Low-energy πN scattering from N and Δ poles and the low-energy theorems

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(Received 10 October 1979)

An investigation is made of the extent to which low-energy πN scattering is determined by the N and Δ poles and the low-energy theorems. The on-shell amplitudes are expanded in (μ_π/M_N) at fixed (ν/μ_π) and (t/μ_π^2), and it is assumed that $M_\Delta - M_N = O(\mu_\pi)$, in contrast to $M_{\rm res} - M_N = O(M_N)$ for all other resonances. The invariant amplitudes are determined using fixed-t dispersion relations and are calculated to second order in (μ_π/M_N). The crossing-odd amplitudes are fully determined by the N and Δ poles and the low-energy theorems. The crossing-even amplitudes receive undetermined background contributions described by three expansion coefficients. If they are fixed using the "experimental" scattering lengths, the resulting model is in fairly good agreement with experiment.

I. INTRODUCTION

The aim of this paper is to investigate to what extent the low-energy πN scattering amplitudes are determined by the N and Δ poles (via fixed-t dispersion relations) and the low-energy theorems.[1-5]

We want to give an approximate description of the low-energy amplitudes in terms of a few fundamental parameters (e.g., $g_{NN\pi}$, f_π, or the σ term). We do not want to calculate the amplitudes as accurately as possible by using the enormous amount of experimental data as an input; we shall call the output of such a phase-shift—or amplitude—analysis "experimental" scattering amplitudes.

The low-energy theorems refer to the scattering of soft pions; they are exact statements for the off-shell πN amplitudes. The on-shell continuation induces corrections whose magnitude can be described in an expansion in μ_π/M_N [see Eqs. (2a) and (2b) in Sec. II].[5-7]

We expand the on-shell amplitudes in μ_π/M_N, keeping the ratios ν/μ_π and t/μ_π^2 fixed; i.e., we consider the amplitudes for varying pion mass and fixed relative distance to the lowest-lying branch-point singularities. Since we want to treat the N and Δ poles on the same footing and keep the Δ contribution as an explicit pole, we must set

$$M_\Delta - M_N = O(\mu_\pi) \qquad (1a)$$

in contrast to

$$M_{\rm res}^2 - M_N^2 = O(1/\alpha') = O(M_N^2) \qquad (1b)$$

for all other resonances. This is also very natural since the physical threshold is equally close to the Δ as to the N pole.

With the μ_π/M_N expansion for fixed ν/μ_π and t/μ_π^2 we can avoid the following difficulties.

(1) Satisfying a low-energy theorem in a fixed-t dispersion calculation necessitates a subtraction. But the low-energy theorems determine the subtraction functions, e.g., $A^{(+)}(\nu=0,t)$, only at $t=2\mu_\pi^2$ and give no information for $t\neq 2\mu_\pi^2$. If one expands the subtraction function in t (and treats the expansion coefficients as free parameters), the expansion diverges at $t=4\mu_\pi^2$ and, if the expansion is done around $t=0$, also at $t=-4\mu_\pi^2$. But since the discontinuity (of $A'^{(\pm)}$ say) for $t\geq 4\mu_\pi^2$, $t=O(\mu_\pi^2)$ is of $O(\mu_\pi^3/M_N^3)$ (see the Appendix), the t expansion of the subtraction function for $A'^{(\pm)}$ makes sense outside the t interval $(-4\mu_\pi^2, +4\mu_\pi^2)$ up to $O(\mu_\pi^2/M_N^2)$ for t/μ_π^2 fixed.

(2) In the low-energy region the contribution of the higher resonances is slowly varying in ν. But again an expansion in ν of the contributions other than N and Δ diverges at threshold unless one restricts oneself to $O(\mu_\pi^2/M_N^2)$ for ν/μ_π fixed (see the Appendix).

Instead of using subtracted dispersion relations (at fixed t) we close the contour of an unsubtracted dispersion relation (at fixed t) at $|\nu|=N$ and expand the background term (contribution of the higher resonances, of the nonresonating background and of the contour integral) in ν and t to second order in μ_π/M_N for $\nu=O(\mu_\pi)$, $t=O(\mu_\pi^2)$. This corresponds to an expansion up to quadratic terms in the pion momenta. The two low-energy theorems determine those two expansion parameters which contribute to leading order in μ_π/M_N.

In lowest order (linear in μ_π/M_N) the S waves in the low-energy region ($|\nu|\lesssim\nu_\Delta$) are fully determined by the low-energy theorems and reproduce Weinberg's scattering lengths.[2] The P waves are fully determined by the N and Δ poles with static kinematics.

In the next higher order (quadratic in μ_π/M_N) the μ_π/M_N expansion allows us to obtain the following exact results.

(1) The correction terms of $O(\mu_\pi^2/M_N^2)$ in the crossing-odd amplitudes are fully determined by the N and Δ poles and by the low-energy theorem, i.e., there are no undetermined background contributions.

(2) The crossing-even amplitudes receive undetermined background contributions described by three expansion coefficients, which we fix by experiment.

Including quadratic terms in μ_π/M_N with the assumptions (1a) and (1b) produces a description for the low-energy amplitudes, which is in fairly good agreement with experiment.

II. THE LOW-ENERGY THEOREMS

The low-energy theorems for πN-πN (e.g., Refs. 1–5) can be formulated as exact statements only for the scattering of soft pions, $q_\mu \to 0$, i.e., for off-shell amplitudes. The order of corrections (in the μ/M expansion) arising from the on-shell continuation can be determined in the context of chiral perturbation theory.[6,7] [The assumption of an approximate chiral symmetry where perturbation theory makes sense is technically equivalent to the assumption of current algebra and PCAC (partial conservation of axial-vector current).[3]] The low-energy theorems for the on-shell amplitudes read (for $C^{(+)}$ the corrections to the relation of Cheng and Dashen[8] are indeed of higher order as claimed by them)

$$\lim_{\nu\to 0} C^{(+)}\left(\nu, \nu_B=-\frac{\nu^2}{2M};\ \mu^2,\mu^2\right) = \frac{\sigma_{NN}}{f_\pi^2}+\frac{1}{M}O\left(\frac{\mu^4/M^3}{M_\Delta-M},\ \frac{\mu^4}{M^4}\ln\frac{\mu^2}{M^2}\right), \quad (2a)$$

$$\lim_{\nu\to 0} \frac{1}{\nu} C^{(-)}\left(\nu, \nu_B=-\frac{\nu^2}{2M};\mu^2,\mu^2\right) = \frac{F_1^V(2\mu^2)}{2f_\pi^2}-\frac{g_\Delta^2}{2M^3}\,\frac{3\mu^2}{M_\Delta-M} +\frac{1}{M^2}O\left(\frac{\mu^2}{M^2},\ \frac{\mu^4/M^2}{(M_\Delta-M)^2}\right), \quad (2b)$$

where

$$C\equiv A+\nu B,\quad \nu=(s-u)/4M,$$

$$\nu_B\equiv(t-2\mu^2)/4M,\quad M=M_N,\quad \mu\equiv\mu_\pi .$$

A and B are the standard invariant amplitudes and $\pm$ refers to $I_t=0,1$. $F_1^V(t)$ is the isovector form factor of the nucleon and M_Δ and $g_\Delta\equiv g_{\Delta N\pi}$ are the mass and the coupling constant of the Δ pole. $\sigma_{NN}=\sigma_{NN}(2\mu^2)$ is the nucleon matrix element of the equal-time commutator of the axial-vector current with its divergence at the momentum transfer $t=2\mu^2$; it is of order μ^2/M:

$$\langle p'|\sigma_{ik}(0)|p\rangle\equiv\delta_{ik}\sigma_{NN}((p'-p)^2)\bar u(p')u(p),$$

$$[A_i^0(x),\partial_\mu A_k^\mu(y)]_{x^0=y^0}\equiv -i\delta^{(3)}(\vec x-\vec y)\sigma_{ik}(y) .$$

The point $\nu=0$, $\nu_B=0$ is the Cheng-Dashen point, which is the crossing point of the s- and u-channel N poles for $m_n=m_p$. It is important to specify the line along which this point is approached: It is the parabola

$$\nu_B=-\nu^2/2M$$

going through the Cheng-Dashen point and the threshold. This parabola is approximately the $\theta_{\text{c.m.}}=90°$ line (see Secs. VI A and VII A).

This information of the low-energy theorems has the following advantages.

(1) It refers to the full amplitudes, and not to amplitudes where the N- and Δ-pole terms have been subtracted. Therefore, there is no need to specify which effective Lagrangian has to be used for the N and the Δ and which propagator for the latter.

(2) It reads the same whether $m_n=m_p$ or $m_n\neq m_p$, i.e., it is irrelevant whether the nucleon pole is right at or away from the Cheng-Dashen point.

(3) It does not contain, to order μ/M in $C^{(\pm)}$, any coupling constants which refer to specific channels (such as g_A^{NN} or $g_{NN\pi}$).

For $m_n\neq m_p$ the line of approach is of course irrelevant. If one used for $m_n=m_p$ the line of approach of Cheng and Dashen,[8] first setting $\nu_B=0$ and then $\nu=0$, the right-hand side of Eq. (2b) gets an additional term $-g_N^2(2M^2)^{-1}$, $g_N\equiv g_{NN\pi}$. Using instead the line of approach $\nu_B\to 0$ for $\nu=0$ would require a large additional term $+g_N^2/M$ in Eq. (2a) while the left-hand side of Eq. (2b) would be infinite.

