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# Nonperturbative method for radiative corrections applied to lepton-proton scattering

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We present a new, nonperturbative method to effect radiative corrections in lepton- (electron or muon) nucleon scattering, useful for existing or planned experiments. This method relies on a spectral function derived in a previous paper, which takes into account both real soft photons and virtual ones and hence is free from infrared divergence. Hard effects are computed perturbatively and then included in the form of "hard factors" in the nonperturbative soft formulas. Practical computations are effected using the Gauss-Jacobi integration method which reduces the relevant integrals to a rapidly converging sequence. For the simple problem of the radiative quasielastic peak, we get an exponentiated form conjectured by Schwinger and found by Yennie, Frautschi, and Suura. We compare also our results with the peaking approximation and with the exact one-photon-emission formula of Mo and Tsai. Applications of our method to the continuous spectrum include the radiative tail of the $\Delta_{33}$ resonance in $e+p$ scattering and radiative corrections to the Feynman scale-invariant $F_2$ structure function for the kinematics of two recent high-energy muon experiments.

## I. INTRODUCTION

Radiative corrections are quite an old subject. Routinely used by experimentalists, these are considered to be so well known (or no longer interesting?) that many experimental groups do not publish the details or even the magnitude of QED effects, usually incorporated in intricate programs. Since a proper handling of radiative corrections is crucial to the extractions of the nonradiative (NR) cross sections and structure functions, it would be wise that these corrections appear transparently along with the published experimental results. In some domains of the experimental variables, typically for small fixed scaling variable and large momentum transfer, the difference between the raw and radiatively corrected data becomes so large that the term "corrections" becomes improper. Although the events belonging to these domains are usually "put aside," it is not clear at which level one must consider radiative corrections as being too large to be trusted.

In this paper, we present a simple and practical nonperturbative method to effect radiative corrections, using the spectral weight function and nonperturbative vertices we introduced in a previous paper,[1] hereafter denoted I. We apply here the method to lepton-nucleon scattering, although our formalism does apply in a straightforward manner to many other QED corrections. By nonperturbative, we mean that soft-photon effects (in fact, we use what might be called the soft-peaking approximation) are taken into account to all orders while the effects of hard photons are computed perturbatively and used as "hard factors," thus modifying the soft cross section. The first advantage of this nonperturbative approach to radiative corrections, besides avoiding infrared divergences at intermediate stages, is that at least the elastic-peak spectrum is integrable[2] whereas it is not in standard perturbation theory. The second and decisive advantage of our method is that the two extra integrations brought about by the radiative corrections can be reduced using the Gauss-Jacobi integration method to a double sum which converges rapidly. This favorable circumstance permits us to make contact with the peaking approximation and our method appears then as a simple generalization.

In Sec. II, we derive the basic formulas which express the soft radiatively corrected cross section and structure functions in terms of the corresponding (NR) ones, assumed to be known. This is done using the soft spectral function $E$ which takes care of *both* real soft photons and virtual radiative corrections and thus is free from infrared divergence.

Section III is devoted to the simplest case of radiative corrections, the elastic contribution to the structure functions. In particular, the almost elastic lepton-nucleon scattering cross section, where the emitted photons are necessarily soft, is obtained in closed form and compared with corresponding known results. The radiative tail of the elastic peak is discussed next, using the Gauss-Gegenbauer integration method, and the results are compared to those one gets in the peaking approximation. We compare also our results with the exact one-photon-emission cross section and we emphasize that the discrepancy is due to hard bremsstrahlung which must be treated as a separate contribution.

Section IV deals with radiative corrections to the hadronic continuous spectrum, a reputedly difficult subject. The efficiency of our method is demonstrated by easily getting the radiative tail of the $\Delta_{33}$ resonance and the radiative corrections for two recent high-energy $\mu+p$ experiments.

In Appendix A we present the Gauss-Gegenbauer and more generally, the Gauss-Jacobi integration methods which are central to almost all computations of this paper. Appendix B is devoted to the computation of the lepton tensor needed by the exact one-photon-emission contribution and get a result equivalent to that found in the literature.

## II. THE SOFT PART OF RADIATIVE CORRECTIONS

In this section, we derive the soft-photon radiatively corrected $W_1$ and $W_2$ structure functions in terms of the corresponding nonradiative (NR) ones, assumed to be known.

### A. Radiatively corrected structure functions

The cross section for lepton- (electron or muon) proton inclusive scattering in the one-photon-exchange approximation, including the soft part of photon emission from the lepton vertex and corresponding virtual radiative corrections to all orders in perturbation theory, is given by[1]

$$\sigma=\frac{2\alpha^2}{p}\int\frac{d^3p'}{2E'}\frac{d^4q}{(q^2)^2}W^{\mu\nu}_{\rm NR}(P,q)X_{\mu\nu}(p,p')\times\hat{E}(p,p';p-q-p')\,. \tag{2.1}$$

Here $p$, $E$ and $p'$, $E'$ are the momenta and energies of the incident and scattered lepton, $P$ the proton momentum, $q$ the momentum of the exchanged photon, and $\alpha$ the fine-structure constant. $W^{\mu\nu}_{\rm NR}$ is the usual nonradiative proton structure tensor, $X_{\mu\nu}$ the lepton tensor defined below, and $\hat{E}$ the soft-photon spectral function. This function, which describes both real and virtual soft photons, is related to the spectral weight function $E_\lambda$ defined in I by

$$\hat{E}=E_\lambda F_{es}{}^2\,, \tag{2.2}$$

where $F_{es}$ is the soft part of $F_e=F_1+(\alpha/2\pi)F_2$, $F_1$ and $F_2$ being the electric and magnetic lepton vertices.

