Covariant 2+2 formulation of the initial-value problem in general relativity

R. A. d'Inverno and J. Smallwood*
Faculty of Mathematics, Southampton University, SO9 5NH, England
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A covariant 2 + 2 formalism is developed in which space-time is decomposed into a family of spacelike two-surfaces and their orthogonal timelike two-surface elements. The resulting 2 + 2 breakup of the Einstein vacuum field equations is then used to investigate covariant formulations of spacelike, characteristic, and mixed initial-value problems. In each case the so-called conformal two-structure (essentially the conformal metric of a family of spacelike two-surfaces) is identified as the freely specifiable initial data. The formalism makes clear the geometrical significance of both the initial data and the various choices of gauge variables. A Lagrangian formulation is included which supports the role of the conformal two-structure as dynamical variables of the pure gravitational field.

I. INTRODUCTION

The covariant formulation of the spacelike Cauchy problem in general relativity, using a space +time or "3 +1" formalism, where space-time is considered as being foliated by spacelike hypersurfaces rigged by the normal timelike vector field, has been extensively studied—initially by Stachel[1] and later by, in particular, York and others.[2] Although there has been considerable success with this approach, there would appear to be two problems associated with using a 3+1 formulation. The main problem is that it cannot be used to analyze characteristic or mixed initial-value problems, where it is necessary to consider foliations of null (as opposed to spacelike) hypersurfaces. This stems from the fact that, unlike spacelike hypersurfaces, null hypersurfaces are not Riemannian submanifolds and consequently do not have induced on them any natural affine structure. (A fuller discussion of this problem can be found, for example, in Ref. 3.) The other problem is related to the difficulty of identifying the gravitational degrees of freedom (that is, essentially, in defining two quantities per space-time point which uniquely characterize a solution)[4] and the associated dynamical variables of the theory. Now in the 3 +1 formulation, one starts by prescribing initial data (essentially the first and second fundamental forms) on an initial spacelike hypersurface, and then the six evolution equations are used to generate a solution forward in time. These initial data are not freely specifiable but are subject to differential constraints. Considerable progress has been made in the analysis of these constraint equations by the introduction of conformal three-geometry techniques. This leads to an identification of the gravitational degrees of freedom as the conformal three-geometry of a family of spacelike hypersurfaces, but only after an arbitrary choice of basis has been subtracted out at each point. One way of removing this arbitrariness is to construct a transverse trace-free tensor, the Bach tensor, which has the property of being invariant under conformal transformations and diffeomorphisms of the three-geometry on each slice. Although this quantity uniquely characterizes a solution, it depends upon spacelike derivatives of the conformal metric and hence has a nonlocal interpretation. It is therefore hard to see how the Bach tensor could be interpreted as explicitly embodying the dynamical variables of the theory, for, in particular, one cannot explicitly isolate a subset of the six evolution equations as dynamical equations for propagating the Bach tensor. Closely related to this is the fact that in a Hamiltonian formulation of the Cauchy problem, the 3 +1 approach leads to a constrained Hamiltonian (with the five functions in the conformal three-geometry acting as constrained configuration coordinates).[3] This property may inhibit progress towards a quantization program, since in at least one approach to the problem one requires the two dynamical variables explicitly isolated, and this in turn leads to an unconstrained Hamiltonian. We hope to demonstrate that a 2 +2 formulation adopted in this paper may overcome the two problems outlined above.

In a recent paper[4] it was suggested that the so-called "conformal two-structure"—essentially the conformal metric of a family of spacelike two-surfaces—might be considered as the dynamical variables of the theory. Using coordinate-dependent techniques this prescription is shown to work, formally at least, in characteristic, mixed, and spacelike initial-value problems. The work motivates the introduction in this paper of a covariant "2 +2" formalism, in which we consider spacetime as being foliated by spacelike two-surfaces, rigged by a pair of normal vector fields which span timelike two-surfaces (which are in general anholonomic). Then by a suitable choice of these rigging fields subfamilies of the spacelike two-surfaces can be regarded as foliating hypersurfaces in space-time, which may be either spacelike, timelike, or null. This formalism allows us to deal conveniently with all three types of initial-value problem, namely Cauchy, mixed, and characteristic. In the last two cases, it is because we are working directly with the geometry of Riemannian two-manifolds that the problems of the degeneracy of the intrinsic geometry of null hypersurfaces are essentially bypassed.

In Sec. II we analyze the 2+2 formalism in detail, in particular introducing various fundamental geometrical quantities and then expressing the field equations in terms of these quantities. After some general remarks about the initial-value problem in Sec. III, we then use this formalism in Sec. IV to discuss a covariant 2+2 formulation of the spacelike Cauchy problem. In Sec. V we formulate covariantly the characteristic and mixed initial-value problems. It is seen that in the latter case, by a suitable choice of gauge quantities, we can obtain the covariant analogs of these problems as considered in a coordinate-dependent manner by Sachs,[5] Gambini and Restuccia,[6] and Tamburino and Winicour.[7] In each case, it is seen that the dynamical variables are the conformal two-structure and that they are propagated by the same subset of the field equations, namely the dynamical equations. In Sec. VI, a Lagrangian formulation is considered, and in particular it is shown that variation of the Lagrangian with respect to the dynamical variables does indeed lead to the dynamical equations.

All our considerations are purely local and we restrict our attention to the vacuum field equations. More importantly, our results are purely formal in character; that is to say, we do not prove results concerning uniqueness, existence, and stability of solutions, but rather simply provide iterative integration schemes which hopefully generate solutions from given initial data for certain classes of space-times. We follow Schouten's[8] conventions and notation in the main, and in addition we take the metric of space-time V to have signature $(+---)$, and we use an arbitrary coordinate system x^α, $\alpha=0,1,2,3$. In fact, many of our results are valid also in a general space-time basis (E^α, E_α), but we use a coordinate basis, since this simplifies the discussion at some points.

II. 2+2 DECOMPOSITION OF SPACE-TIME

A. Foliation of space-time

A foliation of V by two-surfaces $\{S\}$ ($\{S\}$ is a foliation of codimension 2) is defined by a pair of closed one-forms $n^a = n^a_\alpha dx^\alpha$ $(a=0,1)$. Now n^0 and n^1 define, respectively, foliations of V, $\{\Sigma_1\}$, and $\{\Sigma_0\}$ into hypersurfaces, and each $S \varepsilon \{S\}$ can be thought of as the intersection of some $\Sigma_0 \varepsilon \{\Sigma_0\}$ and $\Sigma_1 \varepsilon \{\Sigma_1\}$. Since n^a are closed, we have

$$ {}^{(4)}\nabla_{[\alpha} n^a_{\beta]} = 0 , \quad (2.1)$$

where ${}^{(4)}\nabla_\alpha$ is the covariant derivative in V. Equation (2.1) implies that (locally) there are scalar functions ϕ^a such that

$$ n^a_\alpha = {}^{(4)}\nabla_\alpha \phi^a . \quad (2.2)$$

Then $\{\Sigma_0\}$ and $\{\Sigma_1\}$ arise as the level surfaces of ϕ^1 and ϕ^0, respectively. Each $\Sigma_{(a)} \varepsilon \{\Sigma_{(a)}\}$ is itself foliated by a subset of $\{S\}$, and we denote these subsets by $\{S\}_{(a)}$ (where parentheses around a Latin index indicate that we are referring to a fixed value of that index), as indicated in Fig. 1.

We restrict our attention to foliations $\{S\}$ into spacelike two-surfaces. A necessary and sufficient condition that $\{S\}$ be spacelike is that at each point of V, n^a_α should span a timelike two-surface element, say T. We denote the totality of these elements by $\{T\}$. $\{T\}$ must necessarily be orthogonal to $\{S\}$. The reciprocal basis of $\{T\}$, namely $n_a = n^\alpha_a \partial/\partial x^\alpha$, is uniquely defined by

$$ n^\alpha_a n^b_\alpha = \delta^b_a , \; n^\alpha_a = \eta_{ab} n^{b\alpha} , \quad (2.3)$$

where η_{ab} is a symmetric 2×2 scalar matrix with inverse η^{ab}. The vectors n^α_a and n^a_α form a dyad basis of $\{T\}$, with dyad (Latin) indices raised and lowered by η^{ab} and η_{ab}, respectively. In general, n^α_a are not closed under the Lie bracket operation, so the elements of $\{T\}$ do not define a foliation of V; they are rather anholonomic timelike two-

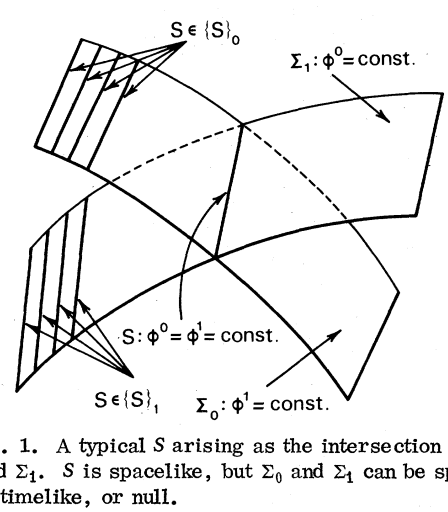

FIG. 1. A typical S arising as the intersection of some Σ_0 and Σ_1. S is spacelike, but Σ_0 and Σ_1 can be spacelike, timelike, or null.

surface elements, or fields of nonintegrable timelike two-planes. Equations (2.3) imply that $n^{\alpha}_{(a)}$ is tangent to $\{\Sigma_{(a)}\}$, and also that it is the orthogonal connecting vector of neighboring members of $\{S\}_{(a)}$. Each element $\eta_{(a)(a)}$ of η_{ab} defines a "lapse" function; the metrical separation of nearby members of $\{S\}_{(a)}$, parameter distance $\delta\phi^{(a)}$ apart, is given by

$$({}^{(4)}g_{\alpha\beta}n^{\alpha}_{(a)}n^{\beta}_{(a)})^{1/2}\delta\phi^{(a)}=\eta_{(a)(a)}{}^{1/2}\delta\phi^{(a)}$$
$$[\text{not summed over }(a)]\,, \quad (2.4)$$

as illustrated in Fig. 2. The elements η^{00} and η^{11} determine the metrical properties of $\{\Sigma_1\}$ and $\{\Sigma_0\}$, respectively, since

$$\eta^{(a)(a)}={}^{(4)}g^{\alpha\beta}n^{(a)}_{\alpha}n^{(a)}_{\beta}\,. \quad (2.5)$$

Projection operators B^{α}_{β} and C^{α}_{β}, which project tensors of V into $\{S\}$ and $\{T\}$, respectively are defined by[8]

$$\delta^{\alpha}_{\beta}=B^{\alpha}_{\beta}+C^{\alpha}_{\beta}\,,\quad C^{\alpha}_{\beta}=n^{\alpha}_{a}n^{a}_{\beta}\,. \quad (2.6)$$

It is straightforward to show from (2.1) and (2.3) that

$$[n_0,n_1]^{\alpha}=\pounds_{n_0}n^{\alpha}_{1}\equiv-2\overline{\Omega}^{\alpha}\,,=-2B^{\alpha}_{\beta}\overline{\Omega}^{\beta}\,. \quad (2.7)$$

Thus the commutator of n^{α}_{0} and n^{α}_{1} is a vector tangent to $\{S\}$, and $\{T\}$ is holonomic if and only if $\overline{\Omega}^{\alpha}$ vanishes. The most general vector connecting neighboring members of $\{S\}_{(a)}$ is $e^{\alpha}_{(a)}$, where

$$e^{\alpha}_{a}=n^{\alpha}_{a}+b^{\alpha}_{a}\,,\quad b^{\alpha}_{a}=B^{\alpha}_{\beta}b^{\beta}_{a} \quad (2.8)$$

(see Fig. 2). We shall restrict our attention to those b^{α}_{a} for which the resulting e^{α}_{a} commute. By virtue of (2.7), such b^{α}_{a} always exist for a given foliation $\{S\}$ but are not unique. From (2.3) and (2.8) we have

$$e^{\alpha}_{a}n^{b}_{\alpha}=\delta^{b}_{a}\,, \quad (2.9)$$

and so for a given choice of b^{α}_{a} we may write, by virtue of (2.9) and the fact that e^{α}_{a} commute,

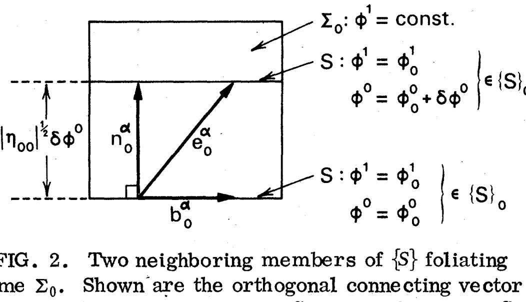

FIG. 2. Two neighboring members of $\{S\}$ foliating some Σ_0. Shown are the orthogonal connecting vector n^{α}_{0}, the general connecting vector e^{α}_{0}, and shift vector b^{α}_{0} of $\{S\}$. Metrical separation is $|\eta_{00}|^{1/2}\delta\phi^0$.

$$e_a=e^{\alpha}_{a}\frac{\partial}{\partial x^{\alpha}}=\frac{\partial}{\partial\phi^{a}}\,.$$

The integral curves $\mathfrak{C}_{(a)}$ of each $e^{\alpha}_{(a)}$ are parametrized by $\phi^{(a)}$ and define a one-to-one correspondence between points on different members of $\{S\}$. Put another way, if we start at some point ${}^0P$ on a given initial ${}^0S\varepsilon\{S\}$, then traveling parameter distances ϕ^0 and ϕ^1 from ${}^0P$ along curves of $\mathfrak{C}_0$ and $\mathfrak{C}_1$, respectively, we always arrive at the same point P on a particular two-surface S. The vectors e^{α}_{a} are thus the natural ones with which to propagate quantities through V. If we assume that the equation of ${}^0S$ is $\phi^0=\phi^1=0$, then the value of some geometric object Φ_{Λ} (indices suppressed) on S is given, in terms of Φ_{Λ} and its Lie derivatives with respect to e^{α}_{a} evaluated on ${}^0S$, by a generalized Taylor expansion[8]

$$\Phi_{\Lambda|_S}=\exp\{\phi^a\pounds_{e_a}\}\Phi_{\Lambda|{}^0S}\,. \quad (2.10)$$

We may think of the foliation and fibration of V as being generated in a rather different way. Suppose that we are given in V a pair of commuting vector fields e^{α}_{a} and a two-dimensional cross section S of e^{α}_{a}. The fibrations $\mathfrak{C}_a$ of V are then just the integral curves of e^{α}_{a} (as before), but the foliation $\{S\}$ is obtained by Lie dragging S with e^{α}_{a} so as to fill up V. The covariant normals n^{a}_{α} and the normal rigging vectors n^{α}_{a} can then be determined from $\{S\}$. The two viewpoints are essentially equivalent (cf., for example, in the 3+1 decomposition of space-time, the approaches of York[2] and Stachel[1]).