The corrections due to the Δ resonance[5] on the right-hand side of Eqs. (2) cannot be neglected (in contrast to Ref. 7) because we set $M_\Delta-M=O(\mu)$.

To calculate the amplitude $C^{(-)}$ to second order in μ/M, i.e., modulo $1+o(\mu/M)$, it would not be necessary to take the difference $F_1^V(2\mu^2)-1=O((\mu^2/M^2)\ln(\mu^2/M^2))$ (Ref. 9) into account. But since the charge radius of the nucleon is numerically so large that the correction $(2f_\pi^2)^{-1}[F_1^V(2\mu^2)-1]$ is almost as large as the correction due to the Δ pole[5] in Eq. (2b), we will at this very point not systematically neglect all terms of $O(\mu^2/M^2)$ and not replace $F_1^V(2\mu^2)$ by unity.

In the following we work with the amplitude

$$A'\equiv A+\nu(1-t/4M^2)^{-1}B$$

rather than with C. In order μ^2/M^2 one must distinguish between A' and C. $C^{(+)}$ is directly connected to the σ term [see Eq. (2a)], while A' has a particularly simple partial-wave expansion in this order (see Sec. III). In the limit $\nu \to 0$, $\nu_B = -\nu^2/2M$ only the N pole contributes to $A'^{(+)} - C^{(+)}$, and to $O(\mu^2/M^2)$ the only contribution to $(1/\nu)(A'^{(-)} - C^{(-)})$ comes from the Δ pole. Therefore, the low-energy theorems can be formulated for A'. From Eqs. (2) follow

$$\lim_{\nu\to 0} A'^{(+)}\left(\nu, \nu_B = -\frac{\nu^2}{2M}\,;\, \mu^2, \mu^2\right) = \frac{\sigma_{NN}}{f_\pi^2} - \frac{g_N^2}{2M}\frac{\mu^2}{M^2} + \frac{1}{M} O\left(\frac{\mu^4/M^3}{M_\Delta - M}, \frac{\mu^4}{M^4}\ln\frac{\mu^2}{M^2}\right), \tag{3a}$$

$$\lim_{\nu\to 0} \frac{1}{\nu} A'^{(-)}\left(\nu, \nu_B = -\frac{\nu^2}{2M}\,;\, \mu^2, \mu^2\right) = \frac{F_1^V(2\mu^2)}{2f_\pi^2} - \frac{g_\Delta^2}{2M^3}\frac{2\mu^2}{M_\Delta - M} + \frac{1}{M^2} O\left(\frac{\mu^2}{M^2}, \frac{\mu^4/M^2}{(M_\Delta - M)^2}\right). \tag{3b}$$

As the numerical value for f_π we use

$$f_\pi = 0.66\,\mu\ .$$

As the σ term cannot be reliably calculated theoretically, we use an "experimental" value, which has been computed from experimental data via dispersion relations and a short extrapolation from $t = 0$ to $t = 2\mu^2$. We use the value of the "compilation of coupling constants and low-energy parameters"[10]:

$$\sigma_{NN} = 62 \pm 10\ \text{MeV}\ .$$

With the Δ parameters of Sec. IV and the isovector charge radius of the nucleon[11] $r_{1V} = 0.76$ fm, the first two terms of the right-hand side of Eq. (3b) combine to $(2f_\pi^2)^{-1}(1 + 0.001)$, while taken separately the deviation of $F_1^V(2\mu^2)$ from unity and the Δ term represent each a 10% correction to $1/2f_\pi^2$. As a numerical statement Eq. (3b) becomes

$$\lim_{\nu\to 0} \frac{1}{\nu} A'^{(-)}\left(\nu, \nu_B = -\frac{\nu^2}{2M}\,;\, \mu^2, \mu^2\right) = \frac{1}{2f_\pi^2} + \frac{1}{M^2} O\left(\frac{\mu^2}{M^2}, \frac{\mu^4/M^2}{(M_\Delta - M)^2}\right). \tag{3c}$$

III. THE CONNECTION BETWEEN THE INVARIANT AMPLITUDES AND THE PARTIAL-WAVE AMPLITUDES TO ORDER μ^2/M^2

In the expansion in μ/M for fixed ν/μ and t/μ^2 the partial-wave amplitudes are of order

$$\begin{aligned} f_{0+}^{(+)} &= M^{-1}O(\mu^2/M^2),\\ f_{0+}^{(-)}, f_{1\pm}^{(\pm)} &= M^{-1}O(\mu/M),\\ f_{l\pm}^{(\pm)} &= \begin{cases} M^{-1}O(\mu^2/M^2), & l = 2\\ M^{-1}O(\mu^3/M^3), & l \geq 3 \end{cases} \end{aligned} \tag{4}$$

where $(\pm)$ stands for $I_t = 0, 1$. The order of the S waves is given by the low-energy theorems. The order of the higher partial waves is determined by the direct- and crossed-channel N and Δ poles and the t-channel discontinuity. For an estimate of the latter see the Appendix. Note that the crossing-odd S wave and the P waves are of the same order, namely linear in μ/M.

The partial-wave expansions of the invariant amplitudes

$$A' \equiv A + \nu(1 - t/4M^2)^{-1}B$$

and B become very simple to order μ^2/M^2:

$$\begin{aligned} A'/4\pi &= (1 + \omega/M)[f_{0+} + z(f_{1-} + 2f_{1+}) + (\tfrac{3}{2}z^2 - \tfrac{1}{2})(2f_{2-} + 3f_{2+})][1 + O(\mu^2/M^2)],\\ B/4\pi &= 2M/q^2[(f_{1-} - f_{1+}) + 3z(f_{2-} - f_{2+})][1 + O(\mu^2/M^2)], \end{aligned} \tag{5}$$

with ω the laboratory energy of the pion, $z = \cos\theta_{\text{c.m.}}$ and q the c.m. momentum. The partial-wave projections are then given by

$$\begin{aligned} f_{0+} &= (1 + \omega/M)^{-1}\left[\frac{A'(90°)}{4\pi} + \frac{q^4}{6}\left(\frac{\partial^2}{\partial(t/2)^2}\frac{A'}{4\pi}\right)_s\Big|_{90°}\right][1 + O(\mu^2/M^2)],\\ a_{A'} &\equiv q^{-2}(f_{1-} + 2f_{1+}) = (1 + \omega/M)^{-1}\left(\frac{\partial}{\partial(t/2)}\frac{A'}{4\pi}\right)_s\Big|_{90°}[1 + O(\mu^2/M^2)],\\ a_B &\equiv q^{-2}(f_{1-} - f_{1+}) = (2M)^{-1}\frac{B(90°)}{4\pi}[1 + O(\mu^2/M^2)]\ . \end{aligned} \tag{6}$$

The unitarity-cusp effects of the partial-wave amplitudes are of order

$$\mathrm{Im} f_l/\mathrm{Re} f_l \approx \delta_l \approx q|f_l| = O(\mu^2/M^2), \quad l=0,1$$

and can be neglected in our approximation. Experimentally one has $\delta_0 \lesssim 8° = 0.14$ for the S waves below $T = 100$ MeV.

IV. THE N AND Δ CONTRIBUTION TO THE FIXED-t DISPERSION RELATION

The amplitudes $A'^{(\pm)}$ and $B^{(\pm)}$ are calculated by Cauchy integrals in the ν plane at fixed t:

$$F_\pm(\nu,t) = \frac{2}{\pi}\int_0^N d\nu'(\nu'^2-\nu^2)^{-1}\binom{\nu'}{\nu}\mathrm{Im}\,F_\pm(\nu',t) + (2\pi i)^{-1}\oint_{|\nu'|=N} d\nu'(\nu'-\nu)^{-1}F_\pm(\nu',t)\,, \tag{7}$$

where $F_\pm$ stands for $A'^{(\pm)}$ and $B^{(\mp)}$.

The N and Δ contributions are calculated modulo $1+O(\mu^2/M^2)$ for fixed (ν/μ) and (t/μ^2). This implies in particular that the Δ can be approximated by a zero-width resonance, since

$$\Gamma \sim q^3 = O(\mu^3)$$

and

$$\nu_\Delta - \nu - iM_\Delta\Gamma/(2M) = (\nu_\Delta - \nu)[1+O(\mu^2/M^2)]\,.$$

We obtain from Eq. (5)

$$\begin{aligned}
A'^{(\pm)}|_N &= \binom{\nu_B}{\nu}\frac{g_N^2}{2M}\frac{2\nu_B}{\nu_B^2-\nu^2}\,,\\
B^{(\pm)}|_N &= \binom{\nu}{\nu_B}\frac{g_N^2}{2M}\frac{2}{\nu_B^2-\nu^2}\,,\\
A'^{(\pm)}|_\Delta &= 2\binom{2\nu_\Delta}{-\nu}\frac{g_\Delta^2}{2M}\left(1+\frac{\nu_\Delta}{M}\right)\frac{1}{2M}\frac{t+2q_\Delta^2}{\nu_\Delta^2-\nu^2}\,,\\
B^{(\pm)}|_\Delta &= -\binom{2\nu}{-\nu_\Delta}\frac{g_\Delta^2}{2M}\frac{2}{\nu_\Delta^2-\nu^2}\,,
\end{aligned} \tag{8}$$

with $\nu_B = \nu|_{s=M^2}$, $\nu_\Delta = \nu|_{s=M_\Delta^2}$, and $q_\Delta^2 = q^2|_{s=M_\Delta^2}$.