In the soft-peaking approximation which is discussed in detail in I, the explicit form of the spectral function $\hat{E}$ is [see Eqs. (3.41) and (4.77) of Ref. 1]

$$\hat{E}(p,p';K)=(\alpha\bar{A})^2e^{\alpha\hat{F}(r)}\times\int_0^\infty d\sigma\int_0^\infty d\sigma'(\sigma\sigma')^{\alpha\bar{A}-1}\times\delta^4(K-\sigma l-\sigma' l')\,, \tag{2.3}$$

where $K$ is the total four-momentum of the emitted photons. The other quantities are defined as follows. Let $Q_E{}^2$ be the "experimental" squared momentum transfer

$$Q_E{}^2=-(p-p')^2=2(EE'-pp'\cos\theta-m^2)\overset{m}{\simeq}4EE'\sin^2\tfrac{1}{2}\theta\,, \tag{2.4}$$

where $\theta$ is the scattering angle and $m$ is the lepton mass. The symbol $\overset{m}{\simeq}$ emphasizes the high-energy limit $Q_E{}^2\gg m^2$. The kinematical variable $r$, convenient whenever it is desirable not to neglect the lepton mass, is defined through the equation

$$\frac{m^2}{Q_E{}^2}=\frac{r}{(1-r)^2}\overset{m}{\simeq}r\,. \tag{2.5}$$

In terms of $r$, the usual infrared exponent[2] $\alpha A=2\alpha\bar{A}$ reads

$$\alpha A=2\alpha\bar{A}=\frac{2\alpha}{\pi}\left[\frac{(1+r^2)}{1-r^2}\ln r^{-1}-1\right]\overset{m}{\simeq}\frac{2\alpha}{\pi}\left(\ln\frac{Q_E{}^2}{m^2}-1\right). \tag{2.6}$$

The infrared-finite function $\hat{F}(r)$, which is used to normalize $\hat{E}$, is given by

$$\begin{aligned}\hat{F}(r)&=F(r)+2B+\frac{1}{\pi}-A\ln\frac{\lambda e^\gamma}{Q_E}\\&=-\frac{(1+5r)}{2\pi(1-r)}\ln r-2\left(\bar{A}+\frac{2}{\pi}\right)\ln(1-r)\\&\quad-\frac{(1+r^2)}{1-r^2}\frac{\pi}{6}+\frac{(1+r^2)}{\pi(1-r^2)}\mathrm{Li}_2(r^2)\\&\overset{m}{\simeq}\frac{1}{\pi}\ln\frac{Q_E}{m}-\frac{\pi}{6}.\end{aligned} \tag{2.7}$$

The infrared-divergent function $F(r)$ has been introduced in I to normalize the spectral weight function[3] $E_\lambda$, $B$ is the standard[2] infrared-divergent part of the vertex function, $\mathrm{Li}_2(y)$ is Euler's dilogarithm function[4] closely related to the Spence function $\phi$ and defined by

$$\mathrm{Li}_2(y)=-\int_0^y\frac{dt}{t}\ln(1-t)=-\phi(-y)-\frac{\pi^2}{12}\,. \tag{2.8}$$

In Eq. (2.3), $l$ and $l'$ are the light-cone momenta[5]

$$l=\frac{p-rp'}{1+r}\overset{m}{\simeq}p,\quad l'=\frac{p'-rp}{1+r}\overset{m}{\simeq}p' \tag{2.9}$$

fulfilling the equations

$$l^2=l'^2=0 \tag{2.10a}$$

and

$$(l-l')^2=(p-p')^2=-Q_E{}^2. \tag{2.10b}$$

We shall denote by $\tilde{E}$ and $\tilde{E}'$ the time components of $l$ and $l'$ in the laboratory reference frame,

$$\tilde{E}=\frac{E-rE'}{1+r}\overset{m}{\simeq}E\,,\quad \tilde{E}'=\frac{E'-rE}{1+r}\overset{m}{\simeq}E'. \tag{2.11}$$