B. Induced metrical structure in $\{S\}$ and $\{T\}$

The metric of V has only two nonzero projections, namely

$$g_{\gamma\beta}\equiv B^{\mu\nu}_{\gamma\beta}{}^{(4)}g_{\mu\nu}\,,\quad {}'g_{\gamma\beta}=C^{\mu\nu}_{\gamma\beta}{}^{(4)}g_{\mu\nu}\,; \quad (2.11)$$

then $g_{\gamma\beta}$ and ${}'g_{\gamma\beta}$ are the induced metrics of $\{S\}$ and $\{T\}$, respectively. They have contravariant form

$$g^{\gamma\beta}=B^{\gamma\beta}_{\mu\nu}{}^{(4)}g^{\mu\nu}\,,\quad {}'g^{\gamma\beta}=C^{\gamma\beta}_{\mu\nu}{}^{(4)}g^{\mu\nu}\,, \quad (2.12)$$

respectively, and satisfy

$$g_{\beta\epsilon}g^{\gamma\epsilon}=B^{\gamma}_{\beta}\,,\quad {}'g_{\beta\epsilon}{}'g^{\gamma\epsilon}=C^{\gamma}_{\beta}\,. \quad (2.13)$$

From (2.3), (2.6), and (2.11) we can see that

$${}'g_{\gamma\beta}=\eta_{cb}n^{c}_{\gamma}n^{b}_{\beta}\,,\quad \eta_{cb}=n^{\gamma}_{c}n^{\beta}_{b}{}'g_{\gamma\beta}\,; \quad (2.14)$$

that is, η_{cb} are the dyad (scalar) components of ${}'g_{\gamma\beta}$. Similarly, η^{cb} are the dyad components of ${}'g^{\gamma\beta}$. Covariant derivatives in $\{S\}$ and $\{T\}$ are defined by projection:

$$\nabla_{\gamma}v^{\alpha}=B^{\mu\alpha}_{\gamma\pi}{}^{(4)}\nabla_{\mu}v^{\pi}\quad\text{for any } v^{\alpha}=B^{\alpha}_{\beta}v^{\beta}\,, \quad (2.15a)$$

$${}'\nabla_{\gamma}w^{\alpha}=C^{\mu\alpha}_{\gamma\pi}{}^{(4)}\nabla_{\mu}w^{\pi}\quad\text{for any } w^{\alpha}=C^{\alpha}_{\beta}w^{\beta} \quad (2.15b)$$

and we define for any scalar λ in V,

$$\nabla_\alpha \lambda = B_\alpha^{\mu}{}^{(4)}\nabla_\mu \lambda . \qquad (2.15c)$$

With the above definitions it is easy to show that

$$\nabla_\gamma g_{\beta\alpha} = {}'\nabla_\gamma {}'g_{\beta\alpha} = 0 , \qquad (2.16)$$

and hence the connections induced in $\{S\}$ and $\{T\}$ by (2.15) are the connections of the induced metrics. Taking dyad components of Eq. (2.15b) we get

$$n_c^\gamma\, n_\alpha^a {}'\nabla_\gamma w^\alpha = \mathcal{L}_{n_c} w^a + \Gamma^a_{cb} w^b \equiv {}'\nabla_c w^a , \qquad (2.17)$$

where $w^a \equiv n_\alpha^a w^\alpha$ are the dyad components of w^α and

$$\Gamma^a_{cb} = \Gamma^a{}_{(cb)} = \tfrac{1}{2}\eta^{ad}(\mathcal{L}_{n_c}\eta_{db} + \mathcal{L}_{n_b}\eta_{dc} - \mathcal{L}_{n_d}\eta_{cb}) \qquad (2.18)$$

are the dyad components of the induced connection in $\{T\}$. We may define Riemann tensors in $\{S\}$ and $\{T\}$ as follows. In $\{S\}$ we define

$$R_{\delta\gamma\beta}{}^\alpha v^\beta = 2\nabla_{[\delta}\nabla_{\gamma]}v^\alpha , \quad R_{\delta\gamma\beta}{}^\alpha n_b^\beta = 0 , \qquad (2.19)$$

which is precisely the definition we would expect. However, the Riemann tensor in $\{T\}$ has a more complicated definition due to the anholonomicity of $\{T\}$. It is given by[8]

$$\begin{aligned} {}'R_{\delta\gamma\beta}{}^\alpha w^\beta &= 2{}'\nabla_{[\delta}{}'\nabla_{\gamma]}w^\alpha + 2n_a^\alpha\, \Omega_{\delta\gamma}{}^\epsilon \nabla_\epsilon w^a , \\ B_\mu^\beta {}'R_{\delta\gamma\beta}{}^\alpha &= 0 , \end{aligned} \qquad (2.20)$$

where

$$\begin{aligned} \Omega_{\delta\gamma}{}^\epsilon &= \Omega_{[\delta\gamma]}{}^\epsilon = B_\alpha^\epsilon n_\delta^d n_\gamma^c\, \Omega_{dc}{}^\alpha , \\ \Omega_{dc}{}^\alpha &\equiv -\tfrac{1}{2}\mathcal{L}_{n_d} n_c^\alpha \end{aligned} \qquad (2.21)$$

is the anholonomic object of $\{T\}$, which vanishes if and only if $\{T\}$ is holonomic. $\Omega_{dc}{}^\alpha$ has only one independent component, $\Omega_{01}{}^\alpha = \overline{\Omega}^\alpha$ [cf. Eq. (2.7)]. The extra term in (2.20), involving the anholonomic object, is required to keep the right-hand side linear algebraic in w^α. Taking dyad components of (2.20), we get

$$\begin{aligned} {}'R_{dcb}{}^a w^b &\equiv n_d^\delta n_c^\gamma n_\alpha^a {}'R_{\delta\gamma\beta}{}^\alpha w^\beta \\ &= 2{}'\nabla_{[d}{}'\nabla_{c]}w^a + 2\Omega_{dc}{}^\epsilon \nabla_\epsilon w^a , \end{aligned}$$

where

$${}'R_{dcb}{}^a = \mathcal{L}_{n_d}\Gamma^a_{cb} - \mathcal{L}_{n_c}\Gamma^a_{db} + \Gamma^a_{de}\Gamma^e_{cb} - \Gamma^a_{ce}\Gamma^e_{db} . \qquad (2.22)$$

From (2.22) we can show that ${}'R_{dc[ba]}$ has all the usual symmetries of the Riemann tensor of a metric two-space (although of course ${}'R_{dcba}$ are just scalars), but in addition we can show that

$$\begin{aligned} {}'R_{dc(ba)} = -\Omega_{dc}{}^\epsilon \nabla_\epsilon \eta_{ba} \Rightarrow {}'R_{cb} - \tfrac{1}{2}\eta_{cb}{}'R \\ = -\Omega^e{}_c{}^\epsilon \nabla_\epsilon \eta_{be} , \end{aligned} \qquad (2.23)$$

where

$${}'R_{cb} = {}'R_{ecb}{}^e , \quad {}'R = {}'R_e{}^e .$$

C. Extrinsic curvatures and decomposition of dyad derivatives

We start by considering the curvature tensors of valence 3 of $\{S\}$ and $\{T\}$, defined by Schouten[8] as

$$H_{\delta\gamma}{}^\alpha = B_{\delta\gamma}^{\lambda\mu}{}^{(4)}\nabla_\lambda B_\mu^\alpha , \qquad (2.24a)$$

$$L_{\delta\gamma}{}^\alpha = C_{\delta\gamma}^{\lambda\mu}{}^{(4)}\nabla_\lambda C_\mu^\alpha . \qquad (2.24b)$$

The contravariant indices of $H_{\delta\gamma}{}^\alpha$ and $L_{\delta\gamma}{}^\alpha$ lie in $\{T\}$ and $\{S\}$, respectively, and the covariant indices in $\{S\}$ and $\{T\}$. Then, taking the dyad components of (2.24) we get

$$\begin{aligned} H_{(\delta\gamma)}{}^a \equiv h_{\delta\gamma}{}^a &= -\tfrac{1}{2}B_{\delta\gamma}^{\mu\nu}\eta^{ae}\mathcal{L}_{n_e} g_{\mu\nu} \\ &\Rightarrow h^{\delta\gamma}{}_a = \tfrac{1}{2}\mathcal{L}_{n_a} g^{\delta\gamma} , \end{aligned} \qquad (2.25a)$$

$$H_{[\delta\gamma]}{}^a = 0 , \qquad (2.25b)$$

$$L_{(dc)}{}^\alpha \equiv l_{dc}{}^\alpha = -\tfrac{1}{2}\nabla^\alpha \eta_{dc} , \qquad (2.25c)$$

$$L_{[dc]}{}^\alpha = -\Omega_{dc}{}^\alpha . \qquad (2.25d)$$

We define $h_{\delta\gamma a}$ as the extrinsic curvatures, or second fundamental forms of $\{S\}$. With regard to its defining equation (2.25a) we note that the Lie derivative with respect to n_a^α of a contravariant vector field tangent to $\{S\}$ is itself tangent to $\{S\}$, whereas the same is not true of a corresponding covariant vector field.

We can now decompose the covariant derivatives of the dyad vectors n_α^a and n_a^α in terms of their projections into $\{S\}$ and $\{T\}$. We get

$${}^{(4)}\nabla_\gamma n_\beta^a = -h_{\gamma\beta}{}^a + 2L_e{}^a{}_{(\gamma} n_{\beta)}^e - \Gamma^a_{cb} n_\gamma^c n_\beta^b , \qquad (2.26a)$$

$${}^{(4)}\nabla_\gamma n_b^\alpha = -h_\gamma{}^\alpha{}_b - L_b{}^a{}_\gamma n_a^\alpha + L_{cb}{}^\alpha n_\gamma^c + \Gamma^a_{cb} n_\gamma^c n_a^\alpha . \qquad (2.26b)$$

We are now able to decompose the covariant derivative of any arbitrary tensor of V in a convenient way. Suppose we have a tensor with every index lying in either $\{S\}$ or $\{T\}$. (An arbitrary tensor can always be expressed as a sum of such tensors.) For example, suppose for some $T_{\gamma\beta}$ we have

$$T_{\gamma\beta} = B_\gamma^\mu n_\beta^b T_{\mu b} .$$

The only projections of the covariant derivative of $T_{\gamma\beta}$ which involve derivatives of $T_{\gamma\beta}$ are $B_{\delta\gamma}^{\lambda\mu} n_b^\nu{}^{(4)}\nabla_\lambda T_{\mu\nu}$ and $n_d^\lambda B_\gamma^\mu n_b^\nu{}^{(4)}\nabla_\lambda T_{\mu\nu}$. All other projections follow immediately from (2.26). We can then show that

$$B_{\delta\gamma}^{\lambda\mu} n_b^\nu{}^{(4)}\nabla_\lambda T_{\mu\nu} = \nabla_\delta T_{\gamma b} + L_b{}^e{}_\delta T_{\gamma e} \qquad (2.27a)$$

and

$$n_d^\lambda B_\gamma^\mu n_b^\nu{}^{(4)}\nabla_\lambda T_{\mu\nu} = {}'\nabla_d T_{\gamma b} + h_\gamma{}^\epsilon{}_d T_{\epsilon b} , \qquad (2.28a)$$

where

$${}'\nabla_d T_{\gamma b} \equiv B_\gamma^\mu \mathcal{L}_{n_d} T_{\mu b} - \Gamma^e_{db} T_{\gamma e} . \qquad (2.29a)$$

Similar formulas hold for contravariant tensors, namely

$$B^{\lambda\gamma}_{\delta\mu} n^{b}_{\nu}\, {}^{(4)}\nabla_\lambda T^{\mu\nu} = \nabla_\delta T^{\gamma b} - L_e{}^b{}_\delta T^{\gamma e}\,, \tag{2.27b}$$

$$n^\lambda_d B^\gamma_\mu n^b_\nu\, {}^{(4)}\nabla_\lambda T^{\mu\nu} = {}'\nabla_d T^{\gamma b} - h_\epsilon{}^\gamma{}_d T^{\epsilon b}\,, \tag{2.28b}$$

and

$${}'\nabla_d T^{\gamma b} \equiv \pounds_{n_d} T^{\gamma b} + \Gamma^b_{de} T^{\gamma e}\,. \tag{2.29b}$$

Equations (2.29) extend the definition of $'\nabla_a$ to tensors with both dyad indices and tensor indices lying in $\{S\}$.

