\bar\epsilon}\bar r^2d\bar r\int d\bar\Omega\left[\frac{\partial}{\partial\bar t}O((x^{\bar\alpha})^2(F^{\bar\beta\bar\gamma})^2)\right]_{\bar t=0}. \tag{50}$$

Consider the second integral first. One can show from (48) that in the neighborhood of the test charge the most singular terms in its integrand have the structure (odd number of $n^{\bar k}$)/$\bar r^3$; these vanish by symmetry when integrated over solid angles. The less singular terms will vanish when the limit $\bar\epsilon\to 0$ is taken. Thus only the first integral above contributes. Substituting the expressions in (48) and performing the angular integrations, one obtains

$$\int_{\bar r\le\bar\epsilon}\left[\frac{\partial}{\partial\bar t}T^{\bar z\bar 0}_{\text{em}}\right]_{\bar t=0}d^3\bar x=\tfrac{2}{3}\lambda e^2\int_0^{\bar\epsilon}\frac{d\bar r}{\bar r^2}=\tfrac{2}{3}e^2\left(\int_0^{\bar\epsilon}\frac{d\bar r}{\bar r^2}\right)\frac{M}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}\,, \tag{51}$$

after dropping terms which vanish in the limit $\bar\epsilon\to 0$.

Next, since

$$T^{\bar z\bar j}_{\text{em}}=\frac{1}{4\pi}\left[\tfrac{1}{2}(E_{\bar k}E_{\bar k}+B_{\bar k}B_{\bar k})\delta^{\bar z\bar j}-E_{\bar z}E_{\bar j}-B_{\bar z}B_{\bar j}\right]+O((x^{\bar\alpha})^2(F^{\bar\beta\bar\gamma})^2)\,, \tag{52}$$

term (24e) becomes

$$\int_{\bar r=\bar\epsilon}[T^{\bar z\bar j}_{\text{em}}]_{\bar t=0}d^2S_{\bar j}=\frac{1}{4\pi}\bar\epsilon^2\int_{\bar r=\bar\epsilon}d\bar\Omega\, n^{\bar j}\left[\tfrac{1}{2}(E_{\bar k}E_{\bar k}+B_{\bar k}B_{\bar k})\delta^{\bar z\bar j}-E_{\bar z}E_{\bar j}-B_{\bar z}B_{\bar j}\right]_{\bar t=0}$$
$$+\bar\epsilon^2\int_{\bar r=\bar\epsilon}d\bar\Omega\, n^{\bar i}[O((x^{\bar\alpha})^2(F^{\bar\beta\bar\gamma})^2)]_{\bar t=0}\,. \tag{53}$$

As before, the second integral vanishes in the limit $\bar\epsilon\to 0$ because the terms which are singular enough to contribute contain an odd number of unit vectors and thus vanish by symmetry when the angular integration is performed. The first integral yields

$$\int_{\bar r=\bar\epsilon}[T^{\bar z\bar j}_{\text{em}}]_{\bar t=0}d^2S_{\bar j}=(\tfrac{2}{3}\alpha+\lambda)e^2\frac{1}{\bar\epsilon}+(\gamma+\mu)e^2$$
$$=\frac{2e^2}{3\bar\epsilon}\,\frac{M}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}-\frac{Me^2}{b^3}\,\frac{1}{(1+M/2b)^6}\,, \tag{54}$$

after dropping terms which do not contribute in the limit $\bar{\epsilon}\to 0$.

Finally, one can check that the last term (24f) does not contribute to the external force for the same reasons that the other terms arising from the $O((x^{\bar{\alpha}})^2)$ corrections to $g_{\bar{\alpha}\bar{\beta}}$ did not.