The infrared-finite lepton tensor $X^{\mu\nu}$, defined by leaving out the factor $F_{es}{}^2$ from the tensor $X^{\mu\nu}_{(\lambda)}$ defined in I is given by

$$X^{\mu\nu}(p,p')=V_H{}^2\Big[2(m^2-p\cdot p')g^{\mu\nu}+2(p^\mu p'^\nu+p^\nu p'^\mu)-\frac{\alpha}{\pi}\frac{F_2}{F_e}(p+p')^\mu(p+p')^\nu$$
$$+\Big(\frac{\alpha}{\pi}\Big)^2\Big(\frac{F_2}{F_e}\Big)^2\frac{(p\cdot p'+m^2)}{8m^2}(p+p')^\mu(p+p')^\nu\Big], \tag{2.12}$$

where $V_H$ is the hard part of $F_e$ defined below while the conventional nonradiative proton structure tensor reads

$$W^{\rho\sigma}_{\rm NR}(P,q)=\frac{1}{M^2}\Big(P^\rho-\frac{\nu}{q^2}q^\rho\Big)\Big(P^\sigma-\frac{\nu}{q^2}q^\sigma\Big)W_{2\,\rm NR}(q^2,\nu)-\Big(g^{\rho\sigma}-\frac{q^\rho q^\sigma}{q^2}\Big)W_{1\,\rm NR}(q^2,\nu), \tag{2.13}$$

where $\nu=P\cdot q$. Contracting the lepton tensor and the proton structure tensor leads to

$$\frac{W^{\mu\nu}_{\rm NR}X_{\mu\nu}}{V_H{}^2}=W_{2\,\rm NR}\Big[4EE'-Q_E{}^2-\frac{\alpha}{\pi}\Big(\frac{F_2}{F_e}\Big)(E+E')^2+\Big(\frac{\alpha}{\pi}\Big)^2\Big(\frac{F_2}{F_e}\Big)^2(E+E')^2\frac{(Q_E{}^2+4m^2)}{16m^2}\Big]$$
$$+W_{1\,\rm NR}\Big[2(Q_E{}^2-2m^2)+\frac{\alpha}{\pi}\Big(\frac{F_2}{F_e}\Big)(Q_E{}^2+4m^2)-\Big(\frac{\alpha}{\pi}\Big)^2\Big(\frac{F_2}{F_e}\Big)^2\frac{(Q_E{}^2+4m^2)^2}{16m^2}\Big]. \tag{2.14}$$

To arrive at this equation, we have assumed that $q^\mu X_{\mu\nu}=q^\nu X_{\mu\nu}=0$, which is in fact strictly valid only in the soft-photon limit $K\to 0$. To go beyond soft-photon radiative corrections, we show in Sec. III how to include approximately hard effects, by comparison with the exact one-photon bremsstrahlung cross section. Using this hard-factor correction, gauge invariance will be enforced when $K$ is no longer soft, at least to lowest-order perturbation theory.

In I, the following forms of $F_e$ and $F_2$ were derived[6]:

$$F_e=F_{es}\Big[1+\frac{\alpha}{\pi}\Big(\frac{1+r}{2(1-r)}\ln r^{-1}-1\Big)+O(\alpha^2)\Big]$$
$$\equiv F_{es}V_H \tag{2.15}$$

and

$$F_2=F_{es}\frac{2r}{(1-r^2)}\ln r^{-1}[1+O(\alpha)], \tag{2.16}$$

where

$$F_{es}=\Gamma(1+\alpha\bar{A}_1)e^{\alpha[B+(2\pi)^{-1}+r\bar{A}_1]}. \tag{2.17}$$

In the last equation, $\bar{A}_1$ is an analytic function whose real part is $\bar{A}$. Although $F_2/F_e$ is proportional to $m^2/Q_E{}^2$, we shall keep track of the terms involving this ratio, in view of possible applications for $\mu$-proton scattering at low momentum transfer. Using the explicit forms of $F_e$ and $F_2$, Eq. (2.14) becomes

$$W^{\mu\nu}_{\rm NR}X_{\mu\nu}=V_H{}^2(4EE'-Q_E{}^2)\times\Big[W_{2\,\rm NR}(1+rC_2)+2W_{1\,\rm NR}\frac{(Q_E{}^2-2m^2)}{4EE'-Q_E{}^2}(1+rC_1)\Big]$$
$$\overset{m}{\simeq}V_H{}^2(4EE'-Q_E{}^2)(W_{2\,\rm NR}+2W_{1\,\rm NR}\tan^2\theta/2). \tag{2.18}$$

The last factor in this equation is familiar from electron-proton scattering where the electron mass is usually neglected. The $C_1$ and $C_2$ functions result from the magnetic-form-factor contribution and are given by

$$C_1=\frac{1+r}{(1-r)(1-4r+r^2)}\Big[-\frac{\alpha}{\pi}\ln r-\frac{1}{8}\Big(\frac{\alpha}{\pi}\Big)^2\frac{(1+r)}{(1-r)}\ln^2 r\Big] \tag{2.19a}$$

and

$$C_2=\frac{(E+E')^2}{4EE'-Q_E{}^2}\Big[\frac{2\alpha}{\pi}\frac{\ln r}{1-r^2}+\Big(\frac{\alpha}{\pi}\Big)^2\frac{\ln^2 r}{4(1-r)^2}\Big]. \tag{2.19b}$$

The function $V_H{}^2$ is the square of the "hard part" of the elastic vertex function

$$V_H{}^2=1+\frac{\alpha}{\pi}\Big(\frac{1+r}{1-r}\ln r^{-1}-2\Big)+O(\alpha^2). \tag{2.20}$$

It is convenient to express Eq. (2.1) using the Mott cross section as reference unit,

$$F_0 = \frac{\alpha^2 p'(4EE' - Q_E{}^2)}{p Q_E{}^4}$$
$$\overset{m}{\simeq} \frac{\alpha^2 \cos^2\theta/2}{4E^2 \sin^4\theta/2} \,. \tag{2.21}$$