Note that in our approximation, Eq. (5), the residues of the P-wave poles of A' are linear in t and vanish at 90°, i.e., they are proportional to $(t+2q_{\mathrm{res}}^2)$. In the case of the nucleon this gives $t+2q_N^2 = t-2\mu^2 = 2\nu_B\,2M$.

For the $NN\pi$ coupling constant $g_N^2/4\pi$ we use the value 14.5. For the Δ parameters g_Δ^2 and M_Δ we use the values of Höhler *et al.*[12]:

$$g_\Delta^2/4\pi = 5.3, \quad M_\Delta = 1220 \text{ MeV}\,.$$

They obtained these values by considering

$$X_B(\nu,t) \equiv \pi^{-1}\int_\mu^{\nu_0} d\nu'(\nu'-\nu)^{-1}\mathrm{Im}\,B^{(-)}_{3,3}(\nu',t)\,,$$

where $\mathrm{Im}\,B^{(-)}_{3,3}$ is the contribution of the 3-3 P wave to $\mathrm{Im}\,B^{(-)}$. Comparison of X_B calculated with the experimental phase shifts and the zero-width approximation for X_B

$$X_B^{zw} = \frac{g_\Delta^2}{2M}\frac{1}{\nu_\Delta-\nu}$$

in the region $0 \lesssim \nu \lesssim \mu$ determines g_Δ^2 and M_Δ. The analogous procedure with the amplitude $A^{(-)}$ instead of $B^{(-)}$ yields somewhat different values for g_Δ^2 amd M_Δ. Furthermore they are slightly dependent on the t value, at which the comparison is done. The quoted values are averaged values. This $\Delta N\pi$ coupling constant is 30% smaller than the one calculated from Γ_Δ in the narrow resonance approximation.

V. THE BACKGROUND CONTRIBUTION

We expand the contribution of the higher resonances, the nonresonating background, and contour integral to the invariant amplitudes in power series in ν and ν_B. The assumption $M_{\mathrm{res}} - M = O(M)$ together with PCAC and current algebra imply that it is sufficient to consider the following terms, since we want to go just one order beyond the lowest order in μ/M [see Appendix, Eq. (A5)]:

$$\begin{aligned}
A'^{(+)}/4\pi|_{\mathrm{BG}} &= \alpha_L^{(+)} + \alpha_P^{(+)}[\nu_B + \nu^2/2M] + \alpha_s^{(+)}\nu^2\,,\\
A'^{(-)}/4\pi|_{\mathrm{BG}} &= \alpha_L^{(-)}\nu\,,\\
B^{(+)}/4\pi|_{\mathrm{BG}} &= 0\,,\\
B^{(-)}/4\pi|_{\mathrm{BG}} &= \beta^{(-)}.
\end{aligned} \tag{9}$$

The expansion coefficients are of order $O(1)$. Remember that the N and Δ contribution to the invariant amplitudes are of order

$$A'^{(\pm)}:\ M^{-1}O(\mu/M)\,,$$
$$B^{(\pm)}:\ M^{-2}O[(\mu/M)^{-1}]\,.$$

To lowest order only $\alpha_L^{(\pm)}$ contribute.

The background contributions $\alpha_L^{(\pm)}$ are determined by the low-energy theorems and are therefore not free parameters:

$$\begin{aligned}
\alpha_L^{(+)} &= \frac{1}{4\pi}\frac{\sigma_{NN}}{f_\pi^2} - \frac{\mu^2}{2M^3}\frac{g_N^2}{4\pi} - \frac{2}{M^2}\frac{g_\Delta^2}{4\pi}\xi_\Delta\left(1+\frac{\mu^2}{\omega_\Delta M}\right)\left(\omega_\Delta + \frac{\mu^2}{2M}\right),\\
\alpha_L^{(-)} &= \frac{1}{4\pi}\frac{1}{2f_\pi^2} - \frac{1}{2M^2}\frac{g_N^2}{4\pi} + \frac{1}{M^2}\frac{g_\Delta^2}{4\pi}\xi_\Delta\left(1+\frac{\mu^2}{\omega_\Delta M}\right),
\end{aligned} \tag{10}$$

where

$$\xi_\Delta \equiv (1+\omega_\Delta/M)/(1+2\omega_\Delta/M)\,.$$

Since $[\nu_B+\nu^2/(2M)]_{90°}=MO(\mu^3/M^3)$, $\alpha_P^{(+)}$ contributes only to the P wave $a_{A'}^{(+)}$ and not to the S wave $f_{0+}^{(+)}$. $\alpha_s^{(+)}$, on the other hand, contributes only to the S wave $f_{0+}^{(+)}$. We shall later determine the free parameters $\alpha_s^{(+)}$, $\alpha_P^{(+)}$, and $\beta^{(-)}$ by a fit to the "experimental" scattering lengths $a_S^{(+)}$, $a_{A'}^{(+)}$ $(\omega=\mu)$, and $a_B^{(-)}$ $(\omega=\mu)$.

It should be noted that there is no one-to-one relation between the rate, with which the dispersion integrals of the invariant amplitudes converge, and their number of background parameters: The dispersion integral of $(d/dt)A'^{(-)}$ and of $B^{(-)}$ converge with the same rate, and $(d/dt)A'^{(-)}$ has no background term while $B^{(-)}$ has one.

VI. THE πN AMPLITUDES IN LOWEST ORDER: $O(\mu/M)$

Kinematics in lowest order is static kinematics, where the kinetic energy of the nucleon is neglected, the laboratory and center-of-mass systems are identical, and $\nu=\omega$.

A. The S waves (linear model)

In lowest order the S waves are given by

$$f_{0+}=A'(90°)/4\pi$$

and we need not distinguish between A' and C nor between the 90° line and the parabola

$$\nu_B=-\nu^2/(2M)=-\omega^2/(2M)\,.$$

The S waves in lowest order have no left-hand singularities from the u-channel resonances, the $S^{(+)}$ wave is constant and the $S^{(-)}$ wave is linear in ω.

In the $S^{(+)}$ wave the N contribution is zero, while the Δ contribution is of $O(\nu_\Delta/M)$ and therefore in qualitative disagreement with the low-energy theorem, which reads, neglecting order $O(\mu^2/M^2)$,

$$f_{0+}^{(+)}=0 \tag{11}$$

since $\sigma_{NN}/f_\pi^2=M^{-1}O(\mu^2/M^2)$. Therefore, the Δ contribution must be canceled by the background (higher resonances, nonresonating background, contour integral).

In the $S^{(-)}$ wave the N and Δ give linear contributions of the correct order $O(\mu/M)$. The background also gives a linear contribution of $O(\mu/M)$ and is sizable. The sum of the slopes is fixed by current algebra and PCAC:

$$f_{0+}^{(-)}(\omega)=\frac{1}{4\pi}\;\frac{\omega}{2f_\pi^2}\,. \tag{12}$$

Equations (11) and (12) reproduce Weinberg's scattering lengths[2] up to the factor $(1+\mu/M)$, which is negligible in this order.

B. The P waves

In lowest order, the contribution of the higher resonances, the nonresonating background, and the contour integral vanishes, and the P waves are given by the N and Δ poles alone (with static kinematics):

$$a_{3,3}=\frac{4}{9}\;\frac{\lambda_N}{\omega_B+\omega}+1\,\frac{\lambda_\Delta}{\omega_\Delta-\omega}+\frac{1}{9}\;\frac{\lambda_\Delta}{\omega_\Delta+\omega}\,,$$

$$a_{1,3}=a_{3,1}=-\frac{2}{9}\;\frac{\lambda_N}{\omega_B+\omega}+\frac{4}{9}\;\frac{\lambda_\Delta}{\omega_\Delta+\omega}\,, \tag{13}$$

$$a_{1,1}=1\,\frac{\lambda_N}{\omega_B-\omega}+\frac{1}{9}\;\frac{\lambda_N}{\omega_B+\omega}+\frac{16}{9}\;\frac{\lambda_\Delta}{\omega_\Delta+\omega}\,,$$

where $a_{2I,2J}\equiv q^{-2}f_{J,l=1}^{(I)}$ and $\lambda_i\equiv(3/4M^2)g_i^2/4\pi$. Since in this order $\omega_B=0$, we have

$$a_{1,1}=4a_{1,3}=4a_{3,1}\,. \tag{14}$$

For the linear combinations of P waves defined in Eq. (6) we have

$$a_{A'}^{(-)}=a_B^{(+)}=\frac{\lambda_N}{3}\;\frac{2\omega}{\omega_B^2-\omega^2}-2\,\frac{\lambda_\Delta}{3}\;\frac{2\omega}{\omega_\Delta^2-\omega^2}\,,$$

$$a_{A'}^{(+)}=4a_B^{(-)}=+4\,\frac{\lambda_\Delta}{3}\;\frac{2\omega_\Delta}{\omega_\Delta^2-\omega^2}\,. \tag{15}$$

The N contribution to $a_{A'}^{(+)}$ and $a_B^{(-)}$ is proportional to $2\omega_B/(\omega_B^2-\omega^2)$ and therefore zero in this order.

