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Strong anomaly and $\eta \to 3\pi$ decay

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The amplitude for $\eta \to 3\pi$ is calculated in terms of the amplitude for $\eta' \to \eta\pi\pi$ using the strong anomaly and the partial conservation of the U(1) current. The isospin violation, an $(m_u - m_d)/(m_u + m_d)$ effect, is large. The η' decay constant is estimated from $\psi \to \eta'\gamma$. The result for $\eta \to 3\pi$ is in good agreement with experiment.

A long-standing difficulty for current algebra and approximate chiral symmetry has been the calculation of the electromagnetic decay rate $\eta \to 3\pi$. If the strong Hamiltonian conserves isospin the rate vanishes when either a charged or neutral pion is soft.[1] Experimentally, only the former is observed in the Dalitz plot.[2] The conventional remedy[3] is to allow isospin violation in the strong Hamiltonian

$$\mathcal{H}' = m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s = \bar{q}\mathfrak{M}q \tag{1}$$

and treat the $m_u - m_d$ term by chiral perturbation theory. Then the rate $A(\eta \to \pi^+\pi^-\pi^0)$ can be related to the electromagnetic mass difference of the pseudoscalars by reducing in two pions; the resulting decay rate is a factor of 3 too small.[4] However, by reducing in only the π^0 the matrix element $A(\eta \to \pi^+\pi^-\pi^0)$ can be related[5] through the strong anomaly[6] to a matrix element of the gluon operator $F\tilde{F}$.

This brings this long-standing problem into contact with two other long-standing problems. The first, recently discussed by Gross, Treiman, and Wilczek[7] (GTW) is the nature of isospin breaking[8] in quantum chromodynamics (QCD). They point out that if isospin violation is an accidental symmetry, in the sense that for the light quarks the ratio $(m_d - m_u)/(m_d + m_u)$ is truly of order one, then it would be important to see this effect in ways that are subject to accurate experimental verification. GTW point out that most common isospin-violating processes can be shown through use of the strong anomaly to be related to $(m_d - m_u)/m$ (strong), and the rest have larger experimental uncertainties; we will see here that the isospin violation of $\eta \to 3\pi$ is related through the use of the strong anomaly to $(m_d - m_u)/(m_d + m_u)$, and is consistent with the standard value of 0.29 of this ratio.

The second long-standing problem, recently discussed by Veneziano and Witten (VW),[9] is the role of the η' in connection with the strong anomaly and the U(1) problem. In this note we shall use their recently proposed hypothesis for a partially conserved U(1) current[9] (PCU_1C) and standard current-algebra techniques to relate the amplitudes $A(\eta \to \pi^+\pi^-\pi^0)$ to the amplitude $A(\eta' \to \eta\pi^+\pi^-)$. According to this hypothesis, framed in the $1/N$ expansion, it is the operator $F\tilde{F}$ which one should use for the η' interpolating field. We evaluate the corresponding decay constant $F_{\eta'}$, following Goldberg[10] and Novikov, Shifman, Vainshtein, and Zakharov,[11] from the processes $\psi \to \eta\gamma$ and $\psi \to \eta'\gamma$. The end result of our calculation is

$$\frac{A(\eta \to \pi^+\pi^-\pi^0)}{A(\eta' \to \eta\pi^+\pi^-)} = \frac{1}{\sqrt{3}} \frac{m_d - m_u}{m_d + m_u} \frac{F_\eta}{F_\pi} \frac{m_\eta^2}{m_{\eta'}^2} \left(\frac{m_\psi^2 - m_\eta^2}{m_\psi^2 - m_{\eta'}^2}\right)^{3/2} \times \left(\frac{\Gamma(\psi \to \eta'\gamma)}{\Gamma(\psi \to \eta\gamma)}\right)^{1/2}, \tag{2}$$

when the η' and π^0 are at zero momentum. Taking the electromagnetic-mass-difference estimate of GWT for the isospin violation, $(m_d - m_u)/(m_u + m_d) \approx 0.29$, the experimental decay ratio[12] $\Gamma(\psi \to \eta'\gamma)/\Gamma(\psi \to \eta\gamma) = 5.75 \pm 1.42$, and F_η/F_π from the photonic decay rates, we find

$$\frac{A(\eta \to \pi^+\pi^-\pi^0)}{A(\eta' \to \eta\pi^+\pi^-)} = 0.11 \pm 0.03\ , \tag{3}$$

where the error reflects theoretical uncertainty for the ratios m_u/m_d and F_η/F_π, and experimental uncertainty for the ψ decays. Note we neglect the small mixing terms considered in Ref. 10.

