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Approximate dynamical symmetry in lattice quantum chromodynamics

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We discuss the phenomenological implications of an approximate SU(6)×SU(6)×U(1) symmetry of hadron physics which remains after dynamical symmetry breaking in the strong-coupling lattice gauge theory. This symmetry is similar to but differs in an essential fashion from previous versions of SU(6)×SU(6) or $SU(6)_W$. The difference resolves some of the problems of the older schemes—for example, although we obtain the "good" result $\mu_p/\mu_n=-3/2$, we avoid the "bad" result $g_A/g_V=-5/3$. We find that mesons are better approximated as irreducible representations of an $SU(6)_W$ than static SU(6). Vector mesons are pseudo-Goldstone bosons in our scheme, which explains why the sum rules for their masses should be written in terms of mass squared, like those of the pseudoscalars.

I. INTRODUCTION

Recently we showed that spontaneous breaking of continuous chiral symmetries and the associated massless Goldstone bosons arise naturally within the context of a confining lattice gauge theory.[1] In particular, we concluded that the usual chiral symmetry of quantum chromodynamics (QCD) with three flavors of massless quarks must, in the strong-coupling regime, break spontaneously so that only the SU(3) symmetry of the vector charges is realized in the "normal" or Wigner mode. A renormalization-group argument was offered to make the strong-coupling calculation relevant to the hadronic regime. This result provides the theoretical basis for understanding, within the framework of QCD, the success of predictions based upon the joint assumptions of current algebra and partial conservation of axial-vector current[2]: relations such as the Adler-Weisberger g_A/g_V sum rule, the Goldberger-Treiman relation, and the Adler self-consistency conditions. This paper is devoted to further discussion of features of hadron phenomenology which emerge from this same analysis.

The focus of our earlier paper (hereafter denoted as paper I) was on the dynamical origins of spontaneous symmetry breaking and on the iterative block-spin techniques used to analyze this phenomenon. As discussed there, the renormalization of the effective Hamiltonian as the lattice spacing is increased is conjectured to take us into the strong-coupling regime when the lattice spacing becomes of the order of a hadron radius. In this regime, states containing gluon excitations become energetically expensive and hence we neglected them in our study of the low-lying spectrum. Thus we considered only color-singlet fermion configurations at each site of the effective lattice on this distance scale. We derived an effective Hamiltonian for this sector which was found to have several properties important for the discussion in this paper:

(1) The effective theory is that of a quantum spin system with interactions occurring only between sites separated along any one lattice direction; the interactions fall off rapidly, with the cube of the separation of the two sites.

(2) For QCD with three flavors of quarks we can identify an SU(12) of charges defined on the lattice which commute with the part of the Hamiltonian describing just the interactions between nearest-neighbor sites. In terms of the local densities of these charges the nearest-neighbor Hamiltonian is antiferromagnetic in character. We find that the ground state is not invariant under the full symmetry group, and infer that chiral symmetry is spontaneously broken.

(3) These SU(12) charges also commute with all those terms in the Hamiltonian which involve sites separated by an *odd* number of links—these terms reinforce the antiferromagnetic pattern. We will write the Hamiltonian as $H^{\rm eff}=H_0^{\rm eff}+V^{\rm eff}$, where $H_0^{\rm eff}$ is the largest piece of $H^{\rm eff}$ which commutes with the SU(12) and $V^{\rm eff}$ is the remaining part of $H^{\rm eff}$.

(4) The effects of including $V^{\rm eff}$ may be understood via perturbative analysis. In particular, of the 72 Goldstone bosons which appear in the spectrum of $H_0^{\rm eff}$, 63 acquire masses, leaving massless only the nonet originating in spontaneous breaking of the U(3)× U(3), under which $V^{\rm eff}$ as well as $H_0^{\rm eff}$ is invariant. (We have speculated in I on the fate of the ninth of these particles, which is the old U(1) problem—we will not discuss it further.) We note that the situation here is quite different from that in free-fermion field theory, where a similar division of $H=H_0+V$ can be made, again based on the SU(12) invariance of nearest-neighbor terms. However, in the free-

field case the breaking term can in no way be regarded as a perturbation. We will discuss this contrast in more detail later.

In Sec. II we will construct the SU(12) of charges and give explicitly the division of the strong-coupling lattice Hamiltonian into $H_0^{\rm eff}+V^{\rm eff}$. We will also discuss how these charges differ from the algebra of current components introduced in 1965 by Bardakci *et al.*,[3] and discussed by Dashen and Gell-Mann,[4] as a relativistic generalization of the SU(6) scheme of Bég and Pais.[5] (In particular, none of our charges are integrals over spatial components of the currents.) We will then review the results of paper I as applied to the study of $H_0^{\rm eff}$ as well as $H_0^{\rm eff}+V^{\rm eff}$.

In Sec. III we will discuss the applications of this symmetry structure, with the focus on the differences with previous SU(6) or $SU(6)_W$ treatments[6] that arise because of the different form of the generators of our algebra. In particular, while we obtain successful predictions of the earlier studies—for example, $\mu_p/\mu_n=-\frac{3}{2}$ for the proton-to-neutron moment ratio—we avoid the bad result $g_A/g_V=-\frac{5}{3}$ for the ratio of the axial-vector to vector charges. We also find that mesons are better approximated as irreducible representations of an $SU(6)_W$ than the static SU(6). Finally, our analysis shows why sum rules for the masses of the vector as well as the pseudoscalar mesons are best expressed in terms of mass squared—all these mesons are pseudo-Goldstone bosons[7] in our picture.