Equation (13) can also be written down directly by noting that (in this order) the spin–orbital-angular momentum structure is the same as the isospin structure [compare Eq. (6)]:

$$q^{-2}(f_{J=1/2,l=1}-f_{J=3/2,l=1})=(2M)^{-1}B/4\pi\,, \tag{16a}$$

$$F(I_s=\tfrac{1}{2})-F(I_s=\tfrac{3}{2})=3F^{(-)}\,, \tag{16b}$$

which are both odd under s-u crossing, and

$$q^{-2}(f_{J=1/2,l=1}+2f_{J=3/2,l=1})=\left(\frac{\partial}{\partial(t/2)}\;\frac{A'}{4\pi}\right)_\nu\,, \tag{17a}$$

$$F(I_s=\tfrac{1}{2})+2F(I_s=\tfrac{3}{2})=3F^{(+)}\,, \tag{17b}$$

which are both even under s-u crossing. These crossing relations determine the SU(2) crossing matrix, whose elements appear as factors in Eq. (13).

VII. THE πN AMPLITUDES IN NEXT HIGHER ORDER: $O(\mu^2/M^2)$

Here we go one order beyond Weinberg's treatment of S-wave scattering lengths and the static treatment of the P waves and calculate the partial waves including $O(\mu^2/M^2)$. The subscript 90 (e.g., ν_{B90}) denotes "value on the 90° line":

$$\nu_{B90}=\nu_B(\omega,z=0)\,.$$

A. The S waves

In this order the difference between the 90° line and the parabola of Sec. VI A is not negligible,

$$\nu_{B90}+\nu^2/(2M)=MO(\mu^3/M^3)\,,$$

and the S waves show left-hand singularities, which are, as a result of the expansion, poles and not cuts. The other corrections to the lowest-order expressions for the S waves [Eqs. (11) and (12)] are an overall factor $(1+\omega/M)^{-1}$, a factor $\xi_\Delta\equiv(1+\omega_\Delta/M)/(1+2\omega_\Delta/M)$ in the Δ residue, the additional background term $\alpha_s^{(+)}\nu^2$ in $f_{0+}^{(+)}$, and the terms $d^{(\pm)}$ that eliminate the D-wave contribution to $A'(90°)$ [see Eqs. (5) and (6)]:

$$f_{0+}^{(-)}(\omega)=\frac{\nu_{90}}{1+\omega/M}\,(\alpha_L^{(-)}+S_N-S_\Delta)+d^{(-)}\,, \qquad (18a)$$

$$f_{0+}^{(+)}(\omega)=\frac{1}{1+\omega/M}\,(\alpha_L^{(+)}+\alpha_s^{(+)}\nu_{90}^2+\nu_{B90}S_N+2\nu_{\Delta90}S_\Delta)+d^{(+)}\,, \qquad (18b)$$

where

$$S_N=\frac{1}{2M^2}\,\frac{g_N^2}{4\pi}\left(1+\frac{1}{M}\,\frac{\mu^2-\omega^2}{\omega}\right),$$

$$S_\Delta=\frac{1}{M^2}\,\frac{g_\Delta^2}{4\pi}\,\xi_\Delta\left(1+\frac{1}{M}\,\frac{\mu^2-\omega^2}{\omega_\Delta+\omega}\right),$$

$$d^{(\pm)}=(\mp)\,\frac{q^4}{6}\left[\frac{1}{2M^3}\,\frac{g_N^2}{4\pi}\,\frac{1}{\omega^2}+\begin{pmatrix}+2\\-1\end{pmatrix}\frac{1}{M^3}\,\frac{g_\Delta^2}{4\pi}\,\frac{1}{(\omega_\Delta+\omega)^2}\right].$$

B. The P waves

The calculation of the P waves modulo $1+O(\mu^2/M^2)$ gives the following corrections to the lowest-order expressions [Eq. (15)]: (i) The replacement of ω, ω_B, and ω_Δ by ν_{90}, ν_{B90}, and $\nu_{\Delta90}$, (ii) the nucleon contribution to $a_{A'}^{(+)}$ and $a_B^{(-)}$, (iii) the background contribution to $a_{A'}^{(+)}$, $a_{A'}^{(-)}$, and $a_B^{(-)}$, (iv) additional terms which appear from taking the derivative of the denominators of the u-channel poles in the calculation of $a_{A'}^{(\pm)}$, since in this order

$$\left(\frac{\partial A'}{\partial(t/2)}\right)_s\neq\left(\frac{\partial A'}{\partial(t/2)}\right)_\nu\,,$$

(v) a factor $1+\omega_\Delta/M$ in the Δ residue in $a_{A'}^{(\pm)}$, and (vi) a factor $(1+\omega/M)^{-1}$ in every leading term of $a_{A'}^{(\pm)}$. We obtain

$$a_{A'}^{(-)}=(1+\omega/M)^{-1}\left[\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu}{\nu_B^2-\nu^2}-\frac{2}{4M^2}\,\frac{g_\Delta^2}{4\pi}\left(1+\frac{\omega_\Delta}{M}\right)\frac{2\nu}{\nu_\Delta^2-\nu^2}\right]_{90}+\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu_B}{(\nu_B+\nu)^2}\bigg|_{90}-\frac{1}{4M^2}\,\frac{g_\Delta^2}{4\pi}\,\frac{2}{M}\,\frac{t/2+q_\Delta^2}{(\nu_\Delta+\nu)^2}\bigg|_{90}+\frac{\alpha_L^{(-)}}{2M}\,,$$

$$a_B^{(+)}=\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu}{\nu_B^2-\nu^2}\bigg|_{90}-\frac{2}{4M^2}\,\frac{g_\Delta^2}{4\pi}\,\frac{2\nu}{\nu_\Delta^2-\nu^2}\bigg|_{90}\,, \qquad (19)$$

$$a_{A'}^{(+)}=(1+\omega/M)^{-1}\,\frac{4}{4M^2}\,\frac{g_\Delta^2}{4\pi}\left(1+\frac{\omega_\Delta}{M}\right)\frac{2\nu_\Delta}{\nu_\Delta^2-\nu^2}\bigg|_{90}+\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu_B}{\nu_B^2-\nu^2}\bigg|_{90}-\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu_B}{(\nu_B+\nu)^2}\bigg|_{90}-\frac{2}{4M^2}\,\frac{g_\Delta^2}{4\pi}\,\frac{2}{M}\,\frac{t/2+q_\Delta^2}{(\nu_\Delta+\nu)^2}\bigg|_{90}+\frac{\alpha_P^{(+)}}{2M}\,,$$

$$a_B^{(-)}=\frac{1}{4M^2}\,\frac{g_N^2}{4\pi}\,\frac{2\nu_B}{\nu_B^2-\nu^2}\bigg|_{90}+\frac{1}{4M^2}\,\frac{g_\Delta^2}{4\pi}\,\frac{2\nu_\Delta}{\nu_\Delta^2-\nu^2}\bigg|_{90}+\frac{\beta^{(-)}}{2M}\,.$$

For the nonresonating P waves we obtain

$$a_{3,1}-a_{1,3}=\frac{1}{4M^3}\left(2\frac{g_\Delta^2}{4\pi}-\frac{g_N^2}{4\pi}\right)-\frac{\alpha_L^{(-)}}{2M}=\text{const}\,,$$

$$a_{1,3}=-\frac{2}{3}\frac{1}{4M^2}\frac{g_N^2}{4\pi}\frac{1}{(\nu_B+\nu)_{90}}+\frac{4}{3}\frac{1}{4M^2}\frac{g_\Delta^2}{4\pi}\frac{1}{(\nu_\Delta+\nu)_{90}}+\frac{1}{4M^3}\frac{g_N^2}{4\pi}+\frac{1}{6M}(\alpha_P^{(+)}+2\alpha_L^{(-)}-2\beta^{(-)})\,, \tag{20}$$

$$a_{1,1}=\frac{1}{4M^2}\frac{g_N^2}{4\pi}\frac{3}{\omega_B-\omega}+\frac{1}{3}\frac{1}{4M^2}\frac{g_N^2}{4\pi}\frac{1}{(\nu_B+\nu)_{90}}+\frac{16}{3}\frac{1}{4M^2}\frac{g_\Delta^2}{4\pi}\frac{1}{(\nu_\Delta+\nu)_{90}}$$
$$+\frac{1}{4M^3}\frac{g_N^2}{4\pi}+\frac{1}{6M}(\alpha_P^{(+)}+2\alpha_L^{(-)}+4\beta^{(-)})\,.$$

In $O(\mu^2/M^2)$ three of the six S and P waves, namely $f_{0+}^{(-)}$, $a_{A'}^{(-)}$, and $a_B^{(+)}$, are still fully determined by the N and Δ poles and the low-energy theorem. In particular, the $S^{(-)}$ wave is determined beyond the linear model. The other three partial waves each acquire one background parameter. For the two P waves $a_{A'}^{(+)}$ and $a_B^{(-)}$, they describe a constant background contribution, so that the energy dependence of all four P waves is predicted without free parameter.