We now compare this calculation to experiment. Recently, Binnie *et al.*[13] and Abrams *et al.*[13] have measured the η' full width. With the well-known branching ratios to $\eta\pi^+\pi^-$ this determines $A(\eta'\to\eta\pi^+\pi^-)$. The numerator is determined from the $\eta\to\pi^+\pi^-\pi^0$ rate taking into account the current-algebra variation of the matrix element with π^0 energy. The result is (with the π^0 at zero momentum)

$$\frac{A(\eta\to\pi^+\pi^-\pi^0)}{A(\eta'\to\eta\pi^+\pi^-)}=0.11\pm0.02\ , \tag{4}$$

where the error reflects the uncertainty of the experimental rates. The agreement of our result Eq. (2) with experiment is within errors.[14]

Proceeding with the details of our calculation, the amplitude for $\eta\to\pi^+\pi^-\pi^0$ is[5] after reducing in a soft π^0, and using the interaction $\mathcal{H}'$,

$$A(\eta\to\pi^+\pi^-\pi^0)=\left(1-\frac{2E_0}{m_\eta}\right)\frac{m_d-m_u}{F_\pi}\times\left\langle\pi^+\pi^-\left|\int d^4x(\bar{u}\gamma_5u+\bar{d}\gamma_5d)\right|\eta\right\rangle\ , \tag{5}$$

where the current-algebra zero at vanishing charged-pion four-momentum has been inserted with linear dependence on E_0.

Our notation is

$$\begin{aligned}&A_\mu^i(x)=\bar{q}\gamma_\mu\gamma_5\lambda^i q\ ,\\ &\partial_\mu A_\mu^i(x)=i\bar{q}\gamma_5\{\lambda^i,\mathfrak{M}\}q+\delta_{i0}\frac{3\alpha_s}{4\pi}\operatorname{Tr}F\tilde{F}(\tfrac{2}{3})^{1/2}\ ,\\ &\langle 0|A_\mu^3|\pi(p)\rangle=iF_\pi p_\mu\ ,\\ &\langle 0|A_\mu^8|\eta(p)\rangle=iF_\eta p_\mu\ .\end{aligned} \tag{6}$$

The matrix element on the right-hand side of Eq. (5) is a zero-momentum matrix element of the operator[5]

$$\begin{aligned}\bar{u}\gamma_5u+\bar{d}\gamma_5d=&\frac{m_d-m_u}{m_u+m_d}(\bar{u}\gamma_5u-\bar{d}\gamma_5d)\\ &-\frac{i}{m_u+m_d}\partial_\mu A_\mu^{(2)}\\ &+\frac{2i}{m_u+m_d}\frac{2\alpha_s}{8\pi}\operatorname{Tr}F\tilde{F}\ .\end{aligned} \tag{7}$$

The first term is higher order in m_d-m_u. The second term vanishes by Sutherland-Veltman-Adler arguments.[1] We thus find, using this result in Eq. (5)

$$\lim_{k\to0}\langle\pi^+(p^+)\pi^-(p^-)\pi^0(k)|\eta(q)\rangle=\frac{2i}{F\pi}\left(\frac{m_d-m_u}{m_u+m_d}\right)\int d^4x\left\langle\pi^+(p^+)\pi^-(p^-)\left|\frac{2\alpha_s}{8\pi}\operatorname{Tr}F\tilde{F}\right|\eta(q)\right\rangle\ . \tag{8}$$

Following VW, with related treatments of Refs. 10 and 11 and Bardeen and Zacharov,[15] we find

$$\begin{aligned}m_{\eta'}^2F_{\eta'}&\equiv\langle 0|\partial_\mu A_\mu^0|\eta'\rangle\\ &=(\tfrac{2}{3})^{1/2}\langle 0|2i(m_u\bar{u}\gamma_5u+m_d\bar{d}\gamma_5d+m_s\bar{s}\gamma_5s)|\eta'\rangle+(\tfrac{2}{3})^{1/2}\left\langle 0\left|\frac{3\alpha_s}{4\pi}\operatorname{Tr}F\tilde{F}\right|\eta'\right\rangle\\ &=(\tfrac{2}{3})^{1/2}\left\langle 0\left|\frac{3\alpha_s}{4\pi}\operatorname{Tr}F\tilde{F}\right|\eta'\right\rangle+m_{\eta'}^2\tilde{F}_{\eta'}\ .\end{aligned} \tag{9}$$