II. SYMMETRIES–FAMILIAR AND OTHERWISE

A. Basic formalism

The starting point of our discussion is the lattice QCD Hamiltonian[1]

$$H_{\rm QCD}=H_{\rm GF}+\Lambda\sum_{\vec{\jmath},n,\hat{\mu}}\left\{-i\delta'(n)\psi^\dagger_{\vec{\jmath}}\alpha_{\hat{\mu}}\left[\prod_{l=0}^{n-1}U(\vec{\jmath}+l\hat{\mu},\hat{\mu})\right]\times\psi_{\vec{\jmath}+n\hat{\mu}}\right\},\tag{2.1}$$

where $H_{\rm GF}$ stands for the pure gauge-field part of the Hamiltonian; Λ is the lattice cutoff. $\psi_{\vec{\jmath}}$ is a fermion field carrying color, Dirac, and flavor indices. The vector $\vec{\jmath}$ runs over the sites of a three-dimensional spatial lattice, $\hat{\mu}$ runs over the unit vectors in the x, y, and z directions, $\alpha_{\hat{\mu}}$ stands for the corresponding Dirac matrices, and $U(\vec{\jmath},\hat{\mu})$ is the gauge-field operator associated with the link joining the sites $\vec{\jmath}$ and $\vec{\jmath}+\hat{\mu}$. The $U(\vec{\jmath},\hat{\mu})$ are 3×3 matrices of operators and act on the color indices of ψ. Finally, the function $\delta'(n)$ is

$$\delta'(n)=-(-1)^n/n\tag{2.2}$$

and is introduced, as discussed elsewhere,[8] in order to allow the treatment of fermion theories with continuous chiral symmetries.[9] Dirac, color, and flavor indices will be suppressed where no confusion results.

$H_{\rm QCD}$ possesses the full chiral U(3)× U(3) symmetry of the continuum theory with three flavors of massless quarks. The lattice charges which generate these symmetries are given by

$$Q_a^\pm=\sum_{\vec{\jmath}}\psi^\dagger_{\vec{\jmath}}M_a^\pm\psi_{\vec{\jmath}}\,.\tag{2.3}$$

The matrix $M_a^\pm$ is one of the 18 matrices

$$M_a^\pm=\frac{1\pm\gamma_5}{2}\lambda_a\,,$$

where γ_5 is the usual Dirac matrix, and the λ_a's are the nine 3×3 Hermitian generators of U(3). We will now show that half of the fermion terms in (2.1) commute with the larger symmetry group U(12). The extra charges differ from the U(3) × U(3) generators (2.3) in a crucial way: They *cannot* be identified with continuum expressions of the form $\int d^3x\,\psi(x)M\psi(x)$. This point will be discussed later in more detail.

Here we will first show how the additional charges arise by considering the nearest-neighbor terms in the fermionic part of (2.1)

$$H_0'=\sum_{\vec{\jmath}\hat{\mu}}-i\psi^\dagger_{\vec{\jmath}}\alpha_{\hat{\mu}}U(\vec{\jmath},\hat{\mu})\psi_{\vec{\jmath}+\hat{\mu}}\delta'(1)\,.\tag{2.4}$$

In addition to the U(3)× U(3) charges (2.3) there are more general operators of the form

$$Q^\lambda=\sum_{\vec{\jmath}}\psi^\dagger_{\vec{\jmath}}M^\lambda(\vec{\jmath})\psi_{\vec{\jmath}}\tag{2.5}$$

which commute with H_0'. Here $M^\lambda(\vec{\jmath})$ stands for a $\vec{\jmath}$-dependent 12×12 Hermitian matrix, acting on the Dirac and flavor indices carried by $\psi_{\vec{\jmath}}$. Q^λ commutes with H_0' if the matrices $M^\lambda(\vec{\jmath})$ are chosen to satisfy

$$M^\lambda(\vec{\jmath}+\hat{\mu})=\alpha_\mu M^\lambda(\vec{\jmath})\alpha_\mu\,.\tag{2.6}$$

The solution to (2.6) is

$$M^\lambda(\vec{\jmath})=\alpha_x^{j_x}\alpha_y^{j_y}\alpha_z^{j_z}M^\lambda\alpha_z^{j_z}\alpha_y^{j_y}\alpha_x^{j_x}\tag{2.7}$$

for any 12×12 Hermitian matrix M^λ. It is easy to see that the operators (2.5) now form the Lie algebra of U(12).