VIII. COMPARISON WITH EXPERIMENT

A. The crossing-odd amplitudes: Parameter-free predictions

We make parameter-free predictions for the amplitudes $A'^{(-)}$ and $B^{(+)}$. In Table I we test our predictions at threshold by comparing the S-wave scattering length $a_S^{(-)}$ and the two P-wave scattering lengths $a_{A'}^{(-)}$ $(\omega=\mu)$ and $a_B^{(+)}$ $(\omega=\mu)$ with the experimental values from Ref. 10. The agreement is good.

The correction of $O(\mu^2/M^2)$ to $a_S^{(-)}$ improves the $O(\mu/M)$ prediction. The net correction is small, although the corrections to the individual contributions of the N, Δ, and background are rather large (up to 40% of the "experimental" scattering length). The background contribution, which is fixed by the low-energy theorem, is not the same in $O(\mu/M)$ and $O(\mu^2/M^2)$ because the Δ contribution to $C^{(-)}/\nu$ at the Cheng-Dashen point is different in the two approximations.

The net correction of $O(\mu^2/M^2)$ to the P-wave scattering length $a_{A'}^{(-)}$ is practically vanishing because the corrections to the individual contributions of the N and Δ are rather small and have opposite sign and the background contribution to $a_{A'}^{(-)}$ is only 1%. The prediction for the P-wave scattering length $a_B^{(+)}$ $(\omega=\mu)$ is identical in $O(\mu/M)$ and $O(\mu^2/M^2)$.

B. The crossing-even amplitudes

Parameter-free predictions of the crossing-even amplitudes can only be made in $O(\mu/M)$. The comparison at threshold with the "experimental" scattering lengths[10] is shown in Table II. The $O(\mu/M)$ prediction for the S wave is in good agree-

TABLE I. Comparison of the parameter-free predictions for the S-wave scattering length $a_S^{(-)}$ and the two P-wave scattering lengths $a_{A'}^{(-)}$ $(\omega=\mu)$ and $a_B^{(+)}$ $(\omega=\mu)$ with the "experimental" values (Ref. 10). $a_{A'}\equiv q^{-2}(2f_{1+}+f_{1-})$, $a_B\equiv q^{-2}(f_{1-}-f_{1+})$. The background is determined by the low-energy theorem (LET). $O(\mu/M)$ denotes leading order, $O(\mu^2/M^2)$ includes next-order corrections.

		N	Δ	Background from LET ($\alpha_L^{(-)}$)	Total	Experiment (Ref. 10)
$\mu a_S^{(-)}$	$O(\mu/M)$	+0.162	−0.118	+0.048	$(4\pi)^{-1}\mu/(2f_\pi^2)$ =+0.092	+0.087±0.002
	$O(\mu^2/M^2)$	+0.141	−0.082	+0.027	+0.086	
$\mu^3 a_{A'}^{(-)}(\omega=\mu)$	$O(\mu/M)$	−0.161	−0.030	0	−0.191	−0.177±0.004
	$O(\mu^2/M^2)$	−0.152	−0.038	+0.002	−0.188	
$\mu^3 a_B^{(+)}(\omega=\mu)$	$O(\mu/M)$ $=O(\mu^2/M^2)$	−0.161	−0.030	0	−0.191	−0.190±0.004

TABLE II. Comparison of the leading-order predictions, $O(\mu/M)$, for the S-wave scattering length $a_S^{(+)}$ and the P-wave scattering lengths $a_{A'}^{(+)}$ $(\omega=\mu)$ and $a_B^{(-)}$ $(\omega=\mu)$ with the "experimental" values (Ref. 10). $O(\mu^2/M^2)$ includes next-order corrections. Brackets indicate fitted values. The error of the background contribution determined by the low-energy theorem (LET) comes from the error of σ_{NN}.

		N	Δ	Background from LET ($\alpha_L^{(+)}$)	Background (other)	Total	Experiment (Ref. 10)
$\mu a_S^{(+)}$	$O(\mu/M)$	0	+0.526	−0.526	0	$(4\pi)^{-1}\sigma_{NN}/f_\pi^2=0$	−0.005 ± 0.002
	$O(\mu^2/M^2)$	−0.010	+0.366	−0.356 ± 0.011	(−0.005 ∓ 0.011)	(−0.005 ± 0.002)	
$\mu^3 a_{A'}^{(+)}(\omega=\mu)$	$O(\mu/M)$	0	+0.133	0	0	+0.133	+0.219 ± 0.005
	$O(\mu^2/M^2)$	+0.024	+0.148	0	(+0.047 ± 0.005)	(+0.219 ± 0.005)	
$\mu^3 a_B^{(-)}(\omega=\mu)$	$O(\mu/M)$	0	+0.033	0	0	+0.033	+0.068 ± 0.003
	$O(\mu^2/M^2)$	+0.012	+0.033	0	(+0.023 ± 0.003)	(+0.068 ± 0.003)	

ment with experiment at threshold (but not at all away from threshold), which is a well known result. The predictions for the P-wave scattering lengths are off by a factor 2.

The net correction of $O(\mu^2/M^2)$ to the crossing-even S wave in the low-energy region is large. For example, the σ term, which is an $O(\mu^2/M^2)$ effect, is numerically almost equal to the crossing-odd amplitude at threshold, which is of $O(\mu/M)$:

$$C^{(+)}(\text{Cheng-Dashen point}) = \frac{\sigma_{NN}}{f_\pi^2} \approx \frac{\mu}{2f_\pi^2} \approx C^{(-)}(\text{threshold}) .$$

Neglecting $O(\mu^2/M^2)$ effects in the $S^{(+)}$ wave at low energy ($\omega \lesssim \mu$) means neglecting 100% effects (compared to the dominant $S^{(-)}$ wave). Since the scattering length is small, these large effects must cancel right at threshold.

The background term determined by the low-energy theorem ($\alpha_L^{(+)}$) is corrected by the σ term and by the very large change of the Δ contribution to $C^{(+)}$ at the Cheng-Dashen point. The background contributes to the $S^{(+)}$ wave with an additional energy-dependent term, $\alpha_S^{(+)}\omega^2$. We fix $\alpha_S^{(+)}$ using the experimental scattering length and obtain

$$\alpha_S^{(+)} = -0.006 \pm 0.013 \;\mu^{-3} \;\text{ from } a_S^{(+)} .$$

Since this background term turns out to be compatible with zero, the S wave is given numerically by the low-energy theorem and the N and Δ singularities, which contribute strongly energy-dependent correction terms.

For the P waves the $O(\mu^2/M^2)$ corrections to the N and Δ contributions are quite large (30%), but the background contributions are essential. We fix them using the experimental scattering lengths:

$$\alpha_P^{(+)}/2M = +0.047 \pm 0.005 \;\mu^{-3} \;\text{ from } a_{A'}^{(+)} ,$$

$$\beta^{(-)}/2M = +0.023 \pm 0.003 \;\mu^{-3} \;\text{ from } a_B^{(-)} .$$

C. The nonresonating partial waves above threshold

We compare the predictions for the nonresonating partial-wave amplitudes between threshold and $T = 200$ MeV with the absolute value of the experimental amplitudes.[13-16] Figure 1 shows the S waves $f_{0+}(I_s=\frac{1}{2},\frac{3}{2})$ and Fig. 2 the P waves $a_{1,3}$, $a_{3,1}$ and $a_{1,1}$, where $a_{2I,2J} \equiv q^{-2} f_{J,l=1}(I)$.

Unitarity corrections to the nonresonating amplitudes are expected to be of the magnitude

$$(|f_l| - \mathrm{Re} f_l)/|f_l| \approx \delta_l^2/2 \lesssim 4\% \quad (T \leq 200 \text{ MeV}) .$$

At $T = 200$ MeV kinematic errors in the $O(\mu^2/M^2)$ approximation can be estimated by $\omega^2/M^2 = 14\%$.

The agreement between prediction and experiment is good for the amplitudes S_{11}, P_{13}, and P_{11}, and fair for S_{31} and P_{31}.

The prediction for the S waves is shown with the uncertainty due to the error of the energy-dependent background term ($\alpha_S^{(+)}$), which reflects the uncertainty in the σ term. The predictions for the P waves is shown with the uncertainty due to the "experimental error" of the P-wave scattering lengths $a_{A'}^{(+)}$ and $a_B^{(-)}$. That background contribution which is not determined by the low-energy theorems is numerically essential for the P waves $a_{A'}^{(+)}$ and $a_B^{(-)}$, but negligible for $a_{1,3}$ and $a_{3,1}$ ($\lesssim 1\%$ of the scattering lengths). The variation with energy of the P-wave amplitudes is a parameter-free prediction, which is in good agreement with experiment. The agreement in the case of $a_{1,1}$ is somewhat surprising, since the Roper resonance is not so far away.