D. The alternating quantity $\bar{\epsilon}_{ab}$

We first define

$$\eta = [-\det(\eta_{ab})]^{1/2} \quad \text{and} \quad \bar{\eta}_{ab} = \eta^{-1}\eta_{ab}\,. \tag{2.30}$$

Then from (2.15b) and (2.17) we have immediately

$${}'\nabla_a \eta = 0\,, \tag{2.31}$$

that is, formally, η behaves like a "scalar density of weight 1" with respect to the operator $'\nabla_a$. We next introduce the antisymmetric quantities $\bar{\epsilon}_{ab}$ and $\bar{\epsilon}^{ab}$ defined by

$$\bar{\epsilon}_{01} = -\bar{\epsilon}^{01} = 1\,. \tag{2.32}$$

It follows from (2.30), (2.31), and (2.32) that

$$\bar{\epsilon}^{ab} = \bar{\eta}^{ac}\bar{\eta}^{bd}\bar{\epsilon}_{cd}\,, \tag{2.33}$$

and so $\bar{\epsilon}_{ab}$ behaves like a density of weight -1 with respect to $'\nabla_a$, which leads to

$${}'\nabla_a \bar{\epsilon}_{bc} = 0\,. \tag{2.34}$$

We can now write

$$\Omega_{dc}{}^\alpha = \bar{\epsilon}_{dc}\bar{\Omega}^\alpha\,, \tag{2.35}$$

which implies that

$${}'\nabla_a \bar{\Omega}^\alpha = \pounds_{n_a}\bar{\Omega}^\alpha - \Gamma^e_{ea}\bar{\Omega}^\alpha\,, \tag{2.36}$$

that is $\bar{\Omega}^\alpha$ behaves like a scalar density of weight 1 with respect to $'\nabla_a$.

E. Conformal two-structure

Conformal two-structure is defined in Ref. 4, but only the case of two-surfaces embedded in a three-dimensional manifold is considered in detail. However, as indicated, it is quite straightforward to extend the invariant definition of conformal two-structure to the case of two-surfaces embedded in a four-dimensional manifold. We make that extension here. We start by removing a conformal factor γ from the induced metric of $\{S\}$, and we denote the resulting conformal metric by

$$\tilde{g}_{\gamma\beta} = \gamma^{-1} g_{\gamma\beta}\,, \quad \tilde{g}^{\gamma\beta} = \gamma g^{\gamma\beta}\,.$$

We define the conformal extrinsic curvatures of $\{S\}$ by

$$\tilde{h}^{\gamma\beta}{}_a = \tfrac{1}{2}\pounds_{n_a}\tilde{g}^{\gamma\beta}\,. \tag{2.37}$$

We now look for a conformal factor such that the trace of each conformal extrinsic curvature vanishes. That is such that

$$\tilde{h}_\alpha{}^\alpha{}_a \equiv \tilde{g}_{\alpha\beta}\tilde{h}^{\alpha\beta}{}_a = 0 \tag{2.38}$$

(where all indices on quantities marked by ~ are raised and lowered by the conformal metric). The necessary and sufficient condition for (2.38) is that γ should satisfy

$$\pounds_{n_a}(\eta\gamma)^{-1}\sqrt{-g} = 0\,, \quad g \equiv \det({}^{(4)}g_{\alpha\beta})\,. \tag{2.39}$$

One solution of (2.39), unique up to a constant scalar factor, is

$$\gamma = \eta^{-1}\sqrt{-g}\,. \tag{2.40}$$

For any γ satisfying (2.39) we denote the resulting conformal metric $\tilde{g}_{\gamma\beta}$ as the conformal two-structure. We see by virtue of (2.38) that $\tilde{g}_{\gamma\beta}$ has only two independent components. In fact γ^2, as defined by (2.40), is just the determinant of the metric of $\{S\}$ when the metric is expressed in a basis adapted to $\{S\}$. Note that γ has the same tensorial character as $\sqrt{-g}$. We may now write the extrinsic curvatures as

$$h_{\gamma\beta a} = \gamma\tilde{h}_{\gamma\beta a} + \tfrac{1}{2} g_{\gamma\beta} h_a\,, \tag{2.41}$$

where

$$h_a \equiv h_\epsilon{}^\epsilon{}_a\,. \tag{2.42}$$

We define $'\nabla_a$ on γ by

$${}'\nabla_a\gamma = \pounds_{n_a}\gamma\,. \tag{2.43}$$

F. Projections of the Riemann, Ricci, and Einstein tensors

The equations of Gauss and Codazzi for $\{S\}$ and $\{T\}$ are given, for example, by Schouten.[8] Taking dyad components, we obtain

$$\perp {}^{(4)}R_{\delta\gamma\beta\alpha} = R_{\delta\gamma\beta\alpha} + 2h_{[\delta|\beta|}{}^e h_{\gamma]\alpha e}\,, \tag{2.43a}$$

$$\perp {}^{(4)}R_{\delta\gamma\beta a} = 2\nabla_{[\delta}h_{\gamma]\beta a} + 2L_a{}^e{}_{[\delta}h_{\gamma]\beta e}\,, \tag{2.43b}$$

$${}^{(4)}R_{dcba} = {}'R_{dcba} + 2L_{[d|b|}{}^\epsilon L_{c]a\epsilon} - 2\Omega_{dc}{}^\epsilon L_{ba\epsilon}\,, \tag{2.43c}$$

$$\perp {}^{(4)}R_{dcb\alpha} = 2{}'\nabla_{[d}L_{c]b\alpha} + 2h_\alpha{}^\epsilon{}_{[d}L_{c]b\epsilon} - 2\Omega_{dc}{}^\epsilon h_{\epsilon\alpha b}\,, \tag{2.43d}$$

where we have introduced an abbreviated notation

$$\perp {}^{(4)}R_{\delta\gamma\beta a} = B^{\lambda\mu\nu}_{\delta\gamma\beta}\, n^\tau_a\, {}^{(4)}R_{\lambda\mu\nu\tau}$$

which extends in an obvious manner. The only further independent projection of the Riemann tensor is $\perp{}^{(4)}R_{d\gamma\beta a}$, and this is obtained from the Ricci identity applied to $'g_{\alpha\beta}$. We get, eventually,

$$\perp{}^{(4)}R_{d\gamma\beta a} = {'\nabla}_d h_{\gamma\beta a} + h_{\beta\epsilon d} h_\gamma{}^\epsilon{}_a + \nabla_\gamma L_{da\beta} + L_{de\beta} L_a{}^e{}_\gamma \,. \tag{2.43e}$$

We define the Ricci and Einstein tensors of V by

$$\begin{aligned} {}^{(4)}R^{\gamma\beta} &= {}^{(4)}R^{\epsilon\gamma\beta}{}_\epsilon \,, \\ G^{\gamma\beta} &= {}^{(4)}R^{\gamma\beta} - \tfrac{1}{2}{}^{(4)}g^{\gamma\beta}\,{}^{(4)}R \,. \end{aligned} \tag{2.44}$$

We next split the projections into $\{S\}$ of ${}^{(4)}R^{\gamma\beta}$ and $G^{\gamma\beta}$ into their respective trace and trace-free parts by defining

$$\perp G = \perp G_\epsilon{}^\epsilon \,,$$

$$\gamma^{-1}\perp\tilde{G}^{\gamma\beta} = \perp G^{\gamma\beta} - \tfrac{1}{2} g^{\gamma\beta} \perp G \,,$$

and similarly for ${}^{(4)}R^{\gamma\beta}$. It is then easy to show that the projections of ${}^{(4)}R^{\gamma\beta}$ and $G^{\gamma\beta}$ are related by

$$\perp\tilde{G}^{\gamma\beta} = \perp{}^{(4)}\tilde{R}^{\gamma\beta} \,, \tag{2.45a}$$

$$\perp G = -{}^{(4)}R_e{}^e \,, \tag{2.45b}$$

$$\perp G^{c\beta} = \perp R^{c\beta} \,, \tag{2.45c}$$

$$G^{cb} = {}^{(4)}R^{cb} - \tfrac{1}{2}\eta^{cb}(\perp{}^{(4)}R + {}^{(4)}R_e{}^e) \,, \tag{2.45d}$$

$$G_e{}^e = -\perp{}^{(4)}R \,. \tag{2.45e}$$

In order to obtain explicit expressions for the projections of $G^{\gamma\beta}$, we first of all take the appropriate combinations of various contractions of (2.43), indicated by (2.45). We then substitute in the decompositions of $L_{dc\alpha}$, given by (2.25c), (2.25d), and (2.35), together with that of $h_{\gamma\beta a}$ given by (2.41). We eventually obtain

$$\begin{aligned} \perp\tilde{G}^{\gamma\beta} = {} & {'\nabla}_e \tilde{h}^{\gamma\beta e} - 2\tilde{h}^{\gamma\epsilon e}\tilde{h}^\beta{}_{\epsilon e} - \tilde{h}^{\gamma\beta e} h_e \\ & + \tilde{T}[\nabla^\gamma l^\beta - l^{ef\gamma} l_{ef}{}^\beta - 2\eta^{-2}\bar{\Omega}^\gamma \bar{\Omega}^\beta] \,, \end{aligned} \tag{2.46a}$$

$$\begin{aligned} \perp G = {} & -{'R} - {'\nabla}_e h^e + \tfrac{1}{2} h_e h^e + \tilde{h}^{\epsilon\theta e}\tilde{h}_{\epsilon\theta e} \\ & - \nabla_\epsilon l^\epsilon + l_\epsilon l^\epsilon - 4\eta^{-2}\bar{\Omega}^\epsilon \bar{\Omega}_\epsilon \,, \end{aligned} \tag{2.46b}$$

$$\begin{aligned} \perp G^{c\beta} = {} & \eta^{-2}\bar{\epsilon}^{ce}[{'\nabla}_e \bar{\Omega}^\beta - 2(\tilde{h}^\beta{}_{\epsilon e} + \delta^\beta_\epsilon h_e)\bar{\Omega}^\epsilon] \\ & - \nabla^\epsilon \tilde{h}^\beta{}_{\epsilon e} + \tfrac{1}{2}\nabla^\beta h^c + 2\tilde{h}^\beta{}_{\epsilon e} l^{ce\epsilon} \\ & - \tilde{h}^\beta{}_\epsilon{}^c l^\epsilon - \tfrac{1}{2} h^c l^\beta - {'\nabla}_e l^{ce\beta} + {'\nabla}^c l^\beta \,, \end{aligned} \tag{2.46c}$$

$$\begin{aligned} G^{cb} = {} & \eta^{-2}\bar{\epsilon}^{ce}\bar{\epsilon}^{bf}\,{'\nabla}_{(e} h_{f)} + 2\eta^{-2}\bar{\epsilon}^{e(c} l^{b)}{}_e{}^\epsilon \bar{\Omega}_\epsilon - \tfrac{1}{2} h^c h^b \\ & - \tilde{h}^{\epsilon\theta c}\tilde{h}_{\epsilon\theta}{}^b + \nabla_\epsilon l^{cb\epsilon} - 2 l^{ce\epsilon} l^b{}_{e\epsilon} - l^{cb\epsilon} l_\epsilon \\ & + \eta^{cb}(\tfrac{1}{2}\tilde{h}^{\epsilon\theta e}\tilde{h}_{\epsilon\theta e} + \tfrac{3}{4} h^e h_e - \nabla_\epsilon l^\epsilon \\ & + \tfrac{1}{2} l^{ef\epsilon} l_{ef\epsilon} + \tfrac{1}{2} l^\epsilon l_\epsilon + \eta^{-2}\bar{\Omega}^\epsilon\bar{\Omega}_\epsilon - \tfrac{1}{2} R) \,, \end{aligned} \tag{2.46d}$$

$$G_e{}^e = -{'\nabla}_e h^e + h_e h^e - \nabla_\epsilon l^\epsilon + l^{ef\epsilon} l_{ef\epsilon} + 2\eta^{-2}\bar{\Omega}^\epsilon \bar{\Omega}_\epsilon - R \,, \tag{2.46e}$$

where

$$l^\alpha \equiv l_e{}^{e\alpha} \,,$$

and in (2.46a), we have used the notation

$$\tilde{T}[X^{\gamma\beta}] = \gamma X^{(\gamma\beta)} - \tfrac{1}{2}\tilde{g}^{\gamma\beta} X_\epsilon{}^\epsilon \text{ for any } X^{\gamma\beta} \text{ in } \{S\} \,.$$

As a final preliminary to our analysis of the initial-value problem, we decompose the Bianchi identities in terms of the projections of $G^{\gamma\beta}$. We get

$$\begin{aligned} B^\alpha_\pi{}^{(4)}\nabla_\epsilon G^{\pi\epsilon} = {} & {'\nabla}_e \perp G^{e\alpha} - (2h^\alpha{}_{\epsilon e} + \delta^\alpha_\epsilon h_e)\perp G^{e\epsilon} \\ & + \nabla_\epsilon \perp G^{\alpha\epsilon} - l_\epsilon \perp G^{\alpha\epsilon} + l_{ef}{}^\alpha G^{ef} = 0 \,, \end{aligned} \tag{2.47a}$$

$$\begin{aligned} n^a_\pi{}^{(4)}\nabla_\epsilon G^{\pi\epsilon} = {} & {'\nabla}_e G^{ea} - h_e G^{ea} \\ & + \nabla_\epsilon \perp G^{a\epsilon} - (2L_e{}^a{}_\epsilon + \delta^a_e l_\epsilon)\perp G^{e\epsilon} \\ & + h_{\epsilon\theta}{}^a \perp G^{\epsilon\theta} = 0 \,. \end{aligned} \tag{2.47b}$$