Combining our results in (31), (51), and (54), we find that Eq. (24) gives

$$F^{\bar{z}}_{\rm ext}=\frac{Mm_0}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}+\frac{M\,\delta m}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}-\frac{Me^2}{b^3}\,\frac{1}{(1+M/2b)^6}\,, \quad (55)$$

where

$$\delta m\equiv \tfrac{2}{3}e^2\lim_{\bar{\epsilon}\to 0}\left(\frac{1}{\bar{\epsilon}}+\int_0^{\bar{\epsilon}}\frac{d\bar{r}}{\bar{r}^2}\right)=\tfrac{2}{3}e^2\int_0^{\infty}\frac{d\bar{r}}{\bar{r}^2}\,. \quad (56)$$

The fact that δm is divergent is of course due to the fact that we have treated the test particle as a point charge. However, since the first two terms in (55) have the same dependence on b, we can absorb this divergence by a mass renormalization: we consider $m\equiv m_0+\delta m$ to be the actual, finite mass of the test particle. Then the external force necessary to hold it stationary at $r=b$, $\theta=0$ is

$$F^{\bar{z}}_{\rm ext}=\frac{Mm}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}-\frac{Me^2}{b^3}\,\frac{1}{(1+M/2b)^6}\,. \quad (57)$$

C. Global method

The external force can also be calculated by a global, energy-conservation method. Consider a *Gedankenexperiment* in which the charge is displaced slowly a distance $|\delta b|$ (in isotropic coordinates) toward the hole. Then, according to the freely falling observer, an amount of work $\delta\bar{W}$ is done on the mechanical support, given in his basis by

$$\delta\bar{W}=-F^{\bar{z}}_{\rm ext}\,\delta\bar{b}\,. \quad (58)$$

This work is sent in the form of a bundle of energy to an observer at rest at asymptotic infinity; because of the gravitational red-shift, the energy δE received by this observer is given by

$$\delta E=[-g_{00}(b)]^{1/2}\delta\bar{W}=\frac{1-M/2b}{1+M/2b}\,\delta\bar{W}\,. \quad (59)$$

But by conservation of energy, this energy must be equal to the change $-\delta M$ in the asymptotically measured mass of the system that results from the displacement of the charge. Then using the fact from (14) that $\delta\bar{b}=(1+M/2b)^2\delta b$, we have

$$F^{\bar{z}}_{\rm ext}=\frac{1}{1+M/2b}\,\frac{1}{1-M/2b}\,\frac{\delta M}{\delta b}\,. \quad (60)$$

Now, Carter[12] has shown that the change in mass δM between two nearby, nonrotating black-hole configurations is given by

$$\delta M=\frac{\kappa}{8\pi}\,\delta A-\frac{1}{8\pi}\,\delta\int G^0{}_0\sqrt{-g}\,d^3x+\frac{1}{16\pi}\times\int G^{\mu\nu}h_{\mu\nu}\sqrt{-g}\,d^3x\,, \quad (61)$$

where κ and A are the surface gravity and the area of the black hole, respectively, $G^{\mu\nu}$ is the Einstein tensor, and $h_{\mu\nu}$ is the difference in the metric between the two configurations. The integrals are to be evaluated over the exterior of the black hole. For a slow displacement, no matter or electromagnetic radiation crosses the horizon, so $\delta A=0$. Furthermore, since we are dealing with a *test* charged particle (ignoring the direct effect of the particle's mass or charge on the metric), we can ignore the term involving $h_{\mu\nu}$. Hence, invoking Einstein's equations, we have

$$\delta M=-\delta\int T_0{}^0\sqrt{-g}\,d^3x\,. \quad (62)$$

Let us split the "energy" integral above into a mechanical contribution and an electromagnetic contribution, i.e.,

$$-\int T^0{}_0\sqrt{-g}\,d^3x\equiv U_{\rm mech}+U_{\rm em}\,. \quad (63)$$

Then combining (60), (62), and (63), we have

$$F^{\bar{z}}_{\rm ext}=\frac{1}{1+M/2b}\,\frac{1}{1-M/2b}\,\frac{\delta}{\delta b}\,(U_{\rm mech}+U_{\rm em})\,. \quad (64)$$

The mechanical contribution to the energy is easily found from (17) and (26) to be

$$U_{\rm mech}=m_0\,\frac{1-M/2b}{1+M/2b}\,. \quad (65)$$

To evaluate the electromagnetic contribution, we first note that

$$T_{\rm em}{}^0{}_0=\frac{1}{8\pi}\,g^{00}g^{ij}\partial_iA_0\partial_jA_0\,, \quad (66)$$

as follows from (19), so

$$U_{\rm em}=\frac{-1}{8\pi}\int g^{00}g^{ij}\partial_iA_0\partial_jA_0\sqrt{-g}\,d^3x\,. \quad (67)$$

At this point one can substitute (39) for A_0 and actually carry out the integration explicitly in polar coordinates. The integral diverges because of the singularity in A_0 at the point charge, but it is possible to separate out the divergent behavior and regulate it by excluding from the region of integration a ball of radius a around the test particle.