It is convenient to choose a basis $Q^{\alpha a}$ for this set of charges, where α runs from 0 to 15 and a runs from 0 to 8. The 144 charges $Q^{\alpha a}$ are defined via (2.5) and (2.7) by inserting $M^\lambda=M^{\alpha a}$, with $M^{\alpha a}$ defined as tensor products

$$M^{\alpha a}=\Gamma_\alpha\otimes\lambda_a\,,\tag{2.8}$$

where Γ_α are the 4×4 Dirac matrices and λ_a are the 3×3 generators of U(3). [Our convention will be that $\Gamma_0 = I$ and $\lambda_0 = I(\frac{2}{3})^{1/2}$.] The 144 charges

$$Q^{\alpha a} = \sum_{\vec{j}} \psi^\dagger_{\vec{j}} \alpha_x^{j_x} \alpha_y^{j_y} \alpha_z^{j_z} M^{\alpha a} \alpha_z^{j_z} \alpha_y^{j_y} \alpha_x^{j_x} \psi_{\vec{j}} \tag{2.9}$$

then *generate* the algebra of U(12). An alternative way of writing $Q^{\alpha a}$ is

$$Q^{\alpha a} = \sum_{\vec{j}} Q^{\alpha a}_{\vec{j}} = \sum_{\vec{j}} \psi^\dagger_{\vec{j}} M^{\alpha a} \psi_{\vec{j}} s_{\alpha a}(\vec{j}) , \tag{2.10}$$

where the sign $s_{\alpha a}(\vec{j})$ is defined by

$$\alpha_x^{j_x} \alpha_y^{j_y} \alpha_z^{j_z} M^{\alpha a} \alpha_z^{j_z} \alpha_y^{j_y} \alpha_x^{j_x} = M^{\alpha a} s_{\alpha a}(\vec{j}) . \tag{2.11}$$

It is now a simple and straightforward exercise to show that the U(12) of charges in (2.10) commutes with the larger piece H_0 of H_{QCD} which includes all the terms in the fermionic Hamiltonian which involve separation of ψ and $\psi^\dagger$ by an odd number of links, as well as H_{GF}. The remaning V is simply the sum over even values of n; every term of V has a smaller coefficient than the corresponding term in H_0.

In summary, the only charges of the form (2.5) which commute with $H_{\text{QCD}} = H_0 + V$ are those corresponding to matrices $M^{\alpha a}$ with $s_{\alpha a}(\vec{j}) = 1$, i.e., which commute with all α_μ's. These are, of course, nothing but the generators of ordinary chiral $\text{U}(3)\times\text{U}(3)$ which are associated with the matrices $M^{0a} = \underline{1}\otimes\lambda_a$ and $M^{1a} = \gamma_5\otimes\lambda_a$.

B. A reprise of results in paper I

In the strong-coupling region, states involving flux on any link have a large energy, proportional to g^2. We derived in I an effective Hamiltonian for the flux-free sector of states, which is obtained by doing second-order degenerate perturbation theory in the fermionic terms in H_{QCD}. This Hamiltonian has the form

$$H^{\text{eff}} = \frac{\tilde{\Lambda}}{g^2} \sum_{\vec{j} n \hat{\mu}} \frac{1}{n^3} \sum_{\alpha a} (\psi^\dagger_{\vec{j}} M^{\alpha a} \psi_{\vec{j}})(\psi^\dagger_{\vec{j}+n\hat{\mu}} M^{\alpha a} \psi_{\vec{j}+n\hat{\mu}}) s_{\alpha a}(\hat{\mu}) \tag{2.12}$$

and is applied for an effective lattice spacing $R_H = 1/\tilde{\Lambda}$ that corresponds to a distance on the order of a typical hadron radius. The trivial color dependence has been suppressed in (2.12) for notational simplicity. In the notation of (2.10),

$$H^{\text{eff}} = \frac{\tilde{\Lambda}}{g^2} \sum \frac{1}{n^3} Q^{\alpha a}_{\vec{j}} Q^{\alpha a}_{\vec{j}+n\hat{\mu}} s_{\alpha a}((n+1)\hat{\mu}) \tag{2.13}$$

and the effective Hamiltonian is antiferromagnetic in character.

We can divide H^{eff} into two terms just as we did for the original H_{QCD} from which it is constructed:

$$H^{\text{eff}} = H_0^{\text{eff}} + V^{\text{eff}} , \tag{2.14}$$

where H_0^{eff} commutes with all the 143 charges $Q^{\alpha a}$ forming the algebra of SU(12).[10] Again, H_0^{eff} couples all sites separated by an odd number of lattice links,[11] and V^{eff} contains the remaining (symmetry-breaking) terms involving lattice separations by even numbers of links.

In studying H_0^{eff} we found that the SU(12) symmetry is spontaneously broken: An $\text{SU}(6)\times\text{SU}(6)\times\text{U}(1)$ subalgebra of charges which commute with a flavor-invariant quark mass term is realized in the normal fashion, leading to Wigner multiplets, but the remaining 72 charges are realized in a Nambu-Goldstone mode—in acting on the $\text{SU}(6)\times\text{SU}(6)$-symmetric vacuum their densities create massless particles. The $\text{SU}(6)\times\text{SU}(6)\times\text{U}(1)$ Wigner symmetry is generated by the 71 $Q^{\alpha a}$'s associated with

$$M^{\alpha a} = \underline{1}\otimes\lambda_a , \quad \gamma_0\otimes\lambda_a , \quad \vec{\sigma}\otimes\lambda_a , \quad \gamma_0\vec{\sigma}\otimes\lambda_a , \tag{2.15}$$

with $\underline{1}\otimes\lambda_0 = \underline{1}\otimes\underline{1}$ excluded.