III. THE INITIAL-VALUE PROBLEM

In this section, we examine the following problem. Given a solution to the vacuum field equations $G_{\alpha\beta} = 0$ in V, together with a particular foliation $\{S\}$ and fibrations $\mathfrak{C}_a$, what data, on an initial two-surface ${}^0S$ and either or both of the hypersurfaces ${}^0\Sigma_a$ intersecting in ${}^0S$, are necessary and sufficient to determine that solution in some neighborhood of ${}^0S$? The metric of V can be written as

$$\begin{aligned} {}^{(4)}g^{\alpha\beta} &= \gamma^{-1}\tilde{g}^{\alpha\beta} + \eta^{ab}(e^\alpha_a - b^\alpha_a)(e^\beta_b - b^\beta_b) \\ &= g^{\alpha\beta}(\gamma, \tilde{g}^{\alpha\beta}, \eta^{ab}, b^\alpha_a) \,. \end{aligned}$$

Hence to determine a solution, we must find the ten independent components $(\gamma, \tilde{g}^{\alpha\beta}, \eta^{ab}, b^\alpha_a)$ of ${}^{(4)}g^{\alpha\beta}$. We may choose four of these components arbitrarily (subject to maintaining the correct signature of ${}^{(4)}g^{\alpha\beta}$), since the field equations are invariant under a four-dimensional coordinate gauge group. This gauge choice essentially gives us freedom to specify some (though not all) of the metrical properties of, and relations between, the foliation $\{S\}$ and the fibrations $\mathfrak{C}_a$. For example, if we demand that one of the foliations $\{\Sigma_a\}, \{\Sigma_1\}$, say, be a family of null hypersurfaces, then by (2.5) we must impose the gauge condition

$$\eta^{00} = 0. \tag{3.1}$$

The imposition of this gauge, which allows us to study characteristic and mixed initial-value problems, makes the ensuing analysis fundamentally different from the analysis of spacelike initial-value problems, in which case we demand that $\{\Sigma_1\}$ be spacelike and $\{\Sigma_0\}$ timelike. That is, by (2.5)

$$\eta^{00} > 0, \quad \eta^{11} < 0 \,. \tag{3.2}$$

Having chosen either (3.1) or (3.2), the remaining analysis falls, roughly speaking, into three parts: firstly, the investigation of the role of the Bianchi identities; secondly, the choice of the (remaining) gauge quantities; and thirdly, the construction of a formal integration scheme. The scheme then indicates how the field equations propagate the six field variables into some region of V in the neighborhood of an initial $S\in\{S\}$. This leads naturally to the identification of the freely specifiable initial data, and hence the dynamical variables. Since, as we have said, the analysis is very different under the alternative choices (3.2) and (3.1), we shall consider them separately in Secs. IV and V, respectively. First, we introduce some notation which will facilitate the ensuing discussion.

In what follows, ${}^0S$ denotes an initial spacelike two-surface and ${}^0\Sigma_a$ denote the initial hypersurfaces emanating from ${}^0S$. In addition, we denote the "kth neighboring hypersurface" to ${}^0\Sigma_a$ by ${}^k\Sigma_a$ and the subsets of $\{S\}$ foliating ${}^k\Sigma_a$ by ${}^k\{S\}_a$. Finally, we denote the two-surface defined by the intersection of ${}^i\Sigma_0$ and ${}^j\Sigma_1$ by ${}^{i,j}S$. Then, for example, the "jth neighboring two-surface" to ${}^0S$ within ${}^0\Sigma_0$ is ${}^{0,j}S$. When, in Secs. IV and V, we refer to knowledge of some field variable Φ_Λ, say on ${}^{i_0,j_0}S$, what we mean is that we know

$$\pounds^{(j)}_{e_0}\pounds^{(i)}_{e_1}\Phi_{\Lambda|{}^0S},\quad i=0,1,\ldots,i_0,\quad j=0,1,\ldots,j_0\,.$$

At this stage, we must make clear the limitations of the approach which we are adopting in the next two sections. We shall show how the field equations can be used to calculate successive Lie derivatives with respect to e_a^α of the field variables on ${}^0S$. In order that this process might generate a solution in some region of ${}^0S$, we must assume therefore that the field variables can be expanded in a Taylor series of the form (2.10) in the region of integration under consideration. This condition is very strong, since it assumes analyticity of the initial data and of the field variables. By the same token, our formal integration scheme only guarantees uniqueness in the analytic case.

IV. THE SPACELIKE INITIAL-VALUE PROBLEM

In this section, we shall assume (3.2) holds. We group the field equations as follows using the terminology of Bondi[9]:

$$\left.\begin{array}{l}\perp\tilde{G}^{\alpha\beta}=0,\ \text{dynamical equations}\\ \left.\begin{array}{l}\perp G=0,\\ G^{11}=0,\\ \perp G^{1\alpha}=0,\end{array}\right\}\text{constraint equations}\end{array}\right\}\ \text{main equations},$$

$$\left.\begin{array}{l}G^{0a}=0,\\ \perp G^{0\alpha}=0,\end{array}\right\}\ \text{subsidiary conditions.}$$

First of all, we analyze the Bianchi identities. If we assume that the main equations hold everywhere and that on some arbitrary member of $\{\Sigma_1\}$ the subsidiary conditions hold also, then on this hypersurface the Bianchi identities (2.47) imply that

$$\pounds_{e_0}G^{0a}=\pounds_{e_0}\perp G^{0\alpha}=0\,.$$

Hence, by induction we immediately deduce the following lemma: The subsidiary conditions are satisfied everywhere if they are satisfied on the initial hypersurface ${}^0\Sigma_1$, and the main equations hold everywhere.

One possible set of gauge quantities is $\{\eta^{00},\ \eta^{01},\ b_0^\alpha\}$. These four quantities describe the development of $\{\Sigma_1\}$ from ${}^0\Sigma_1$, and the correspondence between points on different members of $\{\Sigma_1\}$. Now the covector defining $\{\Sigma_1\}$ is n_α^0, and hence the natural vector field orthogonal to $\{\Sigma_1\}$ connecting successive members is

$$N^\alpha\equiv(\eta^{00})^{-1}n^{0\alpha}\,.$$

The lapse function a of $\{\Sigma_1\}$ is defined by

$$a=({}^{(4)}g^{\alpha\beta}n_\alpha^0n_\beta^0)^{-1/2}=(\eta^{00})^{-1/2}$$

and its interpretation is as follows. If two nearby members of $\{\Sigma_1\}$ are separated by a parameter distance $\delta\phi^0$, then the orthogonal proper time interval between them is

$$({}^{(4)}g_{\alpha\beta}N^\alpha N^\beta)^{1/2}\,\delta\phi^0=a\,\delta\phi^0\,.$$

$\{\Sigma_1\}$ is a family of geodesically parallel hypersurfaces if and only if $a=1$. The function η^{01} defines the "angle" between $\{\Sigma_0\}$ and $\{\Sigma_1\}$; they are mutually orthogonal if and only if $\eta^{01}=0$. In that case, the orthogonal connecting vector of $\{\Sigma_1\}$, N^α, and the orthogonal connecting vector of $\{S\}_0$, n_0^α, coincide. The vector b_0^α is the shift of $\{S\}_0$, and by varying η^{01} and b_0^α, the curves of $\mathcal{C}_0$ can be made to set up an arbitrary correspondence between points on different members of $\{\Sigma_1\}$. In particular, if $\eta^{01}=b_0^\alpha=0$ then $\mathcal{C}_0$ coincides with the trajectories of N^α.

There are some additional choices of gauge within ${}^0\Sigma_1$ which correspond to the freedom to specify the development of ${}^0\{S\}_1$ from ${}^0S$ and the correspondence between points on different members of ${}^0\{S\}_1$. This is governed respectively by the lapse $|\eta_{11}|^{1/2}$ and shift b_1^α of ${}^0\{S\}_1$, which we shall thus allow to take arbitrary values within ${}^0\Sigma_1$. There remains one final choice of gauge on ${}^0\Sigma_1$ which corresponds to the specification ${}^0\Sigma_1$ as a hypersurface in V. One possible candidate is $\mathcal{L}_{n_0}\eta_{11}$, the geometrical significance of which may be interpreted as follows. If we set $\nabla_\alpha \eta_{11}=0$ on ${}^0\Sigma_1$, then for a given choice of the remaining gauge variables we can always find a choice of $\mathcal{L}_{n_0}\eta_{11}$ for which the resulting trajectories of n_1^α are geodesics in V. Finally, on ${}^0S$ we may specify $\tilde{g}^{\alpha\beta}$ arbitrarily. Since ${}^0S$ is conformally flat, a choice of $\tilde{g}^{\alpha\beta}$ merely corresponds to a particular choice of basis in ${}^0S$.

To summarize, we choose the following as gauge variables:

$$\{\eta^{00},\ \eta^{01},\ b_0^\alpha\} \text{ in } V,\quad \{\eta_{11},\ \mathcal{L}_{n_0}\eta_{11},\ b_1^\alpha\} \text{ on } {}^0\Sigma_1,$$

$$\tilde{g}^{\alpha\beta} \text{ on } {}^0S\,,$$

and we regard these as freely specifiable, subject, of course, to (3.2).

We now use (2.46) to write the field equations in the following form:

$$\perp\tilde{G}^{\gamma\beta}=0 \Rightarrow {}'\nabla_e \tilde{h}^{\gamma\beta e}=2\tilde{h}^{\gamma\epsilon e}\tilde{h}^{\beta}{}_{\epsilon e}+\tilde{h}^{\gamma\beta e}h_e-\tilde{T}[\nabla^\gamma l^\beta-l^{ef\gamma}l_{ef}{}^{\beta}-2\eta^{-2}\overline{\Omega}^\gamma\overline{\Omega}^\beta],$$

$$\perp G-G_e{}^e=0 \Rightarrow {}'R=\tilde{h}^{\epsilon\theta e}\tilde{h}_{\epsilon\theta e}-\tfrac{1}{2}h^e h_e-l^{ef\epsilon}l_{ef\epsilon}+l^\epsilon l_\epsilon-6\eta^{-2}\overline{\Omega}^\epsilon\overline{\Omega}_\epsilon+R,$$

$$\perp G^{c\beta}=0 \Rightarrow \overline{\epsilon}^{ce\prime}\nabla_e\overline{\Omega}^\beta=2\overline{\epsilon}^{ce}(\tilde{h}^{\beta}{}_{\epsilon e}+\delta_\epsilon^\beta h_e)\overline{\Omega}^\epsilon+\eta^2[\nabla^\epsilon\tilde{h}^{\beta}{}_{\epsilon}{}^{c}-\tfrac{1}{2}\nabla^\beta h^c-2\tilde{h}^{\beta}{}_{\epsilon e}l^{ce\epsilon}+\tilde{h}^{\beta}{}_{\epsilon}{}^{c}l^\epsilon+\tfrac{1}{2}h^c l^\beta+{}'\nabla_e l^{ce\beta}-{}'\nabla^c l^\beta],$$

$$G^{cb}=0 \Rightarrow \overline{\epsilon}^{ce}\overline{\epsilon}^{bf\prime}\nabla_{(e}h_{f)}=+\overline{\epsilon}^{(c|e|}l^{b)}{}_{e\epsilon}\overline{\Omega}^\epsilon+\eta^2[\tfrac{1}{2}h^c h^b+\tilde{h}^{\epsilon\theta c}\tilde{h}_{\epsilon\theta}{}^{b}-\nabla_\epsilon l^{cb\epsilon}+2l^{ce\epsilon}l^{b}{}_{e\epsilon}+l^{cbe}l_\epsilon$$
$$-\eta^{cb}(\tfrac{1}{2}\tilde{h}^{\epsilon\theta e}\tilde{h}_{\epsilon\theta e}+\tfrac{3}{4}h^e h_e-\nabla_\epsilon l^\epsilon+\tfrac{1}{2}l^{ef\epsilon}l_{ef\epsilon}+\tfrac{1}{2}l^\epsilon l_\epsilon+\eta^{-2}\overline{\Omega}^\epsilon\overline{\Omega}_\epsilon-\tfrac{1}{2}R)].$$

We first note that no second derivatives with respect to e_a^α (extrinsic derivatives) of any of the field variables occur on the right-hand side of any of the above equations. Second, we see that no second derivatives with respect to e_0^α ("time" derivatives) occur in the subsidiary conditions. In fact, the leading terms (i.e., those involving second extrinsic derivatives) of the subsidiary conditions $G^{00}=0$, $G^{01}=0$, $\perp G^{0\beta}=0$ are

$$-\mathcal{L}_{n_1}h_1\equiv\gamma^{-1}\mathcal{L}_{e_1}^2\gamma+\cdots,$$

$$-(\tfrac{1}{2}\mathcal{L}_{n_0}h_1+\tfrac{1}{2}\mathcal{L}_{n_1}h_0)\equiv\gamma^{-1}\mathcal{L}_{e_1}\mathcal{L}_{e_0}\gamma+\cdots,$$

$$\mathcal{L}_{n_1}\overline{\Omega}^\alpha\equiv\tfrac{1}{2}\mathcal{L}_{e_1}\mathcal{L}_{e_0}b_1^\alpha+\cdots,$$

respectively. Now let us suppose that in addition to some choice of gauge variables, the following initial data are given:

$$\tilde{h}^{\alpha\beta}{}_0,\ \tilde{h}^{\alpha\beta}{}_1 \text{ on } {}^0\Sigma_1;\quad \gamma,\ h_0,\ h_1,\ \overline{\Omega}^\alpha \text{ on } {}^0S\,. \tag{4.1}$$

Then on ${}^0S$ all the field variables and their first extrinsic derivatives are known. Thus we can solve the subsidiary equations on ${}^0S$ for $\mathcal{L}_{e_1}^2\gamma$, $\mathcal{L}_{e_1}\mathcal{L}_{e_0}\gamma$, and $\mathcal{L}_{e_1}\mathcal{L}_{e_0}b_1^\alpha$, which is equivalent to knowing $\mathcal{L}_{e_1}\gamma$, $\mathcal{L}_{e_0}\gamma$, and $\mathcal{L}_{e_0}b_1^\alpha$, respectively, on ${}^{1,0}S$. This then allows us to solve the subsidiary conditions again on ${}^{1,0}S$. Repeating the above procedure on all successive ${}^{i,0}S$, $i=2,3,\ldots$, we build up a knowledge of γ, $\mathcal{L}_{e_0}\gamma$, and $\mathcal{L}_{e_0}b_1^\alpha$ on ${}^0\Sigma_1$. Assuming that we have solved the subsidiary conditions on ${}^0\Sigma_1$, we now know all the field variables and their first time derivatives on ${}^0\Sigma_1$. By virtue of the Bianchi identities, the subsidiary conditions hold automatically on all other members of $\{\Sigma_1\}$. We now turn our attention to the main equations.