The result of this lengthy calculation is

$$U_{\text{em}}=\frac{e^2}{2a}\,\frac{1-M/2b}{(1+M/2b)^3}+\frac{Me^2}{2b^2}\,\frac{1}{(1+M/2b)^4}\,. \qquad (68)$$

A much quicker route to this result is via an integration by parts in (67); one obtains

$$U_{\text{em}}=\frac{1}{8\pi}\int d^3x\,A_0\partial_i(\sqrt{-g}g^{00}g^{ij}\partial_jA_0)-\frac{1}{8\pi}\int_{r\to\infty}d^2S_i\sqrt{-g}g^{00}g^{ij}A_0\partial_jA_0+\frac{1}{8\pi}\int_{r=M/2}d^2S_i\sqrt{-g}g^{00}g^{ij}A_0\partial_jA_0\,. \qquad (69)$$

The integral over the surface at infinity vanishes because A_0 drops off as $1/r$ at large distances. An easy calculation using (39) shows that the angular integral on the horizon averages to zero. One is left with the first term, which because of (34) becomes

$$U_{\text{em}}=-\tfrac{1}{2}\int J^0A_0\sqrt{-g}\,d^3x\,. \qquad (70)$$

Since $\sqrt{-g}J^0=e\delta^3(x^k-b^k)$, U_{em} is formally equal to $-\frac{1}{2}eA_0$ evaluated at $x^i=b^i$, which is divergent. We regulate it by substituting

$$\sqrt{-g}J^0=\frac{e}{4\pi a^2}\lim_{a\to 0}\delta(|x^k-b^k|-a) \qquad (71)$$

and doing the integration before taking the limit. Physically, this corresponds to giving the charge a finite radius a in isotropic coordinates. To be more correct, one should give the charge a constant finite radius $\bar{a}$ in freely falling coordinates, as in subsection B. In freely falling coordinates, a surface of constant a is not exactly spherical; however, it is straightforward to show that this does not alter the final conclusion. Using the expansion (44), this yields the result (68).

Now, in order to renormalize the mass m_0 to m, we must express the radius of the ball of charge in local freely falling coordinates, because it is this radius that is treated as constant as the charge moves. We thus have from (14)

$$\bar{a}=a(1+M/2b)^2\,. \qquad (72)$$

Then we can write

$$U_{\text{mech}}+U_{\text{em}}=m\,\frac{1-M/2b}{1+M/2b}+\frac{Me^2}{2b^2}\,\frac{1}{(1+M/2b)^4}\,, \qquad (73)$$

with

$$m=m_0+\lim_{\bar{a}\to 0}\frac{e^2}{2\bar{a}}\,. \qquad (74)$$

It follows from (64) that

$$F^{\bar{z}}_{\text{ext}}=\frac{Mm}{b^2}\,\frac{1}{(1+M/2b)^3}\,\frac{1}{1-M/2b}-\frac{Me^2}{b^3}\,\frac{1}{(1+M/2b)^6}\,, \qquad (75)$$

in agreement with the local method.

III. SUMMARY AND CONCLUDING REMARKS

We have computed the external force that is required to hold a test particle of mass m and charge e stationary outside the horizon of a Schwarzschild black hold of mass M. In the frame of a freely falling observer who is instantaneously at rest at the position of the test particle, it is

$$F^{\bar{z}}_{\text{ext}}=\frac{GMm}{r_S^2}\left(1-\frac{2GM}{c^2r_S}\right)^{-1/2}-\frac{GMe^2}{c^2r_S^3}\,, \qquad (76)$$

where we have used (11) to express the force in terms of the test particle's Schwarzschild radial coordinate (now denoted r_S rather than b_S) instead of the isotropic radial coordinate b used in (57) and (75). No approximations have been made other than treating the particle as a test particle and thus ignoring its metric perturbations.