We did not compare our predictions for the subthreshold region with the "experimental" expansion coefficients, because such a comparison is equivalent to the experimental tests given here: The lowest coefficients either are determined by

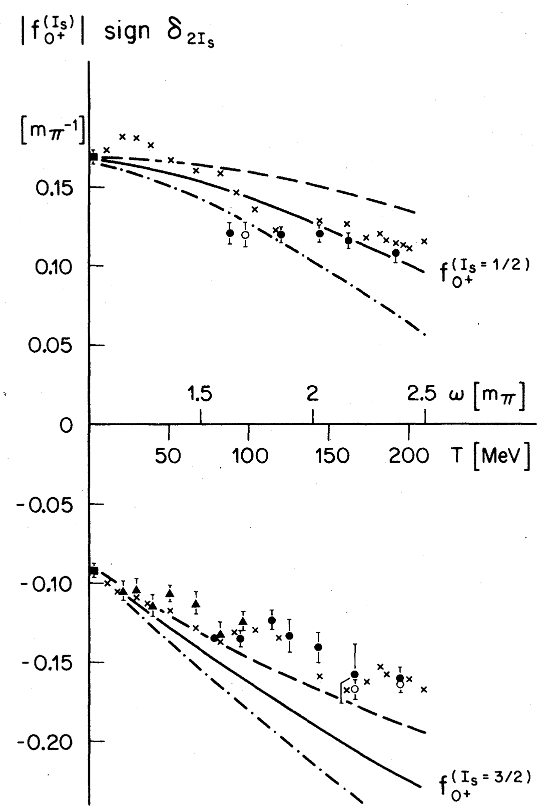

FIG. 1. Comparison of the predictions for the S-wave amplitudes with data of Ref. 13 (●), Ref. 14 (▲), Ref. 15 (○), and Ref. 16 (×). The scattering lengths are taken from Ref. 10 (■). The solid lines give the predictions with the background parameter $\alpha_S^{(+)}$ fitted in Sec. VIII B. The dashed and dot-dashed lines reflect the uncertainty of $\alpha_S^{(+)}$ coming predominantly from the error of σ_{NN}.

the low-energy theorems or fitted to "experiment" and the higher ones are practically given by the Δ contribution (see Ref. 23). The only exceptions are $(d/dt)A'^{(-)}/\nu$ and $(d/dt)B^{(-)}$. The former gives a constant contribution to the P wave $a_{A'}^{(-)}$ and an energy-dependent one to the $S^{(-)}$ wave; the corresponding tests are given in Table I. The coefficient $(d/dt)B^{(-)}$ is directly related to the energy dependence of the P wave $a_B^{(-)}$, and the corresponding tests are given in Fig. 2.

IX. COMPARISON WITH MODELS ON LOW-ENERGY πN SCATTERING

The low-energy theorems (LET's) are a consequence of the Ward identity[3-5]:

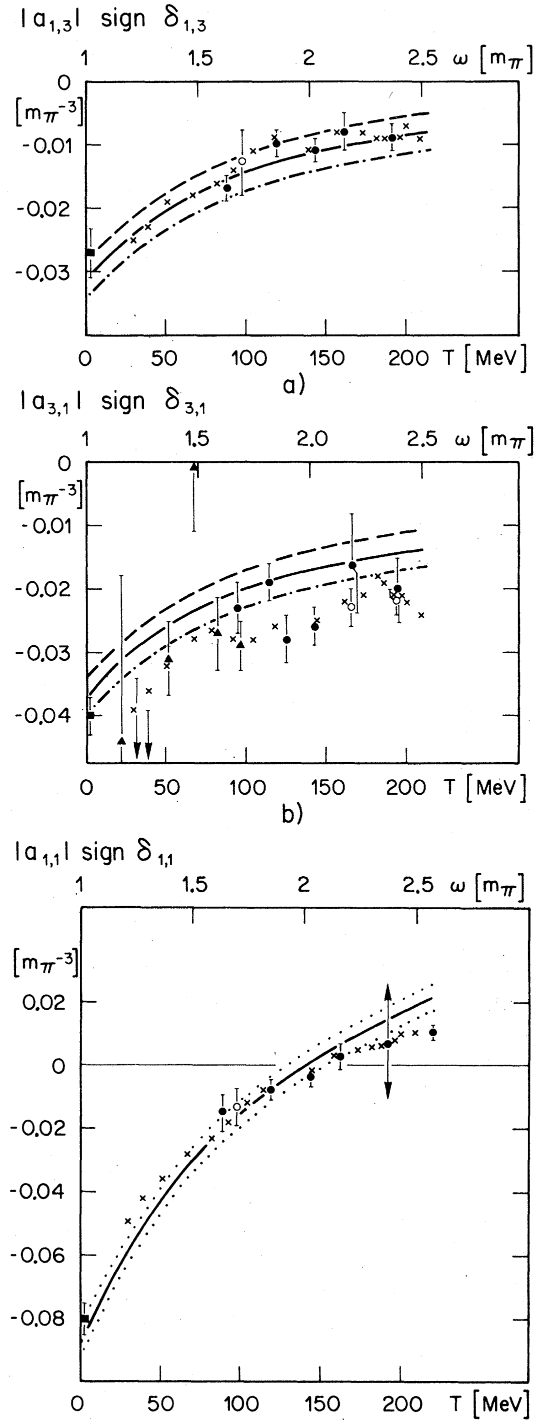

FIG. 2. Comparison of the predictions for the nonresonating P waves $a_{1,3}$, $a_{3,1}$ and $a_{1,1}$ with data of Ref. 13 (●), Ref. 14 (▲), Ref. 15 (○), and Ref. 16 (×). $a_{2I,2J} \equiv q^{-2} f_{J,l=1}^{(I)}$. The scattering lengths are taken from Ref. 10 (■). The solid lines give the predictions with the background parameters $\alpha_P^{(+)}$ and $\beta^{(-)}$ fitted in Sec. VIII B. The dashed, dot-dashed, and dotted lines reflect the errors of the P-wave scattering lengths $a_{A'}^{(+)}$ and $a_B^{(-)}$.

$$T^{ik}_{\pi N}(p,q;p',q') = \frac{1}{f_\pi^2} q'_\mu q_\nu i \int d^4z\, e^{iq'z} \langle p' | T(A^\mu_i(z) A^\nu_k(0))_N | p \rangle + \frac{1}{f_\pi^2}\left(\frac{q^2}{\mu^2} + \frac{q'^2}{\mu^2} - 1\right)\langle p' | \sigma_{ik}(0) | p\rangle + \frac{1}{2f\pi^2}(q+q')_\mu i\epsilon_{ikl}\langle p' | V^\mu_l(0) | p\rangle . \quad (21)$$

Our treatment of πN scattering at low energies is based on dispersion relations for the two invariant πN amplitudes A' and B. We use Eq. (21) only at the special point on the Mandelstam plane (Cheng-Dashen point) where the contribution of the first term on the right-hand side (after subtracting the N and a well defined Δ contribution) is suppressed by order μ^2/M^2 and numerically negligible with respect to the second and the third terms.

In contrast most models which fulfill the low-energy theorems (we know of no model for which the following is not true) use Eq. (21) in the whole low-energy region as a starting point. The assumptions and approximations which distinguish the models do not refer to the πN amplitudes but to the first term in Eq. (21), the axial-vector–nucleon amplitudes. The models separate into two classes:

(a) One class[17-19] in which the invariant $A^\nu N$-$A^\mu N$ amplitudes (after subtracting the N contribution) are calculated by fixed-t dispersion relations and are usually saturated by the Δ pole.[17,19]

(b) The other class[20-23] in which the first term on the right-hand side of Eq. (21) is calculated within field theory using effective Lagrangians. Usually N, Δ, ρ, and σ exchange is considered. In Ref. (23) an additional diffractive term is added.

All these models have the difficulty that the Δ contribution (even in the narrow-width approximation) depends on assumptions on the $\Delta N\pi$ ($\langle\Delta|A^\mu|N\rangle$) coupling and (in the field-theoretic class) on the Δ propagator. Therefore, the Δ contribution to πN scattering differs quite strongly from model to model, especially in the amplitude $B^{(-)}$ [compare discussion in Ref. (12) and the remark in Ref. (18) after Eq. (30)].

The simultaneous use of exchange terms in all three channels in the field-theoretic models raises the question of double counting. If one assumes that the ρ pole is much closer than the higher s- and u-channel resonances, double counting is avoided in $A'^{(\pm)}$ because of the LET.[24] For the $B^{(-)}$ amplitude analogous statements cannot be made.[12]

In most models the amplitudes are approximated just by those contributions which the authors expect to be sizable. In contrast to that, our amplitudes are exact in $O(\mu^2/M^2)$: The contribution of the higher resonances, the nonresonating contribution, and the contour integral in Eq. (7) are completely parametrized by the μ/M expansion. For example, we obtain in a natural way a $c\nu^2$ term in $A'^{(+)}$, which is numerically important but which is missing in most other models.

To $O(\mu^2/M^2)$ the crossing-odd amplitudes $A'^{(-)}$ and $B^{(+)}$ of all models must be identical to ours. Models can only differ with respect to the three background parameters in the crossing-even amplitudes $A'^{(+)}$ and $B^{(-)}$. Either they are also free parameters which must be fitted to experiment, or corresponding terms are missing, or the values of some parameters are determined theoretically from fundamental entities.

The last and physically most interesting case occurs only in $B^{(-)}$ in some models,[19,23] while $B^{(-)}$ in other models[21,22] contains a free parameter corresponding to our $\beta^{(-)}$. In the former models $B^{(-)}$ is approximated just by the Δ contribution and the current-commutator term $G^V_M/2f_\pi^2$. But as shown by Ref. 5, the LET make no constraints on $B^{(-)}$, so that these models are based on the (*ad hoc*) assumption, that the axial-vector–nucleon amplitude can be approximated by the N and Δ contribution alone.