The main equations all contain second time derivatives of the field variables. The leading terms of $G^{11}=0$, $\perp G-G_e{}^e=0$, $\perp G^{1\alpha}=0$, and $\perp\tilde{G}^{\alpha\beta}=0$ are

$$\mathcal{L}_{n_0}h_0\equiv-\gamma^{-1}\mathcal{L}_{e_0}^2\gamma+\cdots,$$

$$\mathcal{L}_{n_0}\overline{\Omega}^\alpha\equiv-\tfrac{1}{2}\mathcal{L}_{e_0}^2 b_1^\alpha+\cdots,$$

$${}'R\equiv\eta^{-2}\mathcal{L}_{e_0}^2\eta_{11}+\cdots,$$

$$\mathcal{L}_{n_e}\tilde{h}^{\alpha\beta e}\equiv\tfrac{1}{2}\eta^{00}\mathcal{L}_{e_0}^2\tilde{g}^{\alpha\beta}+\cdots,$$

respectively. At this stage we have sufficient initial data to solve the main equations for the second time derivatives of all the field variables on ${}^0\Sigma_1$. Solving the main equations on ${}^0\Sigma_1$ then allows us to solve them again on ${}^1\Sigma_1$ and then on all subsequent ${}^k\Sigma_1$, $k=2,3,\ldots$. In this way we build up a solution of the field equations in some neighborhood of ${}^0S$. It is easy to see that the initial data given in (4.1) is necessary and sufficient to determine the solution. Giving $\tilde{h}^{\alpha\beta}{}_0$ and $\tilde{h}^{\alpha\beta}{}_1$ on ${}^0\Sigma_1$ is fully equivalent to giving the conformal two-structure and its first time derivative on ${}^0\Sigma_1$. On ${}^0S$ the functions γ, h_1, and h_0 then determine the complete intrinsic and extrinsic geometry of ${}^0S$, and finally giving $\overline{\Omega}^\alpha$ is equivalent to specifying the first time derivative of the shift vector b_1^α.

V. CHARACTERISTIC AND MIXED INITIAL-VALUE PROBLEMS

In this section, we assume that (3.1) holds. We group the field equations as follows:

$$\left.\begin{array}{ll}\perp\tilde{G}^{\alpha\beta}=0\,, & \text{dynamical equations} \\ \left.\begin{array}{l}\perp G^{0\alpha}=0\,,\\ G^{0a}=0\,,\end{array}\right\} & \text{constraint equations}\end{array}\right\}\text{main equations}\,,$$

$$\left.\begin{array}{l}G^{11}=0\,,\\ \perp G^{1\alpha}=0\,,\end{array}\right\}\text{subsidiary conditions}\,,$$

$$\perp G=0\,,\quad\text{trivial equation}\,.$$

As in Sec. IV, we first analyze the Bianchi identities, and to do so, we assume that the main equations hold everywhere. Then one of Eqs. (2.47b) reduces to

$$n^{0(4)}_{\mu}\nabla_{\nu}G^{\mu\nu}=\tfrac{1}{2}\eta h_1\perp G=0\,. \tag{5.1}$$

Now, $\eta\neq 0$, and the condition $h_1\neq 0$ is necessary and sufficient for the null rays ruling $\{\Sigma_1\}$ to have nonvanishing expansion. We shall assume for the rest of this section that we are dealing with a region in which h_1 is nonzero. Then (5.1) implies that $\perp G=0$. Equation (2.47a) then implies that

$$\mathcal{L}_{n_1}\perp G^{1\alpha}-[2h_{\epsilon}{}^{\alpha}{}_1+(h_1-\mathcal{L}_{n_1}\ln\eta)\delta^{\alpha}_{\epsilon}]\perp G^{1\epsilon}=0\,, \tag{5.2}$$

and thus if $\perp G^{1\alpha}$ vanishes on any cross section of $\{\Sigma_1\}$, then it vanishes everywhere. Under the assumption that $\perp G^{1\alpha}$ does indeed vanish everywhere, the other equation in (2.47b) gives

$$\mathcal{L}_{n_1}G^{11}+(2\mathcal{L}_{n_1}\ln\eta-h_1)G^{11}=0\,. \tag{5.3}$$

Hence G^{11} vanishes everywhere if it vanishes on some cross section of $\{\Sigma_1\}$. Collecting the results of (5.1), (5.2), and (5.3) together, we obtain the usual lemma: The trivial equation is an algebraic consequence of the main equations; the subsidiary conditions hold everywhere if they hold on some hypersurface transvecting $\{\Sigma_1\}$ and the main equations hold everywhere. In practice, we solve the subsidiary conditions on ${}^0\Sigma_0$.

The specific gauge condition (3.1) implies that several dyad components of various quantities vanish identically. In particular, the following results hold:

$$\eta^{00}=0\Rightarrow\eta_{11}=l_{11\alpha}=l^{00\alpha}=\Gamma^0_{1a}=0\,,$$

$$\eta_{01}=(\eta^{01})^{-1}=\eta>0\,,$$

and in fact it is necessary in analyzing the Bianchi identities to take these results into account. It is simplest in analyzing the field equations themselves to write out explicitly the dyad components of the various terms in the equations. In fact, the main equations simplify considerably, and we obtain

$$\eta^2G^{00}\equiv\mathcal{L}_{n_1}h_1-\tfrac{1}{2}(h_1)^2-h_1\mathcal{L}_{n_1}\ln\eta-\tilde{h}^{\epsilon\theta}{}_1\tilde{h}_{\epsilon\theta 1}=0\,,$$

$$-\eta^2G^{01}\equiv\mathcal{L}_{n_1}h_0-\tfrac{1}{2}\eta^{-1}h_1\mathcal{L}_{n_1}\eta_{00}-h_0h_1+\tfrac{1}{4}\eta^{-1}\eta_{00}(h_1)^2-\tfrac{1}{2}\eta^{-1}\eta_{00}\tilde{h}^{\epsilon\theta}{}_1\tilde{h}_{\epsilon\theta 1}+\nabla_{\epsilon}\overline{\Omega}^{\epsilon}$$
$$-\eta^{-2}\overline{\Omega}^{\epsilon}\overline{\Omega}_{\epsilon}-\tfrac{1}{2}\nabla^2\eta+\tfrac{1}{4}\eta^{-1}(\nabla_{\epsilon}\eta)(\nabla^{\epsilon}\eta)+\tfrac{1}{2}\eta R=0\,,$$

$$-\eta^2\perp G^{0\alpha}\equiv\mathcal{L}_{n_1}\overline{\Omega}^{\alpha}-2[\tilde{h}^{\alpha}{}_{\epsilon 1}+\delta^{\alpha}_{\epsilon}(h_1+\tfrac{1}{2}\mathcal{L}_{n_1}\ln\eta)]\overline{\Omega}^{\epsilon}+\eta\nabla^{\epsilon}\tilde{h}^{\alpha}{}_{\epsilon 1}-\tfrac{1}{2}\eta^2\nabla^{\alpha}(\eta^{-1}h_1)+\tfrac{1}{2}\eta\nabla^{\alpha}\mathcal{L}_{n_1}\ln\eta=0\,,$$

$$\tfrac{1}{2}\eta\perp\tilde{G}^{\alpha\beta}\equiv\mathcal{L}_{n_1}\tilde{h}^{\alpha\beta}{}_0-\tfrac{1}{2}\mathcal{L}_{n_1}(\eta^{-1}\eta_{00}\tilde{h}^{\alpha\beta}{}_1)-2\tilde{h}^{\alpha}{}_{\epsilon(0}\tilde{h}^{\beta\epsilon}{}_{1)}+\eta^{-1}\eta_{00}\tilde{h}^{\alpha\epsilon}{}_1\tilde{h}^{\beta}{}_{\epsilon 1}-\tilde{h}^{\alpha\beta}{}_{(0}h_{1)}$$
$$+\eta^{-1}\eta_{00}\tilde{h}^{\alpha\beta}{}_1h_1+\tilde{T}[\nabla^{\alpha}\overline{\Omega}^{\beta}-\eta^{-1}\overline{\Omega}^{\alpha}\overline{\Omega}^{\beta}-\tfrac{1}{2}\nabla^{\alpha}\nabla^{\beta}\eta+\tfrac{1}{4}(\nabla^{\alpha}\eta)(\nabla^{\beta}\eta)]=0\,.$$

We now consider the remaining gauge freedom. Suppose that we have picked an initial two-surface ${}^0S$ and one of the null hypersurfaces emanating from it, which we take to be ${}^0\Sigma_1$. The condition (3.1) ensures the existence of the family of null hypersurfaces $\{\Sigma_1\}$, with initial member ${}^0\Sigma_1$, but this family is nonunique. There remain three further four-dimensional gauge conditions, together with lower-dimensional gauge choices associated with the intrinsic coordinate freedom within ${}^0\Sigma_1$ and ${}^0\Sigma_0$ and with the freedom to specify the embedding of the latter in V. We shall first of all consider the characteristic initial-value problem, where ${}^0\Sigma_0$ is taken to be a null surface, and hence initial data is set on a pair of intersecting null hypersurfaces. This makes the analysis rather easier than in mixed initial-value problems (where ${}^0\Sigma_0$ is taken to be timelike) since the subsidiary conditions have a considerably simpler form when evaluated on a null surface and lead to much simpler integration schemes for the field equations as a whole.

Now, ${}^0\Sigma_0$ is null if and only if

$$\eta^{11}=0\Longleftrightarrow\eta_{00}=0\,,\quad\text{on }{}^0\Sigma_0\,. \tag{5.4}$$

The subsidiary conditions evaluated on a null ${}^0\Sigma_0$ become

$$\eta^2G^{11}=\mathcal{L}_{n_0}h_0-\tfrac{1}{2}(h_0)^2-Uh_0-\tilde{h}^{\epsilon\theta}{}_0\tilde{h}_{\epsilon\theta 0}=0\,,$$

$$\eta^2\perp G^{1\alpha}=\mathcal{L}_{n_0}\overline{\Omega}^{\alpha}-2\overline{\Omega}^{\epsilon}[\tilde{h}^{\alpha}{}_{\epsilon 0}+\delta^{\alpha}_{\epsilon}(h_0+\tfrac{1}{2}\mathcal{L}_{n_0}\ln\eta)]$$
$$+\tfrac{1}{2}\eta\nabla^{\alpha}\mathcal{L}_{n_0}\ln\eta-\eta\nabla^{\epsilon}\tilde{h}^{\alpha}{}_{\epsilon 0}$$
$$+\tfrac{1}{2}\eta^2\nabla^{\alpha}(\eta^{-1}h_0)-\eta\nabla^{\alpha}U=0\,,$$

where

$$U \equiv \Gamma^0_{00} = \pounds_{n_0} \ln\eta - \tfrac{1}{2}\eta^{-1}\pounds_{n_1}\eta_{00}\,. \tag{5.5}$$

Given ${}^0S$ and ${}^0\Sigma_1$, ${}^0\Sigma_0$ is uniquely determined by the condition that it be null, since there are precisely two null surfaces intersecting in any given spacelike two-surface. The vector n_0^α is tangent to the null rays ruling ${}^0\Sigma_0$, and the gauge freedom within ${}^0\Sigma_0$ is associated with the choice of parametrization of these rays and the correspondence between points on different members of ${}^0\{S\}_0$ set up by the curves of $\mathfrak{C}_0$ in ${}^0\Sigma_0$. The latter is determined by the shift vector b_0^α, and with regard to the former we can show that on ${}^0\Sigma_0$,

$$n_0^{\alpha\,(4)}\nabla_\alpha n_0^\beta = U n_0^\beta\,.$$

Hence U controls the parametrization of the null rays in ${}^0\Sigma_0$, and in particular, $U=0$ implies that they are affinely parametrized. We shall regard U and b_0^α as freely specifiable within ${}^0\Sigma_0$. Suppose that $n_0^\alpha = dx^\alpha/du$ in ${}^0\Sigma_0$. Then we can assume without loss of generality that $u=$ constant is the equation of ${}^{0,j}S$. Since ${}^j\Sigma_1$ intersects ${}^0\Sigma_0$ in ${}^{0,j}S$, we see that a particular choice of U determines ${}^j\Sigma_1$ uniquely.