Using Eq. (2.3) into Eq. (2.1), the differential lepton-proton cross section reads

$$\frac{d\sigma}{d\Omega' dE'} = F_0 \left[ W_2(1 + rC_2) + 2W_1 \frac{(Q_E{}^2 - 2m^2)}{(4EE' - Q_E{}^2)}(1 + rC_1) \right], \tag{2.22}$$

where $W_1$ and $W_2$ are the radiatively corrected structure functions given by

$$W_{1,2} = (\alpha\bar{A})^2 e^{\alpha\hat{F}(r)} V_H{}^2 \times \int_0^\infty d\sigma \int_0^\infty d\sigma' (\sigma\sigma')^{\alpha\bar{A}-1} \frac{W_{1,2\,\mathrm{NR}}}{(Q^2/Q_E{}^2)^2}, \tag{2.23}$$

where $Q^2 = -q^2 = -(p - p' - K)^2$. The following kinematics analysis will make more precise the $(\sigma, \sigma')$ integration domain.

### B. Kinematics and integration domains

The nonradiative structure functions depend on $\nu$ and $Q^2$ which should not be confused with the "experimental" invariants

$$\nu_E = P\cdot(p - p') = M(E - E') = M(\tilde{E} - \tilde{E}') \tag{2.24}$$

and $Q_E{}^2$ already defined in Eq. (2.4). The former ones are given by

$$\nu = P\cdot q = P\cdot(p - p' - K) \tag{2.25}$$

and

$$Q^2 = -q^2 = -(p - p' - K)^2 . \tag{2.26}$$

From Eq. (2.3), the momentum $q$ reads

$$q = p - p' - K = l(1 - \sigma) - l'(1 + \sigma') . \tag{2.27}$$

Incidentally, we note that $\sigma$ and $\sigma'$ are simply related to the light-cone components of $K$ in the Breit frame $(\vec{p} + \vec{p}\,' = 0)$ through

$$\sigma = \frac{K_+}{Q_E} = \frac{K_0 + K_z}{Q_E}, \quad \sigma' = \frac{K_-}{Q_E} = \frac{K_0 - K_z}{Q_E}, \tag{2.28}$$

where the $z$ axis is taken along the $\vec{p}$ momentum. Taking into account Eqs. (2.10) and (2.11) we get

$$Q^2 = -q^2 = Q_E{}^2(1 - \sigma)(1 + \sigma') \tag{2.29}$$

and

$$\nu = \nu_E - \sigma M\tilde{E} - \sigma' M\tilde{E}' . \tag{2.30}$$

The effective integration domain in the double integral in Eq. (2.23) is obtained by noting that, since the state of lowest mass which can be produced at the hadronic vertex is a proton, we have the kinematical constraints

$$2\nu - Q^2 \geq 0 \text{ and } \nu \geq 0 . \tag{2.31}$$

From Eqs. (2.29) and (2.30), we see that we can work equivalently in the $(\nu, Q^2)$ or $(\sigma, \sigma')$ planes. In the former plane, the effective integration domain is represented in Fig. 1 where the $\sigma = 0$ and $\sigma' = 0$ equations are straight lines

$$\sigma = 0 \Rightarrow Q^2 = \frac{Q_E{}^2}{\tilde{E}'}\left(\tilde{E} - \frac{\nu}{M}\right) \tag{2.32}$$

and

$$\sigma' = 0 \Rightarrow Q^2 = \frac{Q_E{}^2}{\tilde{E}}\left(\tilde{E}' + \frac{\nu}{M}\right). \tag{2.33}$$

Alternatively, in the $(\sigma, \sigma')$ plane, the effective integration domain is represented in Fig. 2 where the equations $2\nu = Q^2$ and $\nu \geq 0$ represent part of a hyperbola whose equation is, from (2.29) and (2.30),

$$\sigma = \frac{(w_E{}^2 - M^2) - \sigma' A_{E'}}{A_E - \sigma' Q_E{}^2}, \tag{2.34}$$

where $A_E$ and $A_{E'}$ are two combinations of the kinematical variables, extensively used in this paper,

$$A_E = 2M\tilde{E} - Q_E{}^2, \tag{2.35}$$

$$A_{E'} = 2M\tilde{E}' + Q_E{}^2, \tag{2.36}$$

and

$$w_E{}^2 = (P + p - p')^2 = 2\nu_E - Q_E{}^2 + M^2. \tag{2.37}$$

From Figs. 1 or 2, it is clear that the integration domain is empty unless

$$w_E{}^2 \geq M^2, \tag{2.38}$$

which is of course expected. In particular, since the integral in Eq. (2.23) is convergent, we see that the cross section for exactly elastic scattering $(w_E{}^2 = M^2)$ vanishes, which is in agreement with the Bloch-Nordsieck theorem.[7]

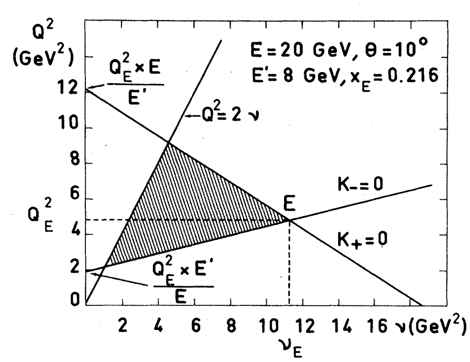


FIG. 1. Integration domain in the $(\nu, Q^2)$ plane for given experimental values of $E$, $\nu_E$, and $Q_E{}^2$.