In studying both H^{eff} and H_0^{eff} using iterative block-spin methods we found that the effects of the inclusion of V^{eff} in H^{eff} can be understood well if V^{eff} is regarded as a relatively small symmetry-breaking correction to H_0^{eff}. When V^{eff} is added to H_0^{eff}, the only subalgebra of the $\text{SU}(6)\times\text{SU}(6)\times\text{U}(1)$ of the Wigner-realized charges which survives as a good symmetry is the SU(3) generated by the charges $Q^{0a} = \underline{1}\otimes\lambda_a$; the only Goldstone charges which stay conserved when the effects of V^{eff} are taken into account are the usual axial-vector charges $Q^{1a} = \gamma_5\otimes\lambda_a$. Hence of the 72 particles which are massless in the theory defined by H_0^{eff} alone, 63 particles acquire masses due to V^{eff}, leaving 8 Goldstone bosons to be identified with the π, K, and η mesons, plus a ninth meson that we could only conjecture as being "seized."[12] Since SU(3) remains as a good Wigner symmetry it will be useful to classify the would-be Goldstone bosons with respect to their SU(3) and angular momentum properties, and we will do this in the first part of Sec. III.

The symmetry-breaking effects of V^{eff} will be treated to first order. Two factors provide the basis for considering this as a reasonable approximation:

(1) The factor $1/n^3$ in (2.13) means that the leading term in V^{eff} is only $\frac{1}{8}$ as strong as the leading term in H_0^{eff}, and that term by term in a rapidly decreasing series, each contribution to V^{eff} is multiplied by a coefficient c relative to the corresponding term in H_0^{eff} with $|c| < 1$.

(2) The "antiferromagnetic" character of H_0^{eff} and its solutions suggests that, as in the analogous solid-state problems,[13] the impact of the long-

range terms in V^{eff} is greatly weakened when studying the low-lying states of the theory, which are formed by bound fermion configurations that form color singlets on each site of the effective lattice. These features are very different from the circumstances that apply in the study of free fermion theory where it is not a valid approximation to treat the long-range terms in a simple perturbative procedure, since there is no remnant of an approximate multiplet structure in the theory obtained by studying the full $H = H_0 + V$. Our calculations with (2.6) in paper I showed that the rapid convergence with lattice separation allows us to simplify further by retaining only the nearest-neighbor terms, $n = 1$, in H_0^{eff} and only the next-nearest terms, $n = 2$, in V^{eff}. The corrections that we shall find in comparing our predictions with experiment may be as large as 30–50%, indicating that the symmetry breaking as formulated in our approach is not quantitatively small, but that nevertheless important residual effects of the SU(6)×SU(6) symmetry in hadron physics can be understood.

III. PHENOMENOLOGICAL CONSEQUENCES

In this section we present some of the most readily derived phenomenological consequences of the approximate symmetry of lattice QCD.

A. Pseudoscalar- and vector-meson masses (the Goldstone bosons)

We begin by examining the effect of the symmetry-breaking part of the Hamiltonian on the masses of the particles which are the Goldstone bosons of H_0^{eff}. According to the results of paper I, summarized in the preceding section, the spectrum of H_0^{eff} includes 72 Goldstone bosons, related to the 72 generators of SU(12) corresponding to those $M^{\alpha a}$ which do not commute with γ_0. Table I gives a list of these 72 particles classified by their "spin" and SU(3) properties, since these properties are preserved by the entire Hamiltonian. (By "spin" we refer to the transformation properties under 90° rotations. We find that particles corresponding to $\gamma_5 \otimes \lambda_a$ are spin singlets and that those corresponding to $\vec{\gamma} \otimes \lambda_a$ transform as a triplet.)

The salient feature of our analysis is that vector as well as pseudoscalar mesons emerge as Goldstone bosons of an approximate symmetry.[14] That the ρ meson is a would-be Goldstone boson of an approximate symmetry has been suggested previously by Caldi and Pagels.[15] The attractive consequence of this classification is that the equations for vector-meson masses arising from the usual treatment of partially conserved quantities naturally involve mass squared, which is well known to give a good understanding of splittings in the vector octet. The masses in Table I are obtained using this formalism for partially conserved currents, that is, from

$$(f^2M^2)^{\alpha a} = \langle 0 | [Q^{\alpha a}, [Q^{\alpha a}, V^{\text{eff}}]] | 0 \rangle , \tag{3.1}$$

where the $f^{\alpha a}$ are defined like the pion decay constant f_π. Since the vacuum is SU(6)×SU(6) symmetric in this approximation, the right side of (3.1) clearly gives effects of first order in the breaking. The $f^{\alpha a}$ are all equal in the zeroth order in V^{eff} and hence differences in the f's enter only as higher-order corrections to M^2. Using the formula (3.1) we obtain the values displayed in Table I where the unknown quantities X and Y contain combinations of reduced matrix elements.

The effects of quark masses can also be included to first order by adding the usual quark mass term to V^{eff},

TABLE I. Goldstone bosons of H_0^{eff}. Mass squared, due to V^{eff}, is shown in terms of two combinations of reduced matrix elements, X and Y. π, K, ρ, ... possess the usual flavor SU(3) classification, and $\tilde{\pi}$, $\tilde{K}$, $\tilde{\rho}$, ... form identical, heavier multiplets. u_1 and $\tilde{u}_1$ are flavor-singlet bosons and u_1 is presumed to "seize."