An additional difficulty for predicting $B^{(-)}$ is the fact that this amplitude depends very sensitively on the assumptions on the Δ-nonpole contributions (e.g., the values of the parameters Z, c). [While in Refs. 19 and 23 the Δ nonpole contributes to $B^{(-)}$ only 12% of $G^V_M/2f_\pi^2$, its contribution is in Ref. 12 about as large as the current-commutator term, so that an equally large background ($\cong\beta^{(-)}$) is needed to cancel it and get agreement with experiment.] The fit of the Δ parameter Z in Ref. 23 is mostly influenced by $B^{(-)}$ and corresponds to our determination of $\beta^{(-)}$.

Reference 19 contains no free parameter for the Δ, but they make assumptions by ignoring some terms in the matrix element $\langle\Delta|A^\mu|N\rangle$. Concerning the approximation of the $A^\nu N$-$A^\mu N$ amplitude by the N and certain Δ contributions alone, they refer to Ref. 18 where this assumption is numerically tested. This is equivalent to a fit of a discrete parameter.

The conclusion for $B^{(-)}$ is that no parameter-free prediction (in the sense discussed above) can be made from the N and Δ poles and the LET.

Our amplitude $A'^{(+)}$ contains two free parameters. Comparing it with those of other models we note that the corresponding terms are either missing or also contain free parameters.

A contribution to $A'^{(+)}$ corresponding to our background ct appears (with a free parameter) in most of the models.[19,21-23] In Ref. 23 this term is interpreted as the t dependence of the σ term and of the σ exchange and is determined by the fit.

A background term of the form $c\nu^2$ in $A'^{(+)}$ appears in Ref. 23, where it also contains a free parameter and is labeled "diffractive." Most other models[17,19,21,22] do not contain such a ν^2 term; the predictions for the S-wave scattering length of Refs. 19 and 21 are bad, while the prediction of Ref. 22 agrees with experiment. This agreement can be explained by the much larger Δ contribution of Ref. 22.

In higher order than $O(\mu^2/M^2)$ other models show the most important difference in $A'^{(-)}$ having a t-dependent background term, which is assumed to be dominated by the t dependence of the current-commutator term. It corresponds to an 8% effect in the S-wave scattering length $a_S^{(-)}$ and a 6% correction of the P-wave scattering length $a_{A'}^{(-)}$. Such a term would improve our prediction for $a_{A'}^{(-)}$ but spoil the one for $a_S^{(-)}$. This shows most clearly the limitations of the μ/M expansion to second order. [In our approach such a term could consistently be introduced as an $O(\mu^2/M^2)$ effect, if one assumes $m_\rho{}^2 = MO(\mu)$.] A much less important difference of higher order is the t dependence of the Δ-nonpole term which induces a correction to $a_{A'}^{(-)}$ up to 2%, depending on the assumptions.

In $B^{(+)}$ other models show very small differences from our amplitude, necessarily of higher order. $B^{(+)}$ in all models listed above is approximated by the N and Δ contribution alone, except for the one of Ref. 23, which contains a diffractive term, contributing 1% to the P-wave scattering length $a_B^{(+)}$. The N contribution is the same in all models, the Δ contribution in the field-theoretic models has a negligible nonpole term of $\frac{1}{2}$% of $a_B^{(+)}$.

This comparison with other models shows the following:

(i) Up to $O(\mu^2/M^2)$ all models contain the same number of free parameters (for $B^{(-)}$ in the sense discussed above) or the corresponding terms are missing. In contrast to other models we used no other dynamical input than the N and Δ poles and the LET and made no other assumptions on the background than Eqs. (1a) and (1b).

(ii) The limitation of the μ/M expansion to second order, that means the specific limitation of our approach, shows up most clearly in two points: (a) The t dependence of the current-commutator term in $A'^{(-)}$ is numerically not negligible (8% of $a_S^{(-)}$). (b) The μ/M expansion of the purely kinematical factors induces errors which we expect to be of order 15%. This is the price we pay for writing explicit and relatively "simple" formulas for the scattering amplitudes. Of course, the kinematics could be calculated exactly; the algebraic effort would just be much larger.

The limitation of all these models comes from neglecting higher-order background terms and unitarity-cusp effects. The latter are most important in the S waves (~10%) and much smaller in the P waves.

At the present time the uncertainties in the "experimental" amplitudes (from amplitude analysis) are also of order 10%, and the models are in fairly good agreement with "experiment". At some future time, when the "experimental" amplitudes will be known to much better accuracy (say 3%), all these models will fail in a detailed comparison with "experiment". Still they will be valuable as a simple explanation of the main features. Incorporating unitarity and higher-order background terms would get us farther and farther away from a simple model determined by a few basic parameters (e.g., g_N, g_Δ, f_π and σ_{NN}) and closer to a precise fit to the experimental data at all energies (amplitude analysis).

It is very difficult to compare the different models by their success in predicting "experimental" numbers, since all authors use different values for the fundamental parameters (e.g., f_π, g_Δ) and fit their free parameters to different experiments. In our opinion, our approach makes predictions of the same quality as other models (if compared with the number of free parameters) with less dynamical input and approximations and and—given the result of the Appendix—with less effort.

X. CONCLUSIONS

We have investigated to what extent the low-energy theorems and the N and Δ poles determine low-energy πN scattering. The low-energy theorems determine only the amplitudes $A'^{(\pm)}$ at the Cheng-Dashen point and only up to unknown corrections which are suppressed by order μ^2/M^2 with respect to the current-commutator terms.

We have expanded the πN amplitudes in μ/M to second order, keeping ν/μ and t/μ^2 fixed and assuming

$$M_\Delta - M = O(\mu)$$

and

$$M_{\rm res} - M = O(M)$$

for all other resonances.

We have calculated the invariant amplitudes A' and B using Cauchy integrals in the complex ν plane at fixed t. This induces a well-defined decomposition of the amplitudes in N, Δ, and background contribution. In the Appendix we have shown that the branch-point singularities of A' and $\mu^2 B$ at $t=4\mu^2$ and $|\nu|=\mu+t/4M$ are of $O(\mu^3/M^3)$ (up to logarithms) and that the background contribution in $O(\mu^2/M^2)$ can be described by five parameters, two of which are determined by the low-energy theorems. The remaining three parameters only contribute to the crossing-even amplitudes.

By using the constraints of the low-energy theorems, by incorporating the contribution of higher resonances and nonresonating background, and by systematically expanding in μ/M we have gone beyond the dispersive model of Chew, Goldberger, Low, and Nambu.[25]

The amplitudes in leading order, $O(\mu/M)$, are fully determined by the low-energy theorems and the N and Δ singularities. In particular, the S waves are given by the low-energy theorems and reproduce Weinberg's scattering lengths, and the P waves are given by the N and Δ poles with static kinematics. Numerically the $O(\mu/M)$ predictions for the crossing-even amplitudes badly disagree with experiment (see Sec. VIII B), except for the $S^{(+)}$ wave right at threshold.

In next order, $O(\mu^2/M^2)$, the crossing-odd amplitudes are still fully determined by the low-energy theorems and the N and Δ poles; the parameter-free predictions agree well with experiment. On the other hand, the crossing-even amplitudes get undetermined background contributions parametrized by three constants. We have fixed them using three "experimental" scattering lengths. With the three background parameters fixed at threshold, we get predictions for the amplitudes above threshold which are in fairly good agreement with experiment.

A comparison with models on low-energy πN scattering has shown the following:

(i) Up to $O(\mu^2/M^2)$ differences can occur only with respect to the three background parameters in the crossing-even amplitudes. In all models $A'^{(+)}$ either also contains two unknown parameters or the corresponding terms are just missing. In some models $B^{(-)}$ also contains a free background parameter. In other models it is approximated just by the N, Δ, and current-commutator terms, an approximation which cannot be justified theoretically (there exists no low-energy theorem for $B^{(-)}$).

(ii) In higher order than $O(\mu^2/M^2)$ the main difference, apart from kinematical effects, is a t-dependent background in $A'^{(-)}$. In all models it is assumed to be dominated by the t dependence of the current-commutator term, giving an 8% (6%) correction to $a_S^{(-)}$ $(a_A^{(-)})$.

(iii) Although it is very difficult to compare different models by their success in predicting "experimental" data, we think that our approach makes predictions of the same quality as other models with less dynamical input and approximations and with less effort.

(iv) All models neglect unitarity-cusp effects, which are more important in the S wave (~10%) than in the P waves.

We did not go beyond $O(\mu^2/M^2)$ because of the following reasons:

(a) The on-shell continuation of the low-energy theorems involves additional and unknown corrections: There are no low-energy theorems in higher order.

(b) Unitarity effects must be considered: The amplitudes have threshold singularities, and the background can no longer be expanded in ν, t beyond threshold. The zero-width approximation for the Δ resonance is no longer valid.

(c) The partial-wave expansion can no longer be truncated after a few terms.

Parametrizing the background to higher order in μ/M than $O(\mu^2/M^2)$ one gets farther away from a simple and predictive model and closer to a fit to the experimental data (amplitude analysis).