Two of the three further choices of four-dimensional gauge quantities are associated with the remaining freedom to choose the correspondence between points on different members of $\{\Sigma_0\}$. [Some of this freedom has already been used up by (3,3).] This is governed by the shift vector b_1^α. In particular, choosing $b_1^\alpha=0$ implies that the curves of $\mathfrak{C}_1$ are the null rays ruling $\{\Sigma_1\}$. The final choice of gauge concerns the freedom to specify the development of $\{\Sigma_0\}$ from ${}^0\Sigma_0$, and this is governed by any one of (at least) three quantities. We consider the three cases separately.

Case (a): η_{00}. This is the lapse function of $\{S\}_0$. If we choose $\eta_{00}=0$, then $\{\Sigma_0\}$ is a family of null hypersurfaces. In general, η_{00} can be chosen arbitrarily, providing it vanishes on ${}^0\Sigma_0$.

Case (b): γ. Suppose that γ is restricted by the equations

$$\pounds_{n_1}h_1 - \tfrac{1}{2}(h_1)^2 = J_1\,,\quad \pounds_{n_0}h_1 = J_2 \quad \text{on } {}^0\Sigma_0\,,$$
$$\nabla_\alpha h_1 = J_3 \quad \text{on } {}^0S\,, \tag{5.6}$$

where J_1, J_2, and J_3 are given functions. Then γ is determined everywhere once it is determined on ${}^0\Sigma_0$. This is the strongest condition we can impose on γ while demanding that ${}^0\Sigma_0$ be a null surface.

Case (c): η. We can show that

$$n_1^{\alpha\,(4)}\nabla_\alpha n_1^\beta = n_1^\beta \pounds_{n_1}\ln\eta\,.$$

Hence any η satisfying $\pounds_{n_1}\ln\eta=0$ implies that the null rays ruling $\{\Sigma_1\}$ are affinely parametrized. For a particular choice of parameter u on ${}^0\Sigma_0$, setting $\eta=1$ picks out a particular affine parameter. In general, we may set η to any arbitrary (nonzero) value, in which case the rays of $\{\Sigma_1\}$ will be arbitrarily parametrized.

We have not yet mentioned the gauge conditions associated with the intrinsic coordinate freedom of the initial surface ${}^0\Sigma_1$. In fact, this is automatically used up in cases (b) and (c), but in case (a) there is still the freedom to parametrize the null geodesics ruling ${}^0\Sigma_1$, and hence [from our discussion in case (c)] η is freely specifiable on ${}^0\Sigma_1$ in case (a). Finally, we may choose $\tilde{g}^{\alpha\beta}$ arbitrarily within ${}^0S$, since ${}^0S$ is conformally flat.

The three gauges we have discussed are generalizations of well-known gauges in which the characteristic initial-value problem has been analyzed using more coordinate-dependent techniques. In particular, if we set

$$b_1^\alpha = 0\,,\quad U = b_0^\alpha = 0 \text{ on } {}^0\Sigma_0\,,$$

then we obtain the so-called light-cone gauge.[10] If in addition we set, in case (a) $\eta_{00}=0$, $\eta=1$ on ${}^0\Sigma_1$, we obtain the covariant formulation of the Sachs double-null problem.[5] In case (b) if we set $J_1=J_2=J_3=0$, we obtain a generalized Bondi gauge first discussed by Gambini and Restuccia.[6] In fact, in this case $\gamma=(x^1)^2 f(x^0,x^A)$ in suitably adapted coordinates, $e_a^\alpha=\delta_a^\alpha$, and $x^A=$ constant along the curves of $\mathfrak{C}_a$. In case (c), setting $\eta=1$ yields the Robinson-Trautman[11] or Newman-Penrose gauge.[12]

Integration schemes for all three cases can be constructed quite straightforwardly, and they follow in essence the schemes in their coordinate dependent counterparts. In each case the initial data required are

$$\tilde{h}^{\alpha\beta}{}_0 \text{ on } {}^0\Sigma_0\,,\quad \tilde{h}^{\alpha\beta}{}_1 \text{ on } {}^0\Sigma_1\,,$$
$$\gamma,\ h_0,\ \bar{\Omega}^\alpha, \text{ and } h_1 \text{ [Cases (a) and (c)]}$$
$$\text{or } \eta \text{ [Case (b)] on } {}^0S\,.$$

Giving $\tilde{h}^{\alpha\beta}{}_0$ and $\tilde{h}^{\alpha\beta}{}_1$ on ${}^0\Sigma_0$ and ${}^0\Sigma_1$, respectively, is entirely equivalent to specifying the conformal two-structure on these initial surfaces. The outline of the integration scheme in case (a) is as follows. On ${}^0\Sigma_0$ the subsidiary condition $G^{11}=0$ is solved for γ, which allows the other subsidiary condition $\perp G^{1\alpha}=0$ to be solved for $\bar{\Omega}^\alpha$, and this determines $\pounds_{n_1}b_0^\alpha$. The main equations are solved on successive ${}^k\Sigma_1$, $k=0,1,2,\ldots$ in the following order: $G^{00}=0$, $\perp G^{0\alpha}=0$, $G^{01}=0$, and $\perp\tilde{G}^{\alpha\beta}=0$. The first of these determines γ on ${}^0\Sigma_1$ and η thereafter. The remaining equations determine $\bar{\Omega}^\alpha$, h_0, and $\tilde{h}^{\alpha\beta}{}_0$, respectively. $\bar{\Omega}^\alpha$ determines

b_0^α on any ${}^k\Sigma_1$; h_0 and $\tilde{h}^{\alpha\beta}{}_0$ on ${}^k\Sigma_1$ determine γ and $\tilde{g}^{\alpha\beta}$ on ${}^{k+1}\Sigma_1$, respectively. Integration schemes for cases (b) and (c) are similar, but rather more complicated, and are outlined in Appendix A.

We shall now consider briefly the mixed initial-value problem, where ${}^0\Sigma_0$ is taken to be a timelike surface. We shall concentrate our attention on a gauge which is a modification of case (b) discussed above. We drop (5.4) and now take as gauge variables on ${}^0\Sigma_0$ the lapse $(\eta_{00})^{1/2}>0$, and shift b_0^α of ${}^0\{S\}_0$, together with h_0, the trace of the extrinsic curvature of ${}^0\{S\}_0$ (considered as hypersurfaces in ${}^0\Sigma_0$). This is the generalization of the Bondi gauge considered by Tamburino and Winicour,[7] in which they set $\eta_{00}=1$, $b_0^\alpha=h_0=0$. If we adopt these latter conditions, then ${}^0\{S\}_0$ are geodesically parallel and maximal with respect to the inner geometry of ${}^0\Sigma_0$. In this gauge we see that γ is determined everywhere. The necessary initial data in this gauge are

$$\tilde{h}^{\alpha\beta}{}_0 \text{ on } {}^0\Sigma_0\,,\quad \tilde{h}^{\alpha\beta}{}_1 \text{ on } {}^0\Sigma_1,\quad \bar{\Omega}^\alpha \text{ and } \eta \text{ on } {}^0S\,.$$

The subsidiary conditions have a very complicated explicit form, nevertheless we may write them formally on each ${}^{0,i}S$, $i=0,1,\ldots,$ as

$$G^{11}=0\Rightarrow \pounds_{n_0}\eta=f^{11}[\tilde{h}^{\alpha\beta}{}_0,\tilde{h}^{\alpha\beta}{}_1,\eta,\pounds_{n_1}\eta_{00},\bar{\Omega}^\alpha,\ \text{gauge variables}],$$

$$\perp G^{1\alpha}=0\Rightarrow \pounds_{n_0}\bar{\Omega}^\alpha=f^{1\alpha}[\tilde{h}^{\alpha\beta}{}_0,\tilde{h}^{\alpha\beta}{}_1,\bar{\Omega}^\alpha,\ \eta,\pounds_{n_0}\eta,\pounds_{n_1}\eta,\ \pounds_{n_1}\eta_{00},\text{gauge variables}],$$

where f^{11} and $f^{1\alpha}$ are some complicated functions of their arguments. The occurrence of derivatives of field variables out of ${}^0\Sigma_0$ in the subsidiary conditions means that they cannot be solved independently of the main equations. The formal integration scheme is as follows: The main equations, in the order $G^{00}=0$, $\perp G^{0\alpha}=0$, $G^{01}=0$, and $\perp\tilde{G}^{\alpha\beta}=0$, are solved on ${}^0\Sigma_1$ for η, $\bar{\Omega}^\alpha$, η_{00}, $\tilde{h}^{\alpha\beta}{}_0$, respectively. The subsidiary equations can then be solved on ${}^{0,0}S$ in the order $G^{11}=0$, $\perp G^{1\alpha}=0$, for $\pounds_{n_0}\eta$ and $\pounds_{n_0}\bar{\Omega}^\alpha$, respectively. This enables the main equations to be solved again on ${}^1\Sigma_1$. In general, solving the main equations on ${}^k\Sigma_1$ allows the subsidiary conditions to be solved on ${}^{0,k}S$, which then allows the main equations to be solved again on ${}^{k+1}\Sigma_1$, $k=0,1,\ldots$.

VI. DERIVATION OF THE FIELD EQUATIONS FROM A LAGRANGIAN

We start by assuming that we have a bare manifold with some fixed foliation $\{S\}$ and fibrations $\mathfrak{C}_a$, defined by n_α^a and e_a^α, respectively (as discussed in Sec. II). We now impose a metric ${}^{(4)}g^{\alpha\beta}$ on this manifold, given by

$${}^{(4)}g^{\alpha\beta}=\gamma^{-1}\tilde{g}^{\alpha\beta}+\eta^{ab}(e_a^\alpha-b_a^\alpha)(e_b^\beta-b_b^\beta)\,. \qquad (6.1)$$

Then $\{\Phi_\Lambda\}\equiv\{\gamma,\tilde{g}^{\alpha\beta},\eta^{ab},b_a^\alpha\}$ are the ten independent field variables. We shall only consider variations in Φ_Λ for which the resulting metric can be written in the same form (6.1). We define

$$\delta\tilde{g}^{\alpha\beta}\equiv\tilde{\delta}^{\alpha\beta},\ \ \delta\eta^{ab}=\delta^{ab},\ \ \delta b_a^\alpha=\delta n_a^\alpha\equiv\not{\delta}_a^\alpha\,. \qquad (6.2)$$

Then we obtain the following:

$$\delta(\tilde{g}^{\alpha\beta}n_\alpha^a)=0\Rightarrow\tilde{\delta}^{\alpha\beta}n_\alpha^a=0\,,$$

$$\delta(\tilde{g}_{\alpha\beta}\pounds_{n_a}\tilde{g}^{\alpha\beta})=0\Rightarrow\tilde{g}_{\alpha\beta}\tilde{\delta}^{\alpha\beta}=0\,,$$

$$\delta(n_\alpha^a n_b^\alpha)=0\Rightarrow n_\alpha^a\not{\delta}_b^\alpha=0\,.$$

Hence $\tilde{\delta}^{\alpha\beta},\delta\gamma,\not{\delta}_a^\alpha,\delta^{ab}$ are ten independent variations of the field variables. We can write

$$\delta\,{}^{(4)}g^{\alpha\beta}=\gamma^{-1}\tilde{\delta}^{\alpha\beta}-g^{\alpha\beta}\gamma^{-1}\delta\gamma+n_a^\alpha n_b^\beta\delta^{ab}+2\eta^{ab}n_a^{(\alpha}\not{\delta}_b^{\beta)}\,. \qquad (6.3)$$

The action function I is defined by

$$I=\int\pounds d^4x\,,$$

where

$$\pounds\equiv\sqrt{-g}\,{}^{(4)}R$$

is the Lagrangian density. Replacing $\pounds$ with

$$\pounds^D=\pounds-\sqrt{-g}\,{}^{(4)}\nabla_\epsilon T^\epsilon\,, \qquad (6.4)$$

where T^α is a vector functional of ${}^{(4)}g^{\alpha\beta}$ and $\partial_\gamma{}^{(4)}g^{\alpha\beta}$, leads to an action function I^D. As defined in (6.4), $\pounds^D$ has the same tensorial character as $\pounds$, namely a scalar density. Variation with respect to ${}^{(4)}g^{\alpha\beta}$ of I is identical to the variation of I^D and leads to the same field equations. We obtain

$$\delta I^D=\int\delta\pounds^D d^4x$$

$$=\int[\pounds_{\alpha\beta}\delta\,{}^{(4)}g^{\alpha\beta}+\sqrt{-g}\,{}^{(4)}\nabla_\alpha Z^\alpha]d^4x\,, \qquad (6.5)$$

where Z^α are linear in $\delta\,{}^{(4)}g^{\alpha\beta}$ and $\partial_\gamma\delta\,{}^{(4)}g^{\alpha\beta}$. Hence,

$$\delta I^D=0\Rightarrow\pounds_{\alpha\beta}\equiv\sqrt{-g}\,G_{\alpha\beta}=0\,. \qquad (6.6)$$