Particles	$M^{\alpha a}$	Spin	SU(3) representation	Parity	Mass squared
π, K, η	$\gamma_5 \otimes \lambda_a$	0	8	−	0
$\rho, K^*, \phi^{(0)}$	$\vec{\gamma} \otimes \lambda_a$	1	8	−	X
$\omega^{(0)}$	$\vec{\gamma}$	1	1	−	$X+Y$
$\tilde{\rho}, \tilde{K}^*, \tilde{\phi}^{(0)}$	$\gamma_0 \vec{\gamma} \otimes \lambda_a$	1	8	−	$2X$
$\tilde{\omega}^{(0)}$	$\gamma_0 \vec{\gamma}$	1	1	−	$2X$
$\tilde{\pi}, \tilde{K}, \tilde{\eta}$	$\gamma_5 \gamma_0 \otimes \lambda_a$	0	8	−	$3X$
u_1	γ_5	0	1	−	0
$\tilde{u}_1$	$\gamma_5 \gamma_0$	0	1	−	$3X$

$$H_m = \sum_{\vec{j}} \psi_{\vec{j}}^{\dagger}(\epsilon_0\gamma_0\lambda_0 + \epsilon_3\gamma_0\lambda_3 + \epsilon_8\gamma_0\lambda_8)\psi_{\vec{j}}$$
$$= \sum (m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s) . \qquad (3.2)$$

This introduces the splittings of the SU(3) multiplets, the quark mass contribution to the meson masses being identical for all four meson nonets. This leads immediately to the result[15]

$$m_{K^*}^2 - m_\rho^2 = m_K^2 - m_\pi^2 + O(\epsilon v) , \qquad (3.3)$$

where v denotes the order of magnitude of SU(6) $\times$ SU(6) breaking and ϵ is of the order of SU(3) breaking. Experimentally we find

$$m_{K^*}^2 - m_\rho^2 = 0.19 \pm 0.01 \text{ GeV}^2 = O(\epsilon) ,$$
$$m_K^2 - m_\pi^2 = 0.23 \text{ GeV}^2 \qquad (3.4)$$

so that (3.3) is correct to 16%.

Some discussion of the states $\tilde{\rho}$ and $\tilde{\pi}$, etc., is in order at this point. Since we have not yet learned how to calculate widths, we do not know whether these states could be expected to have been observed in usual hadronic experiments. For example, the question arises whether the $\tilde{\rho}$ should be identified with the ρ', or whether it is a broader structure. Furthermore, SU(6)$\times$SU(6)-symmetry-breaking corrections are involved here and are much larger than SU(3) corrections as considered in (3.4). Using the ρ and π masses to fix relevant unknown parameters and making first-order estimates we arrive at a prediction

$$m_{\tilde{\rho}}^2 = 1.2 \pm 0.2 \text{ GeV}^2 + O(v^2, \epsilon v) , \qquad (3.5)$$

considerably lighter than the ρ' [$m_{\rho'}^2 = (2.6 \pm 0.5)$ GeV^2]. However, the SU(6)$\times$SU(6) symmetry breaking is quite large, so that a correction of 100% in (3.5) from the terms of $O(v^2)$ is not unreasonable. Note that such corrections in this formula arise from three separate sources: corrections to the ρ mass formula, corrections to the $\tilde{\rho}$ mass formula, and corrections to the equality of f_ρ and $f_{\tilde{\rho}}$. If these corrections accumulate additively then we have, crudely speaking,

$$m_{\tilde{\rho}}^2 \simeq 2m_\rho^2(1 + 3v) .$$

Thus an SU(6)$\times$SU(6) breaking of order 30% could give a 100% error in the estimate of $m_{\tilde{\rho}}^2$. Typically, experimental evidence suggests something like 30% to 60% SU(6) breaking, so that we do not consider it impossible that our $\tilde{\rho}$ is in fact a ρ'. The $\tilde{\pi}$ state cannot be identified with any known particle, but a heavy pseudoscalar resonance decaying into multiple pions might be quite broad and thus difficult to detect. In summary, our calculations are too crude for reliable indications of SU(6)$\times$SU(6) breaking, although they seem qualitatively useful for SU(3)-breaking predictions.

Another point of interest in the mass formulas of Table I is the question[6] of SU(6)$_W$ versus static SU(6) as a symmetry of the hadron spectrum. We can write the symmetry-breaking part of H^{eff} as

$$V^{\text{eff}} = V_x + V_y + V_z , \qquad (3.6)$$

where the individual terms connect sites separated in the x, y, and z directions respectively. We can define three different SU(6)$_{W_i}$, each of which commutes with

$$\tilde{H}_{0i} = H_0^{\text{eff}} + V_i \qquad (3.7)$$

and hence each is a symmetry of a larger part of the full Hamiltonian than is SU(6)$\times$SU(6). Any one SU(6)$_{W_i}$ has degenerate multiplets

$$\{\pi; \rho_j, j \neq i; \tilde{\rho}_i\} ,$$
$$\{\tilde{\pi}; \tilde{\rho}_j, j \neq i; \rho_i\} , \qquad (3.8)$$

where indices i and j indicate helicity states; π and ρ here stand for the full SU(3) pseudoscalar- and vector-meson octets. The term V_i gives equal mass to all particles in the second of these two multiplets, but leaves the first multiplet massless. Hence the ratios $m_\rho^2 : m_{\tilde{\rho}}^2 : m_{\tilde{\pi}}^2 = 1:2:3$ in Table I arise simply from the fact that $m_{\tilde{\pi}}^2$ gets a contribution from all three of the V_i's, whereas each $\tilde{\rho}_j$ gets contributions from the two V_i with $i \neq j$ and each ρ_j gets mass only from V_j. Static SU(6) would place the π and all spin components of the $\tilde{\rho}$ in a degenerate multiplet [see Table I: static SU(6) contains $\vec{\sigma}$ among its charges], which is clearly a considerably worse symmetry than any one of the SU(6)$_{W_i}$.