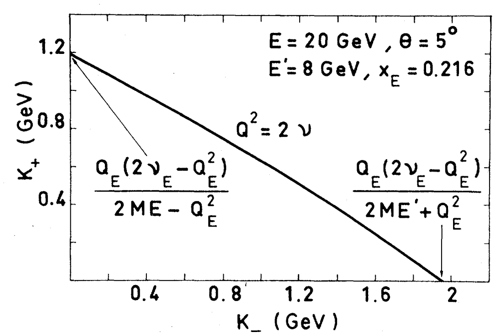


FIG. 2. Integration domain in the $(K_-, K_+)$ plane.

### C. Practical form of $W_1$ and $W_2$

For the applications we shall develop in the following sections, it is convenient to write Eq. (2.23) so that the invariant squared mass produced at the hadron vertex,

$$w^2 = (P+q)^2 = w_E^{\,2} - \sigma A_E - \sigma' A_{E'} + \sigma\sigma' Q_E^{\,2}, \qquad (2.39)$$

is one of the integration variables. From this equation, we can express $\sigma$ in terms of $w^2$ and $\sigma'$ in the form

$$\sigma = \frac{(w_E^{\,2} - w^2)(1-\tau)}{A_E - \tau Q_E^{\,2}(w_E^{\,2} - w^2)/A_{E'}}, \qquad (2.40)$$

where

$$\tau = \frac{\sigma' A_{E'}}{w_E^{\,2} - w^2}. \qquad (2.41)$$

For later reference, we introduce also the following variables:

$$x_+ = 1 - \frac{K_+}{Q_E} = 1 - \sigma = \frac{(2M\tilde{E}' + w^2 - M^2)[1 + B'(w^2)\tau]}{A_E[1 - z(w^2)\tau]} \qquad (2.42)$$

and

$$x_-^{-1} \equiv y_- = 1 + \frac{K_-}{Q_E} = 1 + \sigma' = 1 + B''(w^2)\tau, \qquad (2.43)$$

where the functions $z$, $B'$, and $B''$ are defined by

$$z(w^2) = \frac{Q_E^{\,2}(w_E^{\,2} - w^2)}{A_E A_{E'}}, \qquad (2.44)$$

$$B'(w^2) = \frac{2M\tilde{E}'(w_E^{\,2} - w^2)}{A_{E'}(2M\tilde{E}' + w^2 - M^2)}, \qquad B''(w^2) = \frac{w_E^{\,2} - w^2}{A_{E'}}. \qquad (2.45)$$

Computing the Jacobian for the $(\sigma, \sigma') \to (w^2, \tau)$ change of variables, Eq. (2.23) becomes after straightforward algebra

$$W_{1,2} = \frac{(\alpha\bar{A})^2 e^{\alpha\hat{F}(r)} V_H^{\,2}}{(A_E A_{E'})^{\alpha\bar{A}}} \int_0^1 \frac{d\tau}{[\tau(1-\tau)]^{1-\alpha\bar{A}}} \int_{M^2}^{w_E^{\,2}} \frac{dw^2 W_{1,2\,\mathrm{NR}}(Q^2, w^2)}{(w_E^{\,2} - w^2)^{1-\alpha A}(1 - z(w^2)\tau)^{\alpha\bar{A}}(Q^2/Q_E^{\,2})^2}, \qquad (2.46)$$

where, using Eqs. (2.29), (2.42), and (2.43),

$$Q^2 = Q_E^{\,2} x_+ y_-. \qquad (2.47)$$

Equation (2.46) is the general equation which gives the radiatively corrected structure functions in terms of the nonradiative ones.

In the following sections, we shall develop a suitable method for handling very simply the $\tau$ and $w^2$ integrations. We shall also derive the expressions of the hard factors which extend the validity of Eq. (2.46) in domains where the emitted photons are not necessarily soft and discuss various applications of this equation.

## III. ELASTIC CONTRIBUTION TO THE STRUCTURE FUNCTIONS

In this section, we shall use the formulas derived previously to discuss the elastic contribution to the structure functions, i.e., the contribution from the proton hadronic state. Owing to radiative corrections, this contribution is drastically modified. First, the nonradiative $\delta(E' - E'_{\mathrm{el}})$ peak ($E'_{\mathrm{el}}$ is the elastic energy of the scattered electron) is transformed into a continuous spectrum with a steep peak. The integrated cross section under the peak gives the radiatively corrected "quasielastic" cross section. Second, since photon emission by the lepton lowers the momentum transferred by the exchanged photon, the elastic contribution can be large in the very deep inelastic domain. This is the tail of the elastic peak.