We would expect independent variations of I^D with respect to the different Φ_Λ to give rise to different subsets of the field equations. In fact, substituting (6.3) in (6.5), and remembering the definition of $\pounds_{\alpha\beta}$ in (6.6), we see that

$$0=\delta_{\tilde{g}\alpha\beta}\, I^D=\int \delta_{\tilde{g}\alpha\beta}\, \mathcal{L}^D d^4x=\int (\sqrt{-g}\perp\tilde{G}_{\alpha\beta}\tilde{\delta}^{\alpha\beta}+\sqrt{-g}\,{}^{(4)}\nabla_\alpha Z_1^\alpha)d^4x \Rightarrow \perp\tilde{G}_{\alpha\beta}=0\,, \tag{6.7a}$$

$$0=\delta_\gamma I^D=\int \delta_\gamma \mathcal{L}^D d^4x=\int (-\sqrt{-g}\perp G\gamma^{-1}\delta\gamma+\sqrt{-g}\,{}^{(4)}\nabla_\alpha Z_2^\alpha)d^4x \Rightarrow \perp G=0\,, \tag{6.7b}$$

$$0=\delta_{\eta ab}\, I^D=\int \delta_{\eta ab}\, \mathcal{L}^D d^4x=\int (\sqrt{-g}\, G_{ab}\delta^{ab}+\sqrt{-g}\,{}^{(4)}\nabla_\alpha Z_3^\alpha)d^4x \Rightarrow G_{ab}=0\,, \tag{6.7c}$$

$$0=\delta_{b_a^\alpha}\, I^D=\int \delta_{b_a^\alpha}\, \mathcal{L}^D d^4x=\int (2\sqrt{-g}\perp G_{a\alpha}\eta^{ab}\not{\delta}_b^\alpha+\sqrt{-g}\,{}^{(4)}\nabla_\alpha Z_4^\alpha)d^4x \Rightarrow \perp G_\alpha^a=0\,. \tag{6.7d}$$

The above results allow us to obtain explicit expressions for the various projections of the Einstein tensor. We first obtain an expression for $\mathcal{L}$ in terms of Φ_Λ and their derivatives. From (2.45b) and (2.45e) we see that

$${}^{(4)}R=-(\perp G+G_e{}^e)\,,$$

and hence from (2.46b) and (2.46e) we get

$$\mathcal{L}=\sqrt{-g}\,(2'\nabla_e h^e+2\nabla_\epsilon l^\epsilon-\tfrac{3}{2}h_e h^e-l_\epsilon l^\epsilon-\tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta e}-l_{ef\epsilon}l^{ef\epsilon}-\Omega_{ef\epsilon}\Omega^{ef\epsilon}+'R+R)\,.$$

From this we can write down an equivalent Lagrangian density

$$\mathcal{L}^D=\sqrt{-g}(\tfrac{1}{2}h_e h^e+l_\epsilon l^\epsilon-\tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta e}-l_{ef\epsilon}l^{ef\epsilon}-\Omega_{ef\epsilon}\Omega^{ef\epsilon}+'R+R)\,,$$

where

$$\mathcal{L}-\mathcal{L}^D=2\sqrt{-g}\,{}^{(4)}\nabla_\epsilon(l^\epsilon+n_e^\epsilon h^e)\,.$$

We can now perform the variation of $\mathcal{L}^D$ term by term. We first note that, since variation and ordinary differentiation commute, we have the operator equivalence

$$\delta\mathbf{\pounds}_v=\mathbf{\pounds}_{\delta v}+\mathbf{\pounds}_v\delta \quad \text{for any vector } v^\alpha\,.$$

We then obtain, quite straightforwardly,

$$\delta\sqrt{-g}=\delta(\eta\gamma)=\sqrt{-g}\,\gamma^{-1}\delta\gamma-\tfrac{1}{2}\sqrt{-g}\,\delta_e{}^e\,,$$

$$\delta(\tfrac{1}{2}h_e h^e)=('\nabla_e h^e-h_e h^e)\gamma^{-1}\delta\gamma+\tfrac{1}{2}h_e h_f\delta^{ef}-(\nabla_\epsilon h^e+l_\epsilon h^e)\not{\delta}_e^\epsilon+{}^{(4)}\nabla_\epsilon[n_e^\epsilon h^e\gamma^{-1}\delta\gamma+h^e\not{\delta}_e^\epsilon]\,,$$

$$\delta(l_\epsilon l^\epsilon)=-l_\epsilon l^\epsilon\gamma^{-1}\delta\gamma+\tilde{T}[l_\epsilon l_\theta]\tilde{\delta}^{\epsilon\theta}+(l_\epsilon l^\epsilon-\nabla_\epsilon l^\epsilon)\delta_e{}^e+{}^{(4)}\nabla_\epsilon[l_\epsilon\delta_e{}^e]\,,$$

$$\delta(-l_{ef\epsilon}l^{ef\epsilon})=l_{ef\epsilon}l^{ef\epsilon}\gamma^{-1}\delta\gamma-\tilde{T}[l_{ef\theta}l^{ef}{}_\epsilon]\tilde{\delta}^{\epsilon\theta}-(\nabla_\epsilon l_{ef}{}^\epsilon+l_{ef\epsilon}l^\epsilon+2l_e{}^{d\epsilon}l_{fd\epsilon})\delta^{ef}-{}^{(4)}\nabla_\epsilon(l_{ef}{}^\epsilon\delta^{ef})\,,$$

$$\begin{aligned}\delta(-\tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta e})=&('\nabla_e\tilde{h}_{\epsilon\theta}{}^e+2\tilde{h}_{\epsilon\mu e}\tilde{h}_\theta{}^{\mu e}-\tilde{h}_{\epsilon\theta e}h^e)\tilde{\delta}^{\epsilon\theta}-\tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta}{}_f\delta^{ef}\\&-2(\nabla_\epsilon\tilde{h}^\epsilon{}_\theta{}^e-l_\epsilon\tilde{h}^\epsilon{}_\theta{}^e)\not{\delta}_e^\theta-{}^{(4)}\nabla_\epsilon[n_e^\epsilon\tilde{h}_{\theta\mu}{}^e\tilde{\delta}^{\theta\mu}-2\tilde{h}^\epsilon{}_\theta{}^e\not{\delta}_e^\theta]\,,\end{aligned}$$

$$\begin{aligned}\delta(-\Omega_{ef\epsilon}\Omega^{ef\epsilon})=&-\Omega_{ef\epsilon}\Omega^{ef\epsilon}\gamma^{-1}\delta\gamma+\tilde{T}[\Omega_{ef\epsilon}\Omega^{ef}{}_\theta]\tilde{\delta}^{\epsilon\theta}-2\Omega_e{}^d{}_\epsilon\Omega_{fd}{}^\epsilon\delta^{ef}\\&+2('\nabla_e\Omega^{ef}{}_\epsilon-\Omega^{ef}{}_\epsilon h_e)\not{\delta}_f^\epsilon-2\,{}^{(4)}\nabla_\epsilon[n_e^\epsilon\Omega^{ef}{}_\theta\not{\delta}_f^\theta]\,,\end{aligned}$$

$$\delta'R={}'R_{ef}\delta^{ef}+\eta^{ef}\delta'R_{ef}\,,$$

$$\delta R=-R\gamma^{-1}\delta\gamma+g^{\epsilon\theta}\delta R_{\epsilon\theta}\,.$$

The variations $g^{\epsilon\theta}\delta R_{\epsilon\theta}$ and $\eta^{ef}\delta'R_{ef}$ are rather more difficult to calculate and the procedure is outlined in Appendix B. Eventually, we get

$$\begin{aligned}\eta^{ef}\delta'R_{ef}=&(h_e h^e-'\nabla_e h^e)\delta_f{}^f+('\nabla_{(e}h_{f)}-h_e h_f)\delta^{ef}+2(l_\epsilon h^e-h^f l_f{}^e{}_\epsilon+'\nabla^e l_\epsilon-'\nabla^f l^e{}_{f\epsilon})\not{\delta}_e^\epsilon\\&+{}^{(4)}\nabla_\epsilon[n_e^\epsilon('\nabla^e\delta_f{}^f-'\nabla_f\delta^{ef}+h^e\delta_f{}^f-h_f\delta^{ef}+2l_\theta\not{\delta}^{\theta e}-2l^{ef}{}_\theta\not{\delta}_f^\theta]\,,\end{aligned} \tag{6.8}$$

$$\begin{aligned}g^{\epsilon\theta}\delta R_{\epsilon\theta}=&[\nabla_\epsilon l^\epsilon-l_\epsilon l^\epsilon]\gamma^{-1}\delta\gamma+\tilde{T}[\nabla_\epsilon l_\theta-l_\epsilon l_\theta]\tilde{\delta}^{\epsilon\theta}\\&-{}^{(4)}\nabla_\epsilon[\nabla_\theta(\gamma^{-1}\tilde{\delta}^{\epsilon\theta})+\nabla^\epsilon(\gamma^{-1}\delta\gamma)+l^\epsilon\gamma^{-1}\delta\gamma+l_\theta\gamma^{-1}\tilde{\delta}^{\epsilon\theta}]\,.\end{aligned} \tag{6.9}$$

Putting these results together, we finally obtain

$$\delta_{\tilde{g}^{\epsilon\theta}}\mathcal{L}^D = \sqrt{-g}\{{}'\nabla_e \tilde{h}_{\epsilon\theta}{}^e + 2\tilde{h}_{\epsilon\mu e}\tilde{h}_\theta{}^{\mu e} - \tilde{h}_{\epsilon\theta e}h^e + \tilde{T}[\nabla_e l_\theta - l_{ef\epsilon}l^{ef}{}_\theta - \Omega_{ef\epsilon}\Omega^{ef}{}_\theta]\}\tilde{\delta}^{\epsilon\theta} + \sqrt{-g}\,{}^{(4)}\nabla_\epsilon Z^\epsilon_1 , \tag{6.10a}$$

$$\delta_\gamma\mathcal{L}^D = \sqrt{-g}\{{}'\nabla_e h^e - \tfrac{1}{2}h_e h^e + \nabla_\epsilon l^\epsilon - l_\epsilon l^\epsilon - \tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta e} - 2\Omega_{ef\epsilon}\Omega^{ef\epsilon} + {}'R\}\gamma^{-1}\delta\gamma + \sqrt{-g}\,{}^{(4)}\nabla_\epsilon Z^\epsilon_2 , \tag{6.10b}$$

$$\delta_{\eta^{ef}}\mathcal{L}^D = \sqrt{-g}\{{}'\nabla_{(e}h_{f)} - \tfrac{1}{2}h_e h_f - \tilde{h}_{\epsilon\theta e}\tilde{h}^{\epsilon\theta}{}_f + \nabla_\epsilon l_{ef}{}^\epsilon + 2l_e{}^{d\epsilon}l_{fd\epsilon} - l_{ef\epsilon}l^\epsilon + {}'R_{ef}$$
$$- \eta_{ef}(\tfrac{1}{2}{}'R + \tfrac{1}{2}R + {}'\nabla_d h^d + \nabla_\epsilon l^\epsilon - \tfrac{3}{4}h_d h^d - \tfrac{1}{2}l_\epsilon l^\epsilon + \tfrac{1}{2}\tilde{h}_{\epsilon\theta d}\tilde{h}^{\epsilon\theta d} - \tfrac{1}{2}l_{dc\epsilon}l^{dc\epsilon} + \tfrac{1}{2}\Omega_{dc\epsilon}\Omega^{dc\epsilon})\}\delta^{ef} + \sqrt{-g}\,{}^{(4)}\nabla_\epsilon Z^\epsilon_3 , \tag{6.10c}$$

$$\delta_{b^\epsilon_e}\mathcal{L}^D = 2\sqrt{-g}\{{}'\nabla_f\Omega^{fe}{}_\epsilon - h_f\Omega^{fe}{}_\epsilon - \nabla_\theta\tilde{h}^\theta{}_\epsilon{}^e + \tfrac{1}{2}\nabla_\epsilon h^e + {}'\nabla^e l_\epsilon - {}'\nabla^f l^e{}_{f\epsilon} + l_\theta\tilde{h}^\theta{}_\epsilon{}^e - l_f{}^e{}_\epsilon h^f + \tfrac{1}{2}h^e l_\epsilon\}\delta b^\epsilon_e + \sqrt{-g}\,{}^{(4)}\nabla_\epsilon Z^\epsilon_4 . \tag{6.10d}$$

Comparing (6.10a)–(6.10d) with (6.7a)–(6.7d), respectively, gives us explicit expressions for the various projections of $G_{\alpha\beta}$. It is a simple matter to check that these expressions are identical to the corresponding ones for the projections of $G^{\alpha\beta}$ given by (2.46a)–(2.46d). Explicit expressions can also be calculated for the Z^α from the above.

VII. CONCLUSION

The Cauchy problem is one of the central issues of general relativity. The theory itself gives rise to a set of nonlinear partial differential equations, and consideration of the Cauchy problem throws light on the internal structure of these equations, indicates what information may be used to characterize a solution of the equations, and allows one to formulate what initial data can be freely specified such that a unique, stable solution can be generated from it. At one level then, it gives insight into the role and function of the equations. At a computational level it allows one in principle to generate solutions (numerically and perhaps even analytically) from given initial data. And perhaps more importantly, it provides a possible route towards the quantization of the theory.