The contribution of the reduced-matrix-element combination Y which splits the SU(3) octet and singlet in only the ρ multiplet is an anomaly in this respect. We remark that "magic mixing" for the physical ω and ϕ mesons indicates that Y must be small on the scale of quark mass terms. We find

$$Y \propto (\langle 0 | Q_{\vec{i}}^{\mu a} Q_{\vec{j}}^{\mu a} | 0 \rangle - \langle 0 | Q_{\vec{i}}^{ia} Q_{\vec{j}}^{ia} | 0 \rangle) , \qquad (3.9)$$

where $Q^{\mu a}$ corresponds to $M^{\mu a} = \alpha_\mu \otimes \lambda_a$ and Q^{ia} corresponds to $M^{ia} = \sigma_i \otimes \lambda_a$. The $Q^{\mu a}$ are Goldstone generators and the Q^{ia} are generators of the SU(6)$\times$SU(6); we know of no symmetry reason why the quantity Y should vanish.

B. Wigner symmetries and their consequences

As shown in Sec. II, the generators of our SU(6)$\times$SU(6) [or any of the SU(6)$_{W_i}$ subgroups thereof] are not the integrals over local densities which appear in current algebra, with the exception of the generators of the flavor SU(3), which are the usual quantities. We will now discuss how this difference affects the derivation of certain well-known SU(6) results.

1. $\mu_p/\mu_n = -\frac{3}{2}$

The magnetic moment of a particle is measured by the energy shift in an applied magnetic field:

$$\delta E = \mu_B H = \left\langle B \left| \int d^3x\, \psi^\dagger \vec{\alpha}\, Q\psi \cdot \vec{A} \right| B \right\rangle , \tag{3.10}$$

where, for example, $A_1 = A_3 = 0$, $A_2 = x_1 H$; here $Q = \frac{1}{2}(\lambda_3 + \lambda_8/\sqrt{3})$. The SU(6) ratio of moments is obtained provided (a) the proton and neutron are assumed to be members of a $\underline{56}$ of baryons under the SU(6), and (b) the operators $\psi^\dagger \alpha_\mu Q\psi$ transform as a $\underline{35}$ under SU(6). The ratio $-\frac{3}{2}$ is then simply a ratio of SU(6) Clebsch-Gordan coefficients, and the common reduced matrix element cancels.

If we consider the equivalent lattice calculation of the energy shift, we obtain a very similar formula. The lattice field which creates a magnetic field H in the z direction is given by

$$A_2(\vec{j}) = H j_1 , \quad A_1 = A_3 = 0 .$$

The energy shift in such a field is

$$\delta E = \left\langle B \left| \sum_{\vec{j} n \hat{\mu}} \psi_{\vec{j}}^\dagger \alpha_\mu \left\{ \exp\left[iQ \sum_{m=0}^{n-1} A_\mu (\vec{j} + m\hat{\mu}) \right] - 1 \right\} \right.\right. \times \psi(\vec{j} + n\hat{\mu}) \frac{(-1)^n}{n} \left| B \right\rangle . \tag{3.11}$$

Expanding to lowest order in H to identify the magnetic moment we find

$$\mu_B = \left\langle B \left| \sum_{\vec{j}, n} \frac{j_i}{n} (-1)^n \psi^\dagger(\vec{j}) \alpha_y Q \psi(\vec{j} + n\hat{y}) \right| B \right\rangle . \tag{3.12}$$

The operators which appear in this matrix element each transform as a $\underline{35}$ of our SU(6), so that if the baryons are assumed to lie in a $\underline{56}$ we reproduce the usual result for μ_p/μ_n.

2. $g_A/g_V \neq -\frac{5}{3}$

In usual SU(6) treatments one obtains $g_A/g_V = -\frac{5}{3}$ because both the charge operator, $\int d^3x\, \psi^\dagger Q \psi$, which measures g_V, and the spatial component of the axial current

$$F_i = \int d^3x\, \psi^\dagger(x) \alpha_i \gamma_5 \psi(x) , \tag{3.13}$$

which measures g_A, are assumed to be generators of the SU(6) under which the baryon states are classified. Hence the reduced matrix elements in both the numerator and the denominator are determined to be unity, and the ratio is just a ratio of Clebsch-Gordan coefficients. This result persists in relativistic SU(6)×SU(6) schemes.

In our case, however, although the charge is indeed a generator, the numerator quantity

$$F_i^{\text{lattice}} = \sum_{\vec{j}} \psi_{\vec{j}}^\dagger \alpha_i \gamma_5 \psi_{\vec{j}} \tag{3.14}$$

differs from the corresponding generator by the absence of the sign factors $s_{\alpha a}(\vec{j})$. Hence, although it transforms like a generator (i.e., as a member of a $\underline{35}$), it has a reduced matrix element $X < 1$, so that we find

$$g_A/g_V = -\tfrac{5}{3} X .$$

This is all we can say from symmetry considerations but since PCAC is a property of the theory, the usual Adler-Weisberger method can be used to obtain the correct value for g_A/g_V.