The elastic contribution to the nonradiative structure functions is easily expressed[8] in terms of $G_E$ and $G_M$, the electric and magnetic proton form factors by the well known formulas

$$W_{1\,\mathrm{NR}}^{\mathrm{el}} = \delta\left(\frac{w^2 - M^2}{2}\right)\frac{Q^2}{4M} G_M^{\,2}(Q^2) \equiv M\delta\left(\frac{w^2 - M^2}{2}\right)\hat{W}_1(Q^2) \qquad (3.1)$$

and

$$W_{2\,\mathrm{NR}}^{\mathrm{el}} = M\delta\left(\frac{w^2 - M^2}{2}\right)\frac{[G_E^{\,2}(Q^2) + (Q^2/4M^2)G_M^{\,2}(Q^2)]}{(1 + Q^2/4M^2)} \equiv M\delta\left(\frac{w^2 - M^2}{2}\right)\hat{W}_2(Q^2). \qquad (3.2)$$

We shall call $\hat{W}_1$ and $\hat{W}_2$, which are functions of $Q^2$ only, the reduced structure functions. Owing to the $\delta$ functions in Eqs. (3.1) and (3.2), which express the fact that the scattered proton is on the mass shell, Eq. (2.46) is greatly simplified and we get for the elastic contribution

$$W_{1,2}^{\mathrm{el}}=\frac{(\alpha\bar{A})^2 e^{\alpha\hat{F}(\tau)}V_H{}^2 2M}{(2\nu_E-Q_E{}^2)^{1-\alpha A}(A_E A_{E'})^{\alpha\bar{A}}} \times\int_0^1 \frac{d\tau[\tau(1-\tau)]^{\alpha\bar{A}-1}\hat{W}_{1,2}(Q^2)}{(1-z\tau)^{\alpha\bar{A}}(Q^2/Q_E{}^2)^2}, \tag{3.3}$$

while Eqs. (2.44) and (2.45) simplify to

$$z=\frac{Q_E{}^2(2\nu_E-Q_E{}^2)}{A_E A_{E'}} \tag{3.4}$$

and

$$B'=B''=\frac{2\nu_E-Q_E{}^2}{A_{E'}}, \tag{3.5}$$

where $A_E$ and $A_{E'}$ are defined in Eqs. (2.35) and (2.36). The squared momentum transfer $Q^2$ is still given by Eq. (2.47), while the general expression of $x_+$, Eq. (2.42), simplifies somewhat. In numerical applications, we shall assume the usual expressions of the proton form factors

$$G_E^{\mathrm{proton}}=\frac{G_M^{\mathrm{proton}}}{2.793} =\frac{1}{(1+Q^2/0.71\ \mathrm{GeV}^2)^2}. \tag{3.6}$$

### A. Radiative corrections in quasielastic lepton-proton scattering

By quasielastic scattering we mean that $(2\nu_E-Q_E{}^2)/M^2$ is small, that is, point $E$ in Fig. 1 is near the $Q^2=2\nu$ line, inside the physical region. This is the simplest case of radiative corrections leading to a peak spectrum which can be obtained in closed form. Under the peak, within the one-photon-exchange approximation, this spectrum is expected to be exact to all orders in perturbation theory since the emitted photons are necessarily soft owing to phase-space limitation; there is no hard correction in this domain. Indeed, from Fig. 2 we see that the integration line $w^2=M^2$ is in the vicinity of the origin $K_+=K_-=0$. The elastic scattered energy $E'_{\mathrm{el}}$ is the value of $E'$ for which $2\nu_E-Q_E{}^2$ vanishes. From Eqs. (2.4) and (2.24) we get

$$E'_{\mathrm{el}}\overset{m}{\simeq}\frac{E}{1+2(E/M)\sin^2\theta/2}. \tag{3.7}$$

To avoid writing long formulas which may obscure the physics, we shall limit ourselves, for the rest of this subsection, to the high-energy limit. This simplification is not used in our numerical programs.

For $E'$ close to $E'_{\mathrm{el}}$, we can put $2\nu_E-Q_E{}^2=0$ in Eqs. (3.4) and (3.5) and get $B'=B''=z=0$ and $Q^2=Q_E{}^2$. The $\tau$ integral in Eq. (3.3) is now Euler's $B(\alpha\bar{A},\alpha\bar{A})$ function and the other functions simplify to

$$A_E\sim 2ME'_{\mathrm{el}},\quad A_{E'}\sim 2ME. \tag{3.8}$$

Using, moreover, the kinematical relation

$$\frac{2M}{2\nu_E-Q_E{}^2}\overset{m}{\simeq}\frac{E'_{\mathrm{el}}}{E(E'_{\mathrm{el}}-E')}, \tag{3.9}$$

Eq. (3.3) becomes

$$W_{1,2}^{\mathrm{peak}}=\Gamma^{-1}(\alpha A)\Gamma^2(1+\alpha\bar{A})\frac{\hat{W}_{1,2}(Q_E{}^2)}{E} \times(Q_E/m)^{\alpha/\pi}e^{-\alpha\pi/6}V_H{}^2(Q_E{}^2)\frac{(E/E'_{\mathrm{el}})^{\alpha\bar{A}}}{(1-E'/E'_{\mathrm{el}})^{1-\alpha A}}. \tag{3.10}$$