To date, most work on the Cauchy problem has been focused on the usual Cauchy spacelike problem. Here the 3+1 approach first suggested by Lichnerowicz and extensively investigated by York and others has resulted in some significant advances. In contrast, most of the work on characteristic and mixed initial-value problems which was pioneered by Bondi (and from which much of our current understanding of gravitational radiation has stemmed) has suffered from the disadvantage of it being couched in a rather *ad hoc* coordinate-dependent form. Part of the purpose of this paper is to remove this limitation. The 2+2 formalism provides a framework which is firstly manifestly covariant or, put another way, is couched directly in geometrically meaningful language. Secondly, it attempts to unify all the possible initial-value problems in suggesting in each case that the conformal two-structure may be used to embody the gravitational degrees of freedom of the theory. Next, it makes clear the geometrical significance of the various gauge or coordinate conditions which may and have been employed, and moreover allows one to investigate what happens if various gauge conditions are dropped or altered. Finally, in separating out the conformal two-structure as the unconstrained initial data, and data whose geometrical significance is immediately clear at that, it offers one a starting point for a quantization program.

However, the work is still largely in its early stages. The basic question as to whether the Cauchy problem is properly set with this choice of data is as yet open, although Seifert and Muller Zum Hagen[13] have shown that the problem is well posed in the double-null case. The next task is clearly to determine whether the same is true for the other cases considered here. Of course, the prescription works in the very limited case of analytic solutions, and the probability is that, pathologies aside, it will prove to work in the more realistic case of smooth (C^∞) solutions. If this is indeed so, then the iterative schemes which we have described here may well be of direct use in allowing one to actually compute solutions from given initial data. There are a number of other questions (for example, boundary conditions and the existence of killing fields and the nature of the resulting solutions) which this formalism might be usefully employed to investigate. Our opinion is that conformal two-structure may be a powerful concept and the 2+2 formalism a useful tool for probing a number of central problems in general relativity.

APPENDIX A: REMAINING INTEGRATION SCHEMES

Case (*b*). The subsidiary condition $G^{11}=0$ is solved for γ on ${}^0\Sigma_0$, at which stage the gauge conditions (5.6) determine γ everywhere. The equations $\perp G^{1\alpha}=0$ and $G^{01}=0$ then determine $\bar{\Omega}^\alpha$ and η on ${}^0\Sigma_0$ as follows. On each S, $i=0,1,\ldots$, we solve $\perp G^{1\alpha}=0$ for $\pounds_{n_0}\bar{\Omega}^\alpha$ and $G^{01}=0$ for $\pounds_{n_0}\eta$. [In fact, we solve the latter for $\pounds_{n_1}\eta_{00}$ and then (5.5) determines $\pounds_{n_0}\eta$.] Solution of $G^{01}=\perp G^{1\alpha}=0$ on ${}^{i,0}S$ allows us to solve the equations again on ${}^{i+1,0}S$. At this stage, γ is known everywhere and η, $\bar{\Omega}^\alpha$ are known on ${}^0\Sigma_0$. The main equations are then solved on successive ${}^k\Sigma_1$, $k=0,1,\ldots$ in the order $G^{00}=0$, $\perp G^{0\alpha}=0$, $G^{01}=0$, $\perp\tilde{G}^{\alpha\beta}=0$, for η, $\bar{\Omega}_\alpha$, and $\tilde{h}^{\alpha\beta}{}_0$, respectively.

Case(c). On ${}^0\Sigma_0$, the subsidiary conditions $G^{11}=0$ and $\perp G^{1\alpha}=0$ determine γ and $\bar{\Omega}^\alpha$, respectively. We next solve $G^{01}=0$ on ${}^0\Sigma_0$ for h_1 (using the relationship $\pounds_{n_1}h_0=\pounds_{n_0}h_1-2\nabla_\epsilon\bar{\Omega}^\epsilon$). The integration scheme for the main questions is rather involved. We first solve $G^{00}=0$ on ${}^0\Sigma_1$ for γ, and then $\perp G^{0\alpha}=0$ on ${}^0\Sigma_1$ for $\bar{\Omega}^\alpha$. We now solve $\perp\tilde{G}^{\alpha\beta}=0$ on ${}^{0,0}S$ for $\pounds_{n_1}\tilde{h}^{\alpha\beta}{}_0$, which determines $\tilde{g}^{\alpha\beta}$ on ${}^{1,1}S$. We can then solve $G^{00}=0$ on ${}^{0,1}S$ for $\pounds_{n_1}h_1$, which determines γ on ${}^{2,1}S$. Next we can solve $G^{01}=0$ on ${}^{1,0}S$ for $\pounds_{n_1}\eta_{00}$, which determines η_{00} on ${}^{2,0}S$. This allows us to solve $\perp\tilde{G}^{\alpha\beta}=0$ again on ${}^{1,0}S$. Continuing in this way, we solve $\perp\tilde{G}^{\alpha\beta}=G^{01}=0$ on ${}^0\Sigma_1$ and $G^{00}=0$ on ${}^1\Sigma_1$. We can summarize the above process as follows:

$$\perp\tilde{G}^{\alpha\beta}=0 \text{ on } {}^{0,i}S\to G^{00}=0 \text{ on } {}^{1,i}S\to G^{01}=0 \text{ on } {}^{0,i+1}S\to\perp\tilde{G}^{\alpha\beta}=0 \text{ on } {}^{0,i+1}S,\ i=0,\ldots.$$

We repeat the integration scheme for the main equations on each successive pair of hypersurfaces ${}^k\Sigma_1$, ${}^{k+1}\Sigma_1$, $k=1,2,\ldots$, starting by solving $\perp G^{0\alpha}=0$ on ${}^1\Sigma_1$.

APPENDIX B: CALCULATION OF $g^{\alpha\beta}\delta R_{\alpha\beta}$ AND $\eta^{ab}\delta' R_{ab}$

In order to calculate $g^{\alpha\beta}\delta R_{\alpha\beta}$, it is convenient to introduce an arbitrary coordinate basis into $\{S\}$ which we denote $\partial/\partial' x^A$, with reciprocal basis $d'x^A$. Then, following Schouten,[8] we introduce the connecting quantities B^α_A, B^A_α defined by

$$\frac{\partial}{\partial' x^A}=B^\alpha_A\frac{\partial}{\partial x^\alpha}\ ,\quad dx^A=B^A_\alpha dx^\alpha\ , \tag{B1}$$

from which it follows that

$$B^\alpha_A B^A_\beta=B^\alpha_\beta,\quad B^\alpha_B B^A_\alpha=\delta^A_B.$$

Any tensor $T^\alpha{}_\beta$ in $\{S\}$ has components in the adapted basis given by

$$T^A{}_B=B^{A\beta}_{\alpha B}T^\alpha{}_\beta\Rightarrow T^\alpha{}_\beta=B^{\alpha B}_{A\beta}T^A{}_B.$$

Moreover,

$$\nabla_C v_B\equiv\partial_C v_B-\Gamma^A_{CB}v_A=B^{\gamma\beta}_{CB}\nabla_\gamma v_\beta$$

for any v_α in $\{S\}$, where Γ^A_{CB} are the usual Christoffel symbols build out of g_{AB}. From (B1) we see that we may write

$$B^\alpha_A=\frac{\partial x^\alpha}{\partial' x^A}\ ,$$

and so clearly B^α_A is independent of any variations in Φ_Λ since it depends only upon some arbitrary choice of coordinate basis $\partial/\partial' x^A$ and $\partial/\partial x^\alpha$ in $\{S\}$ and V, respectively. Hence

$$\delta g^{\alpha\beta}=\delta(B^{\alpha\beta}_{AB}g^{AB})=B^{\alpha\beta}_{AB}\delta g^{AB}\Rightarrow\delta g^{AB}=B^{AB}_{\alpha\beta}\delta g^{\alpha\beta},$$

and it then follows that

$$g^{\alpha\beta}\delta R_{\alpha\beta}=g^{AB}\delta R_{AB}.$$

Now R_{AB} has its usual definition in terms of Γ^A_{BC}, namely

$$R_{CB}=\partial_E\Gamma^E_{CB}-\partial_C\Gamma^E_{EB}+\Gamma^E_{EF}\Gamma^F_{CB}-\Gamma^E_{CF}\Gamma^F_{CB}.$$

We can calculate the variation in Γ^A_{CB} and we obtain

$$\delta\Gamma^A_{CB}=\nabla_C(g^{AE}\delta g_{BE})+\nabla_B(g^{AE}\delta g_{CE})-\nabla_E(g^{AE}\delta g_{BC}). \tag{B2}$$

Note in particular that $\delta\Gamma^A_{CB}$ is a tensor in $\{S\}$. We then obtain

$$\delta R_{CB}=\nabla_E\delta\Gamma^E_{CB}-\nabla_C\delta\Gamma^E_{EB}\ , \tag{B3}$$

from which it follows, using (B2) and (B3), that

$$g^{CB}\delta R_{CB}=\nabla_E\nabla^E(g_{CB}\delta g^{CB})-\nabla_C\nabla_B\delta g^{CB}. \tag{B4}$$

However, we can now write this in terms of the basis of V and we obtain

$$g^{\alpha\beta}\delta R_{\alpha\beta}=\nabla_\epsilon\nabla^\epsilon(g_{\alpha\beta}\delta g^{\alpha\beta})-\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta}, \tag{B5}$$

from which (6.9) follows straightforwardly.

In order to calculate $\eta^{cb}\delta' R_{cb}$, we first note that

$$'R_{cb}=\pounds_{n_e}\Gamma^e_{cb}-\pounds_{n_c}\Gamma^e_{eb}+\Gamma^e_{ef}\Gamma^f_{cb}-\Gamma^e_{cf}\Gamma^f_{eb}\,,$$

and hence it is dependent both on η_{ab} and b^α_a. Let us calculate the variation with respect to η^{ab} first. To start, we calculate $\delta\Gamma^a_{bc}$. From the definition (2.18), we can easily show that

$$\delta\Gamma^a_{cb}=-\tfrac{1}{2}{}'\nabla_c\delta^a_b-\tfrac{1}{2}{}'\nabla_b\delta^a_c+\tfrac{1}{2}{}'\nabla^a\delta_{cb}\,;$$

hence $\delta\Gamma^a_{cb}$ is a "tensor" with respect to $'\nabla_a$. Then

$$\delta' R_{cb}={}'\nabla_e\delta\Gamma^e_{cb}-{}'\nabla_c\delta\Gamma^e_{eb}$$

and (B6)

$$\eta^{cb}\delta' R_{cb}={}'\nabla_c{}'\nabla^c\delta^b_b-{}'\nabla_c{}'\nabla_b\delta^{cb}.$$

To calculate the variation with respect to b_a^α, we first calculate $\not\delta\Gamma^a_{cb}$ and obtain

$$\begin{aligned}\not\delta\Gamma^a_{cb} &= \tfrac{1}{2}\eta^{ae}(\pounds_{\delta_c}\eta_{eb} + \pounds_{\delta_b}\eta_{ec} - \pounds_{\delta_e}\eta_{cb}) \\ &= -\not\delta_c^\epsilon l^a{}_{b\epsilon} - \not\delta_b^\epsilon l^a{}_{c\alpha} + \not\delta^{\epsilon\alpha} l_{cb\epsilon}\,,\end{aligned} \tag{B7}$$

so again $\not\delta\Gamma^a_{cb}$ is a tensor with respect to $'\nabla_a$. Next, we can show that

$$\begin{aligned}\not\delta\,'R_{cb} = {}& \pounds_{\delta_e}\Gamma^e_{cb} - \pounds_{\delta_c}\Gamma^e_{eb} + '\nabla_e(\not\delta\Gamma^e_{cb}) \\ & - '\nabla_c(\not\delta\Gamma^e_{eb})\,.\end{aligned} \tag{B8}$$

In order to proceed we need an expression for $\pounds_{\delta_d}\Gamma^a_{cb}$. We use the result that for the commutator of ∇_α and $\pounds_{n_a}$ acting on a scalar:

$$\nabla_\alpha \pounds_{n_a} - \pounds_{n_a}\nabla_\alpha = 2n^e{}_\alpha \pounds_{\Omega_{ae}}\,.$$

Then

$$\begin{aligned}\pounds_{\delta_d}\Gamma^a_{cb} &= \delta_d^\alpha \nabla_\alpha \Gamma^a_{cb} \\ &= 2\not\delta_d^\alpha l^{ae}{}_\alpha \Gamma_{cbe} - \not\delta_d^\alpha \eta^{ae}(\pounds_{n_b} l_{ec\alpha} + \pounds_{n_c} l_{eb\alpha} - \pounds_{n_e} l_{cb\alpha})\end{aligned}$$

and hence

$$\pounds_{\delta_d}\Gamma^a_{cb} = -\not\delta_d^\alpha('\nabla_b l^a_{c\alpha} + '\nabla_c l^a_{b\alpha} - '\nabla^a l_{cb\alpha})\,. \tag{B9}$$

Substituting (B7) and (B9) into (B8) gives

$$\eta^{cb}\not\delta R_{cb} = 2\not\delta_b^\epsilon('\nabla^b l_\epsilon - '\nabla^c l^b{}_{c\epsilon}) + 2'\nabla^b(\not\delta_b^\epsilon l_\epsilon - \not\delta_c^\epsilon l^c{}_{b\epsilon})\,. \tag{B10}$$

Then (B6) and (B10) lead straightforwardly to (6.8).

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*Present address: Department of Engineering Mathematics, Stephenson Building, Claremont Road, Newcastle upon Tyne, NE1 7RU, England.

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