The negative of the second term above is the gravitationally induced electrostatic self-force of the charged test particle, as given in Eq. (1). It is interesting to determine when, if ever, it becomes important physically. If we write

$$F_{\text{ext}}=F_{\text{ext, grav}}(1-\Delta)\,, \qquad (77)$$

where $F_{\text{ext, grav}}$ is the first term in (76) and

$$\Delta=\frac{e^2/mc^2}{r_S}\left(1-\frac{2GM}{c^2r_S}\right)^{1/2} \qquad (78)$$

is the fractional correction due to the self-force, then it is easy to show that the maximum value of Δ (which occurs at $r_S=3GM/c^2$) is $e^2/3\sqrt{3}GMm$. For an electron outside a one-solar-mass black hole, the self-force gives at most an insignificant fractional correction of 4×10^{-19}.

But suppose we have a black hole whose mass is less than $M_{\text{crit}}\equiv e^2/3\sqrt{3}Gm$ (7.3×10^{14} g for electrons as test particles; for the critical mass, the Schwarzschild radius is 38% of the classical electron radius, $e^2/mc^2=2.82\times10^{-13}$ cm). Then one finds that the fractional correction exceeds unity—and thus the repulsive self-force exceeds the attractive gravitational force—over the range of radii $r_-<r_S<r_+$, where [13]

$$r_\pm=\frac{2e^2}{\sqrt{3}mc^2}\cos\left(\tfrac{1}{3}\cos^{-1}\frac{3\sqrt{3}GMm}{e^2}\mp\frac{\pi}{3}\right)\,. \qquad (79)$$

When $M=M_{\text{crit}}$, $r_+=r_-=e^2/\sqrt{3}mc^2=3GM_{\text{crit}}/c^2=1.6\times10^{-13}$ cm. As $M\to0$, $r_-\sim 2GM/c^2$ and $r_+\to e^2/mc^2$. It is amusing to note that at $r=r_+$ the test particle's electrostatic self-force would suffice to support it against the hole's gravity, without the help of any external force; moreover, such an equilibrium situation would be stable against radial perturbations. Unfortunately, in a regime where the black hole's Schwarzschild radius is comparable to or smaller than the classical electron radius, quantum effects will vitiate a classical treatment of the problem. Indeed, it is meaningless to talk of an electron being held fixed at, say, 10^{-13} cm from a miniblack hole, when the Compton wavelength of an electron is two orders of magnitude larger than this.

One is led therefore to wonder about the significance of an effect such as the self-force within the context of a fully quantum-field-theoretic treatment. How much, for example, would it affect the Hawking radiation rate and spectrum? The self-force is the classical manifestation of one of many effects that will presumably appear when the free-field treatments generally used in investigating quantum processes in curved spacetime are extended to include interacting fields. Further work along these lines is in progress.

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[1]C. M. DeWitt and B. S. DeWitt, Physics (N.Y.) 1, 3 (1964).
[2]F. A. Berends and R. Gastmans, Ann. Phys. (N.Y.) 98, 225 (1976).
[3]C. H. MacGruder III, Nature (London) 272, 806 (1978).
[4]A. Vilenkin, Phys. Rev. D 20, 373 (1979).
[5]See, e.g., *Physics and Astrophysics of Neutron Stars and Black Holes: Proceedings of the International School of Physics "Enrico Fermi," Course 65*, edited by R. Giacconi and R. Ruffini (North-Holland, New York, 1978).
[6]B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) 9, 220 (1960).
[7]S. Bazanski, Acta Phys. Pol. 16, 423 (1957).
[8]E. T. Copson, Proc. R. Soc. London A118, 184 (1928).
[9]B. Linet, J. Phys. A 9, 1081 (1976).
[10]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon, New York, 1975), p. 241.
[11]S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
[12]B. Carter, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, New York, 1979), p. 359.
[13]Concerning the trigonometric solution of cubic equations, see, e.g., *Handbook of Tables for Mathematics*, 4th ed., edited by R. C. Weast and S. M. Selby (CRC, Cleveland, 1975), p. 130.