At this point, we shall include the effect of vacuum polarization to lowest order. For electron-nucleon scattering, $V_H{}^2$ is changed to

$$V'(Q_E{}^2)=V_H{}^2(Q_E{}^2)V_P{}^2(Q_E{}^2) \overset{m}{\simeq}1+\frac{5\alpha}{3\pi}\ln\frac{Q_E{}^2}{m^2}-\frac{28\alpha}{9\pi}+O(\alpha^2), \tag{3.11}$$

where we have used Eq. (2.20) and the known expression of vacuum polarization to lowest order,

$$V_P(Q^2)=\left\{1-\frac{\alpha}{3\pi}\left[\left(1-\frac{2m^2}{Q^2}\right)\left(1+\frac{4m^2}{Q^2}\right)^{1/2}\ln\frac{[1+(4m^2/Q^2)]^{1/2}+1}{[1+(4m^2/Q^2)]^{1/2}-1}-\frac{5}{3}+\frac{4m^2}{Q^2}\right]\right\}^{-1}\overset{m}{\simeq}\left[1-\frac{\alpha}{3\pi}\left(\ln\frac{Q^2}{m^2}-\frac{5}{3}\right)\right]^{-1}. \tag{3.12}$$

To get the radiatively corrected cross section in the vicinity of the elastic peak, all we have to do is to substitute Eq. (3.10) into Eq. (2.22). It is convenient to express the result in terms of the elastic cross section without radiative corrections,

$$\left(\frac{d\sigma}{d\Omega'}\right)_0=F_0(E,\theta)\frac{E'_{\mathrm{el}}}{E}(\hat{W}_2+2\hat{W}_1\tan^2\tfrac{1}{2}\theta) \tag{3.13}$$

and we obtain

$$\frac{d\sigma^{\mathrm{peak}}}{d\Omega' dE'} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\Gamma^{-1}(\alpha A)\Gamma^2(1+\alpha\bar{A})}{E'_{e1}} \times (Q_E/m)^{\alpha/\pi} e^{-\alpha\pi/6} \frac{V'(Q_E{}^2)(E/E'_{e1})^{\alpha\bar{A}}}{(1-E'/E'_{e1})^{1-\alpha A}}. \tag{3.14}$$

The important feature of Eqs. (3.14) and (3.10) is the radiative tail, characterized by the integrable $\Gamma^{-1}(\alpha A)(E'_{e1}-E')^{\alpha A-1}$ spectrum, in agreement with YFS work.[2] To illustrate Eq. (3.14), Fig. 3 represents the elastic peak spectrum for $E=10$ GeV and $\theta=34°$. The one-photon-emission cross section is also drawn for comparison.

It may be worthwhile at this point to recall the comparison between these results and standard perturbation theory. In the latter, the differential cross section for one-photon emission is proportional to $(E'_{e1}-E')^{-1}$ which is nonintegrable, whereas the elastic differential cross section is, in lowest order, proportional to $\delta(E'_{e1}-E')$ [in textbooks, only the integrated cross section which we called $(d\sigma/d\Omega')_0$ appears]. To first order, one-virtual-photon radiative corrections modify the above cross section by a factor of $(1+\alpha A\ln\lambda/m)$ which goes to $-\infty$ in the limit $\lambda\to 0$. These difficulties are solved in part by cutting off the spectrum at a distance $\Delta E'$ from $E'_{e1}$. The positive infinite cross section (in the $\lambda\to 0$ limit), integrated between $E'_{e1}-\Delta E'$ and $E'_{e1}$ is added to the elastic cross section to produce a $\lambda$-independent differential cross section proportional to $\delta(E'-E'_{e1})(1+\alpha A\ln\Delta E'/E)$. To avoid in this result an infinite cross section in the limit $\Delta E'\to 0$, which is physically unsatisfactory, Schwinger[9,10] conjectured the exponentiation of this lowest-order result, that is, $(1+\alpha A\ln\Delta E'/E)$ should be merely the first-order expansion of $\exp(\alpha A\ln\Delta E'/E)$, which vanishes in the limit $\Delta E'\to 0$. In our approach, in fact, quite similar to YFS, there is no distinction between elastic and inelastic cross section. The $\delta(E'-E'_{e1})$ and $(E'_{e1}-E')^{-1}$ spectrum are lumped in the continuous integrable spectrum of Eqs. (3.10) and (3.14), where, as we shall see in a moment, Schwinger's conjecture is fulfilled.

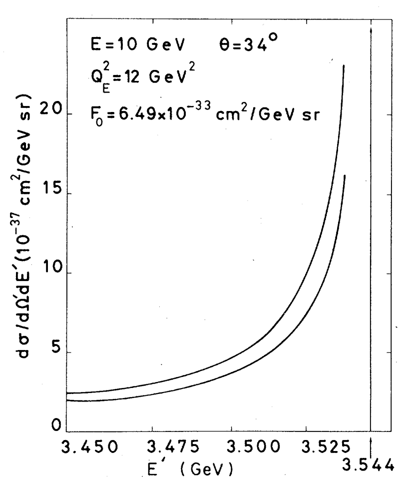


FIG. 3. $e$-$p$ elastic-peak spectrum: comparison between the nonperturbative (lower curve) and the one-photon-emission (upper curve) cross section.