These two results are typical—many of the problems of SU(6) arise because the F_i are assumed to be *generators* of the algebra under which the states are classified, whereas the good results depend only on knowing the transformation properties of certain operators under the algebra. In fact, the work of Melosh[16] and others[17] who introduce "current quarks" and "constituent quarks" involves this realization, and by introducing the transformations between the different quark types they, too, relax the property that the F_i are generators.

3. Mass splittings

In SU(6) treatments the baryons are assumed to fall into a $\underline{56}$. A quark mass term such as (3.2) transforms as a $\underline{35}$. An old problem for SU(6) is that, since there is only one $\underline{56}$ in the product of a $\underline{35}$ with a $\underline{56}$, one cannot reproduce the Gell-Mann–Okubo octet mass formula for baryons using these assumptions (although the spin-$\frac{3}{2}$ decuplet masses are correctly given). The relatively large symmetry-breaking term V has the form of a singlet under either SU(3) or SU(2) rotations, but breaks the SU(6). Hence, it introduces a mixing of the octet spin-$\frac{1}{2}$ components of the $\underline{56}$ with a similar states in the $\underline{70}$ representation of SU(6). If the baryon is a mixture of SU(6) representations then the usual SU(3) Gell-Mann–Okubo formula can be obtained by inserting a quark mass term between baryon states. This amounts to keeping terms of order ϵv as well as those of leading order in evaluating baryon mass splittings. Since the spin-$\frac{3}{2}$ decuplet appears only once in the product of three SU(6) sextets, it is not mixed with anything by the V term and hence the naive SU(6) result is unaltered for this multiplet. In this respect our results are not different from usual SU(6) treatments.[5]

C. Pionic decays and radiative decays

A well-known problem for current algebra with $SU(6)_W$ is presented by meson and baryon decays involving either pion or photon emission. The data are not well reproduced by the theory. Much better fits have been obtained[17] using Melosh's notion of a transformation between current quarks and constituent quarks, which effectively introduces additional operators in the matrix elements over and above those which would appear in a straightforward current-algebra treatment. It is found in several cases that the best fit to data is obtained when the additional operators give the dominant contribution. Our analysis reproduces in zeroth order the usual poor current-algebra results, rather than the improved fits of the current-constituent-quark analysis. However, the additional operators will appear if first-order terms in the symmetry breaking are included. Thus, once again, ignoring higher-order corrections to $SU(6)_W$ symmetry does not give good quantitative results.

IV. DISCUSSION

The new feature of our lattice approach is the U(12) of charges (2.9), among which only the chiral generators (2.3) exhibit a simple continuum limit in terms of charge densities. However, all 144 charges $Q^{\alpha a}$ can be brought to this form by a local unitary transformation

$$\psi_{\vec{j}} = e^{i\phi_{\vec{j}}}\alpha_x^{j_x}\alpha_y^{j_y}\alpha_z^{j_z}\tilde{\psi}_{\vec{j}}\,, \tag{4.1}$$

in terms of which

$$Q^{\alpha a} = \sum_{\vec{j}} \tilde{\psi}_{\vec{j}}^\dagger M^{\alpha a}\psi_{\vec{j}} \equiv \sum_{\vec{j}} Q_{\vec{j}}^{\alpha a} \xrightarrow[\text{(continuum)}]{} \int d^3x\, \tilde{\psi}^\dagger(\vec{x}) M^{\alpha a}\tilde{\psi}(\vec{x})\,. \tag{4.2}$$

We can also rewrite H_0^{eff} in (2.14) as an antiferromagnetic lattice Hamiltonian in terms of these local charge densities:

$$H_0^{\text{eff}} = \frac{\Lambda}{g^2}\sum_{\substack{\vec{j}\hat{\mu}\\ \text{odd}\,n}} \frac{1}{n^3}\sum_{\alpha a} Q_{\vec{j}}^{\alpha a} Q_{\vec{j}+n\hat{\mu}}^{\alpha a}\,. \tag{4.3}$$

There is no corresponding simple form for the original lattice Hamiltonian in the new basis by (4.1), even if we ignore all but the nearest-neighbor interactions. Specifically, due to the algebra of the α_μ,

$$\sum_{\vec{j},\hat{\mu}} \psi_{\vec{j}}^\dagger \frac{\alpha_\mu}{i}(\psi_{\vec{j}+\hat{\mu}} - \psi_{\vec{j}-\hat{\mu}}) = \frac{1}{i}\sum_{\vec{j}} \tilde{\psi}_{\vec{j}}^\dagger[(-1)^{j_y+j_z}(\tilde{\psi}_{\vec{j}+\hat{x}} - \tilde{\psi}_{\vec{j}-\hat{x}}) + (-1)^{j_x+j_z}(\tilde{\psi}_{\vec{j}+\hat{y}} - \tilde{\psi}_{\vec{j}-\hat{y}}) + (-1)^{j_x+j_y}(\tilde{\psi}_{\vec{j}+\hat{z}} - \tilde{\psi}_{\vec{j}-\hat{z}})]\,, \tag{4.4}$$

and there is no way to avoid the sign alternations which were paired away in the strong-coupling H_0^{eff}, as in (4.3).