APPENDIX: THE ORDER OF THE BACKGROUND TERMS IN THE μ/M EXPANSION

The background contribution to the invariant amplitudes (i.e., the contribution of the higher resonances, the nonresonating background, and the contour integral to a Cauchy integral in the complex ν plane at fixed t) has branch-point singularities at $t=4\mu^2$ and $t=4M(\pm\nu-\mu)$.

$$F(\nu,t)|_{\mathrm{BG}}=\sum_{n,m} c_{nm}\nu^{2n}t^m\ , \tag{A1}$$

where F stands for $A'^{(+)}$, $A'^{(-)}/\nu$, $B^{(-)}$, and $B^{(+)}/\nu$, converges for $|t|<4\mu^2$ and $\nu^2<\mu^2(1-\mu/M)^2$. This implies that most of the coefficients diverge in μ as $\mu\to 0$. In this appendix we answer the following questions: Which coefficients are regular in μ as $\mu\to 0$ and what is the order in μ/M of those coefficients that diverge for $\mu\to 0$?

We estimate the order of the coefficients using Cauchy integrals in the complex t plane at fixed ν (as well as Cauchy integrals in the complex ν plane at fixed t) with the N and Δ contribution subtracted. (We choose the radius of the contour integral such that no other resonance contributes to the integral along the cut.) Since the contribution of the contour integral at $|t|\sim M^2$ $(|\nu|\sim M)$ does not diverge in μ as $\mu\to 0$, the coefficients are given by truncated fixed-ν (fixed-t) dispersion relations up to corrections which are regular in μ as $\mu \to 0$.

PCAC permits the approximate calculation of the discontinuities of the invariant amplitudes by two-particle unitarity,[26] i.e., by elastic unitarity in the s and u channel and with $\pi\pi$ intermediate states in the t channel.

1. Expansion in t at fixed ν

We consider first the series

$$F(\nu, t)|_{\mathrm{BG}} = C_0(\nu^2) + C_1(\nu^2)t + \cdots + C_n(\nu^2)t^n + \cdots, \quad \text{(A2)}$$

whose coefficients are given by truncated fixed-ν dispersion relations.

To estimate the discontinuity across the right-hand cut we approximate on the right-hand side of the t-channel unitarity equation the $N\bar{N}$-$\pi\pi$ amplitude by the nucleon Born term and the $\pi\pi$-$\pi\pi$ amplitude by a linear low-energy amplitude[2] plus a typical correction (which contains a D wave)

$$T_{\pi\pi-\pi\pi} = O(1)(\nu/M)^2 \ln[(4\mu^2 + t)/(2M^2 + t)] .$$

Such a correction is obtained from a simple-minded unitarization of the linear $\pi\pi$-$\pi\pi$ amplitude. Considering a D-wave contribution to $\pi\pi$-$\pi\pi$ on the right-hand side of the unitarity equation for $N\bar{N}$-$\pi\pi$ is necessary since S and P waves do not contribute to $B^{(+)}$. The contribution of this correction term to the expansion coefficients of $A'^{(+)}$ is less singular in μ for $\mu \to 0$ than the contribution from the linear $\pi\pi$-$\pi\pi$ amplitude.

The discontinuity across the left-hand cut of $F(\nu, t)$ in the complex t plane is given by the s- and u-channel discontinuities. We estimate them by approximating the πN-πN amplitudes on the right-hand side of the s- (u-) channel unitarity equation by the nucleon Born term alone.

The resulting order of the terms in (A2) is shown in part (a) of Table III. The contribution of the background to the invariant amplitudes can be described up to order (μ^2/M^2) by

$$\begin{aligned} MA'^{(+)}|_{\mathrm{BG}} &= a_0^{(+)}(\nu^2) + a_1^{(+)}(\nu^2)(t/\mu^2) + O(\mu^3/M^3), \\ \mu MA^{(-)}/\nu|_{\mathrm{BG}} &= a_0^{(-)}(\nu^2) + O(\mu^3/M^3)\ln(\mu/M), \\ \mu^2 B^{(-)}|_{\mathrm{BG}} &= b_0^{(-)}(\nu^2) + O(\mu^3/M^3) , \\ \mu^3 B^{(+)}/\nu|_{\mathrm{BG}} &= O(\mu^3/M^3)\ln(\mu/M) . \end{aligned} \quad \text{(A3)}$$

As a second result of this estimate of the t-channel discontinuity we get the order of the πN-πN partial-wave amplitudes in the μ/M expansion ($l \neq 0$): $f_{l\pm} = M^{-1}O(\mu^3/M^3)$ + contributions from N and Δ poles.

2. Expansion in ν^2 at $t = 0$

We expand the coefficients in (A3) in power series in ν^2, i.e., we consider the expansion in ν^2 of $MA'^{(+)}|_{\mathrm{BG}}$, $\mu^2 M(d/dt)A'^{(+)}|_{\mathrm{BG}}$, $\mu MA'^{(-)}/\nu|_{\mathrm{BG}}$, and $\mu^2 B^{(-)}|_{\mathrm{BG}}$ at $t = 0$.

To estimate the imaginary part of these amplitudes by elastic unitarity it is sufficient to consider S and P waves only. This follows from the order of the partial-wave amplitudes

$$\begin{aligned} f_{0+}, f_{1\pm} &= M^{-1}O(\mu/M), \\ f_{2\pm} &= M^{-1}O(\mu^2/M^2), \\ f_{l\pm} &= M^{-1}O(\mu^3/M^3), \quad l \geq 3. \end{aligned} \quad \text{(A4)}$$

TABLE III. Order in μ/M of the background contribution to the invariant amplitudes. (a) Expansion in t at fixed ν. (b) Expansion in ν^2 at $t = 0$. The terms to the right of the line can be neglected in our approximation. See Appendix for details.

(a)

	$C_0(\nu^2)$	$C_1(\nu^2)t$	$C_n(\nu^2)t^n$, $n \geq 2$
$MA'^{(+)}$	$O(1)$	$O(\mu^2/M^2)$	$O(\mu^3/M^3)$
$\mu MA'^{(-)}/\nu$	$O(\mu/M)$	$O(\mu^3/M^3)\ln(\mu/M)$	$O(\mu^3/M^3)$
$\mu^2 B^{(-)}$	$O(\mu^2/M^2)$	$O(\mu^3/M^3)$	$O(\mu^3/M^3)$
$\mu^3 B^{(+)}/\nu$	$O(\mu^3/M^3)\ln(\mu/M)$	$O(\mu^4/M^4)$	$O(\mu^5/M^5)\ln(\mu/M)$

(b)

	c_0	$c_1\nu^2$	$c_n\nu^{2n}$, $n \geq 2$
$MA'^{(+)}$	$O(1)$	$O(\mu^2/M^2)$	$O(\mu^3/M^3)$
$\mu^2 M(d/dt)A'^{(+)}$	$O(\mu^2/M^2)$	$O(\mu^3/M^3)$	$O(\mu^3/M^3)$
$\mu MA'^{(-)}/\nu$	$O(\mu/M)$	$O(\mu^3/M^3)\ln(\mu/M)$	$O(\mu^3/M^3)$
$\mu^2 B^{(-)}$	$O(\mu^2/M^2)$	$O(\mu^3/M^3)$	$O(\mu^3/M^3)$

We make the following approximations on the right-hand side of the unitarity equation: (a) The S waves are given by the low-energy amplitudes [Eqs. (11) and (12)]; and (b) The P waves are estimated by the nucleon Born term alone [see Eq. (13)], since the crossed-channel Δ gives contributions of the same magnitude, and since the direct-channel Δ is absent in the background.

The resulting order of the terms in the expansion in ν^2 of $MA'^{(+)}|_{\mathrm{BG}}$, $\mu^2 M(d/dt)A'^{(+)}|_{\mathrm{BG}}$, $\mu MA'^{(-)}/\nu|_{\mathrm{BG}}$, and $\mu^2 B^{(-)}|_{\mathrm{BG}}$ are given in part (b) of Table III. The contribution of the background to the invariant amplitudes can be described by

$$\begin{aligned} MA'^{(+)}|_{\mathrm{BG}} &= \alpha_0^{(+)} + \alpha_1^{(+)}(\nu/M)^2 + \alpha_2^{(+)} t/M^2 + O(\mu^3/M^3)\,, \\ MA'^{(-)}|_{\mathrm{BG}} &= \alpha_0^{(-)}\nu + O(\mu^3/M^3)\ln(\mu/M)\,, \\ \mu^2 B^{(-)}|_{\mathrm{BG}} &= \beta_0^{(-)} + O(\mu^3/M^3)\,, \\ \mu^2 B^{(+)}|_{\mathrm{BG}} &= O(\mu^3/M^3)\ln(\mu/M)\,, \end{aligned} \tag{A5}$$

with $\alpha_i^{(\pm)} = O(1)$ and $\beta_0^{(-)} = O(\mu^2/M^2)$. The singular background contributions (branch points at $t = 4\mu^2$, etc.) are of order (μ^3/M^3), apart from logarithms. In the paper we have calculated the amplitudes (MA') and $(\mu^2 B)$ including order (μ^2/M^2). If one wants to go beyond (μ^2/M^2), one cannot expand the background in ν and t near and beyond the branch points.

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