Note, however, that we do not drop the $\tilde{F}_{\eta'}$ terms as Refs. 11 and 15 do. The $\mathrm{PCU_1C}$ hypothesis is then that the η' field interpolator is

$$m_{\eta'}^2(F_{\eta'}-\tilde{F}_{\eta'})\varphi_{\eta'}=(\tfrac{2}{3})^{1/2}\frac{3\alpha_s}{4\pi}\operatorname{Tr}F\tilde{F}\ ; \tag{10}$$

where $F_{\eta'}$ is normalized so that in the $1/N$ limit $F_{\eta'}=F_\pi$. Using this hypothesis in Eq. (8) and reducing in the η with $q^2=(p^++p^-)^2$ on the η mass shell, we have

$$\lim_{k\to0}\langle\pi^+(p^+)\pi^-(p^-)\pi^0(k)|\eta(q)\rangle=(\tfrac{2}{3})^{1/2}\frac{(F_{\eta'}-\tilde{F}_{\eta'})}{F_\pi}\frac{m_d-m_u}{m_u+m_d}\int d^4x\,d^4y\,e^{i(p^++p^-)\cdot y}(q^2+m_\eta^2)m_{\eta'}^2\times\langle\pi^+(p^+)\pi^-(p^-)|T(\varphi_{\eta'}(x)\varphi_\eta(y))|0\rangle\ . \tag{11}$$

Now consider the $\eta'\to\eta\pi^+\pi^-$ amplitude at $k=0$ where $p^++p^-=-q$:

$$\lim_{k\to0}\langle\pi^+(p^+)\pi^-(p^-)\eta(q)|\eta'(k)\rangle=-\int d^4x\int d^4y\,e^{i(p^++p^-)\cdot y}m_{\eta'}^2(q^2+m_\eta^2)\langle\pi^+(p^+)\pi^-(p^-)|T(\varphi_\eta(y)\varphi_{\eta'}(x))|0\rangle\ . \tag{12}$$

Comparing Eqs. (11) and (12) we learn

$$\lim_{k\to 0}\langle\pi^+(p^+)\pi^-(p^-)\pi^0(k)\,|\,\eta(q)\rangle=\left(\frac{2}{3}\right)^{1/2}\frac{m_d-m_u}{m_u+m_d}\,\frac{(F_{\eta'}-\tilde{F}_{\eta'})}{F_\pi}\,\lim_{k\to 0}\langle\pi^+(p^+)\pi^-(p^-)\eta(q)\,|\,\eta'(k)\rangle\ . \tag{13}$$

We now estimate $(F_{\eta'}-\tilde{F}_{\eta'})/F_\pi$. From the arguments of GTW, we estimate F_η as follows:

$$m_\eta^2F_\eta=\langle 0\,|\,\partial_\mu A_\mu^8\,|\eta\rangle=\frac{2i}{\sqrt{3}}\langle 0\,|m_u\bar{u}\gamma_5u+m_d\bar{d}\gamma_5d-2m_s\bar{s}\gamma_5s\,|\eta\rangle\ . \tag{14}$$

Extracting the octet part of the divergence we find[16]

$$m_\eta^2F_\eta=\frac{2i}{\sqrt{3}}\left[\tfrac{1}{6}(m_u+m_d+4m_s)\langle 0\,|\bar{u}\gamma_5u+\bar{d}\gamma_5d-2\bar{s}\gamma_5s\,|\eta\rangle+\tfrac{1}{3}(m_u+m_d-2m_s)\langle 0\,|\bar{u}\gamma_5u+\bar{d}\gamma_5d+\bar{s}\gamma_5s\,|\eta\rangle\right]$$

$$\approx\frac{1}{\sqrt{3}}\,\frac{m_u+m_d+4m_s}{m_u+m_d-2m_s}\left\langle 0\left|(\tfrac{3}{2})^{1/2}\,\partial_\mu A_\mu^0-\frac{3\alpha_s}{4\pi}\,\mathrm{Tr}F\tilde{F}\right|\eta\right\rangle\approx\frac{2}{\sqrt{3}}\left\langle 0\left|\frac{3\alpha_s}{4\pi}\,\mathrm{Tr}F\tilde{F}\right|\eta\right\rangle\ , \tag{15}$$

which is identical to the estimate of Ref. 11. Assuming[10,11] now that $\psi\to\eta'\gamma$ and $\psi\to\eta\gamma$ are anomaly mediated we have

$$\frac{\Gamma(\psi\to\eta'\gamma)}{\Gamma(\psi\to\eta\gamma)}=\left(\frac{p_\gamma'}{p_\gamma}\right)^3\left|\frac{\langle 0\,|\,\mathrm{Tr}F\tilde{F}\,|\,\eta'\rangle}{\langle 0\,|\,\mathrm{Tr}F\tilde{F}\,|\,\eta\rangle}\right|^2$$

$$=\left(\frac{m_\psi^2-m_{\eta'}^2}{m_\psi^2-m_\eta^2}\right)^3 2\left(\frac{m_{\eta'}^2}{m_\eta^2}\right)^2\frac{(F_{\eta'}-\tilde{F}_{\eta'})^2}{F_\eta^2}\ . \tag{16}$$