These observations raise the intriguing question of the physical significance of the transformation (4.1). It is immediately clear that (4.1) expresses a lattice periodicity which is explicit when rewritten

$$\psi_{\vec{j}} = e^{i\bar{\phi}_{\vec{j}}} e^{i(\pi/2)\alpha_x j_x} e^{i(\pi/2)\alpha_y j_y} e^{i(\pi/2)\alpha_z j_z}\tilde{\psi}_{\vec{j}} \tag{4.5}$$

with a redefined phase factor

$$\bar{\phi}_{\vec{j}} = -\frac{\pi}{2}(j_x + j_y + j_z) + \phi_{\vec{j}}\,.$$

This suggests that in the continuum theory a transformation of the form (4.1) or (4.5) will be useful if there is a confinement length in the physical problem that can be separated from other slowly varying features of the structure. Such a confinement length presumably occurs in QCD and corresponds to the size of a physical hadron within which color is confined.

Our lattice strong-coupling effective Hamiltonian has allowed us to extract certain physics of a confining theory, without investigating in detail the passage to the continuum limit. We find it remarkable that a well-known approximate dynamical symmetry should emerge from this analysis. In summary, we can say that the SU(6) results which follow from our lattice symmetry are for the most part the well-known results of previous studies. One important difference is that the generators of our symmetry are not the usual integrals over the spatial components of axial-vector currents. Thus we get a different result wherever the old analysis explicitly used the form of the generators, but the same result wherever the analysis merely used the transformation properties of bilinear quark operators. Finally, we remark that the fact that the vector mesons are pseudo-Goldstone bosons in this analysis allows an understanding of why quadratic mass formulas are correct for them as well as for pseudoscalars.

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[1]B. Svetitsky, S. D. Drell, H. R. Quinn, and M. Weinstein, Phys. Rev. D 22, 490 (1980); B. Svetitsky, Ph.D. thesis, Princeton University, 1980 (unpublished).

[2]S. Adler and R. Dashen, *Current Algebra* (Benjamin, New York, 1968).

[3]K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Lett. 14, 48 (1965).

[4]R. F. Dashen and M. Gell-Mann, Phys. Lett. 17, 142 (1965); 17, 145 (1965).

[5]F. Gursey and L. Radicati, Phys. Rev. Lett. 13, 173 (1964); M. A. B. Beg and V. Singh, *ibid.* 13, 418 (1964); M. A. B. Beg, B. W. Lee, and A. Pais, *ibid.* 13, 514 (1964); O. W. Greenberg, *ibid.* 13, 598 (1964); M. A. B. Beg and A. Pais, *ibid.* 14, 51 (1965). For reviews on SU(6) and its relativistic generalizations see B. W. Lee, in *Lecture on Particle Symmetries and Axiomatic Field Theory*, edited by M. Chretien and S. Deser (Gordon and Breach, New York, 1966), Vol. 2, pp. 325–451; A. Pais, Rev. Mod. Phys. 38, 215 (1966); H. Ruegg, W. Rühl, and T. S. Santhanam, Helv. Phys. Acta 40, 9 (1967).

[6]K. Barnes, P. Carruthers, and F. von Hippel, Phys. Rev. Lett. 16, 92 (1965); H. Harari, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (Univ. of Colorado, Boulder, 1966), Vol. 8b.

[7]M. Weinstein, Phys. Rev. D 4, 2544 (1971).

[8]S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 1627 (1976).

[9]$\delta'(n)$ arises if we define the derivative of $\psi_{\vec{J}}$ by

$$(\nabla_\mu \psi)_{\vec{J}} = \Lambda \sum_n \delta'(n) \psi_{\vec{J}+n\hat{\mu}} = \sum_{\vec{K}} iK_\mu \tilde{\psi}(\vec{K}) e^{i\vec{K}\cdot\vec{J}}.$$

It allows the spectrum of a free massive fermion field on a lattice to satisfy the dispersion law $E^2 = K^2 + m^2$ up to $|K_{max}| = \pi\Lambda$.

[10]In deriving (2.12) we have devoted our attention to states containing no baryons, as discussed in I. Thus the invariant U(1) subgroup of U(12) is irrelevant, as the density $Q_{\vec{J}}^{00}$ of its generator is identically zero. We will henceforth deal only with SU(12).

[11]This is clear from (2.13), since $s_{\alpha a}(n\hat{\mu}) = 1$ for n even.

[12]J. Kogut and L. Susskind, Phys. Rev. D 10, 3468 (1974); 11, 3594 (1975).

[13]See for example J. M. Rabin, Phys. Rev. B 22, 2420 (1980).

[14]If H_0 were the full Hamiltonian there would be no way to ensure the Lorentz invariance of the vacuum because of the vector Goldstone bosons. However, the physical vacuum is defined not by H_0, but by $H_0 + V$—we must select from among the degenerate eigenstates of H_0 that state on which V can act as a perturbation. Since V gives mass to all vector particles, this criterion guarantees a Lorentz-invariant vacuum (assuming of course, as is assumed throughout this discussion, that the passage to the continuum limit can be carried out).

[15]D. G. Caldi and H. Pagels, Phys. Rev. D 14, 809 (1976); 15, 2668 (1977).

[16]H. J. Melosh, Ph.D. thesis, California Institute of Technology (unpublished); Phys. Rev. D 9, 1095 (1974).

[17]See, e.g., F. G. Gilman, M. Kugler, and S. Meshkov, Phys. Rev. D 9, 715 (1974); F. Gilman, in *Phenomenology of Particles at High Energies*, proceedings of the Fourteenth Scottish Universities Summer School in Physics, 1973, edited by R. L. Crawford and R. Jennings (Academic, New York, 1974).