Combining (16) with (13) results in Eq. (2).

The ratio $(F_{\eta'}-\tilde{F}_{\eta'})/F_\eta$, an input to our calculation, is determined from $\Gamma(\psi\to\eta'\gamma)/\Gamma(\psi\to\eta\gamma)$ to be 0.62. Let us now consider the relevant photonic decays, $\eta'\to 2\gamma$, $\eta\to 2\gamma$, and $\pi^0\to 2\gamma$, to evaluate the decay constants $F_{\eta'}$ and $\tilde{F}_{\eta'}$,

$$A(\eta'\to 2\gamma):A(\eta\to 2\gamma):A(\pi^0\to 2\gamma)$$

$$=2(\tfrac{2}{3})^{1/2}\frac{F_\pi}{F_{\eta'}}:\frac{1}{\sqrt{3}}\frac{F_\pi}{F_\eta}:1$$

$$=1.41:0.78:1\ \text{(experiment)}$$

yielding $F_{\eta'}=1.16F_\pi$ and $F_\eta=0.74F_\pi$. Thus we find $\tilde{F}_{\eta'}=0.7F_\pi$, indicating that the assumption $\tilde{F}_{\eta'}=0$ made in Refs. 11 and 15 is unjustified. Note, however, that the decay constants are close to their $1/N$ limits $F_{\eta'}=F_\eta=F_\pi$.

We see that despite the large off-shell extrapolations involved in our comparison with experiment, our prediction for $A(\eta\to 3\pi)/A(\eta'\to\eta\pi^+\pi^-)$ is in excellent agreement with experiment, the PCU$_1$C hypothesis, a large value of $(m_d-m_u)/(m_d+m_u)$, and F_η/F_π as deduced from the photonic decay rates.

Perhaps this close numerical agreement is fortuitous considering the large off-mass-shell extrapolation in the amplitudes, and the uncertainties in estimating $F_{\eta'}$ and $\tilde{F}_{\eta'}$. Nonetheless, it may be possible that our calculation is more reliable than that of Ref. 4 in which $\eta\to 3\pi$ is found to be proportional to electromagnetic mass differences. There, two of the mesons are off-shell, and it is not clear that the anomaly term is properly taken into account. Note also that our calculation is completely different in spirit from that of Ref. 4. Our $\eta\to 3\pi$ amplitude is proportional to the strong process $\eta'\to\eta\pi\pi$, with an isospin-violating factor; in Ref. 4 the closest analogous term [Fig. 1(c)] is canceled by a negative metric state.

As for our estimate of $(F_{\eta'}-\tilde{F}_{\eta'})/F_\pi$, admittedly crude here, a more complete analysis of the decay constants and the chiral Ward identities, allowing mixing between η and $\partial_\mu A_\mu^0$ and between η' and $\partial_\mu A_\mu^8$, shows no change in this parameter, and hence no change in our prediction.[17,18]

Finally, we must remark that we do not have an absolute calculation of $\eta\to 3\pi$; rather, we have a relation between it and the strong amplitude $\eta'\to\eta\pi^+\pi^-$, in the classic spirit PCAC (partially conserved axial-vector current) calculations. An absolute calculation would be tantamount in our formulation to an absolute calculation of the strong amplitude $\eta'\to\eta\pi^+\pi^-$, which does not seem possible at this stage of our understanding of QCD.

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[18]It should be pointed out, however, that an alternate estimate of $\tilde{F}_{\eta'}/F_\eta$ can be obtained from Eqs. (9) and (14) by dropping terms of $O(m_u, m_d/m_s)$ and taking η and η' to be pure octet and singlet quark states, with no glue admixture. The result is $\tilde{F}_{\eta'}/F_\eta \cong \frac{1}{6}$, much smaller than our value; the resultant amplitude ratio of Eq. (3) is then a factor of 2 larger than ours. From our point of view this indicates the importance of the glue contribution to η and η' states.