To calculate the integrated cross section under the peak we integrate the differential cross section as given by (3.14) from some threshold energy $E'_0$ up to $E'_{e1}$. After a trivial integration over $E'$ in Eq. (3.14), we get

$$\frac{d\sigma^{\mathrm{peak}}}{d\Omega'} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \left(1-\frac{\pi^2}{24}(\alpha A)^2+\cdots\right) V'(Q_E{}^2)e^{-\delta} \equiv \left(\frac{d\sigma}{d\Omega'}\right)_0 R_c\,, \tag{3.15}$$

where

$$\delta \overset{m}{\simeq} \alpha\bar{A}\ln\frac{E'^3_{e1}}{E(\Delta E')^2} - \frac{\alpha}{2\pi}\ln\frac{Q_E{}^2}{m^2} + \frac{\alpha\pi}{6} \tag{3.16}$$

and

$$\Delta E' = E'_{e1} - E'_0. \tag{3.17}$$

The second factor in Eq. (3.15) comes from the expansion of $\Gamma^{-1}(1+\alpha A)\Gamma^2(1+\alpha\bar{A})$. We note that, leaving aside this factor and $V'(Q_E{}^2)$ which takes into account the hard part of virtual radiative corrections and vacuum polarization which are not expected to exponentiate, Eq. (3.15) is an exponentiated version of Schwinger's formula[9,10]

$$\left(\frac{d\sigma}{d\Omega'}\right) = \left(\frac{d\sigma}{d\Omega'}\right)_0 (1-\delta_S)\,, \tag{3.18}$$

where in our notation

$$\delta_S = \frac{4\alpha}{\pi}\left\{\left[\ln(E/\Delta E') - \tfrac{13}{12}\right]\left[\ln\left((2E/m)\sin(\tfrac{1}{2}\theta)\right) - \tfrac{1}{2}\right] + \tfrac{17}{72} + \tfrac{1}{2}\sin(\tfrac{1}{2}\theta)\phi\left(\cos(\tfrac{1}{2}\theta)\right)\right\}, \tag{3.19}$$

where $\phi$ is the Spence function. To compare with Schwinger's result, we expand Eq. (3.15) in the form of (3.18) and take into account Eqs. (3.16), (3.11), and (2.6) to get

$$\delta' = \frac{2\alpha}{\pi}\left(\ln\frac{E'^{3/2}_{e1}}{E^{1/2}\Delta E'}\right)\left(\ln\frac{Q_E{}^2}{m^2}-1\right) - \frac{13\alpha}{6\pi}\ln\frac{Q_E{}^2}{m^2} + \frac{28\alpha}{9\pi} + \frac{\alpha\pi}{6}\,. \tag{3.20}$$

Since in Schwinger's computation $E \ll M$ is assumed (potential scattering) and therefore, from Eq. (3.7) $E' \simeq E'_{el} \simeq E$, Eqs. (3.19) and (3.20) coincide except for the last term. The origin of this discrepancy, which is very small at high energy, lies in the fact that Schwinger's $k^0_{min}$ and our $\Delta E'$ are introduced in quite different ways. In fact, since we do not introduce any cutoff on photon energy when using the spectral weight function, our approach is completely covariant until we ask the noncovariant question: What is the cross section for $E' \geq E'_0$?

Even the last term of Eq. (3.20) may be understood by noting that the normalization of the spectral weight function $E_1$ in I, through $F(r)$, has been adjusted so that the exact and approximate energy spectral functions coincide in the Breit frame.[1] Thus, $\alpha\pi/6$ is recovered by taking $\theta = \pi$ ($\vec{p}+\vec{p}\,'=0$ in the Breit frame) in the last term of Eq. (3.19) and noting that $\phi(0)=\pi^2/12$.

As a numerical example using Eq. (3.15), let us compute $\delta$ for $E=20$ GeV, $\theta=6°$, and $\Delta E'=10$ MeV. Then $E'_{el}=17.909$ GeV, $Q_E{}^2=3.924$ GeV$^2$, $\alpha\bar{A}=3.6\times10^{-2}$, and $\delta=0.5210$. The radiative-correction factor is thus $R_c=0.592$. Of course, Eq. (3.15) for the elastic peak is trivially inverted to give the uncorrected cross section $(d\sigma/d\Omega')_0$ and, using Eq. (3.13), one separates the $\hat{W}_1$ and $\hat{W}_2$ structure functions. In principle, this is the way the proton form factors [Eq. (3.6)] have been measured.

### B. Radiative tail from the elastic peak

When we leave the immediate vicinity of the elastic peak there is still an *elastic contribution* to the structure functions which is now given by Eq. (3.3). Far from the peak, the emitted photons deviate from "softness" and we must include in this equation hard-photon corrections. The hard correction factors are known from the peaking approximation to the one-photon-emission cross section. To extrapolate these factors for multiphoton emission, it is convenient to work out the integration method of Eq. (3.3) and compare the results with the soft-peaking approximation one gets in lowest-order perturbation theory.

Since the integrand in Eq. (3.3) is weighted by $[\tau(1-\tau)]^{\alpha\bar{A}-1}$, a suitable method of integration is the Gegenbauer-Gauss quadrature described in Appendix A. $\alpha\bar{A}$ being of the order 0.03, it is clear that the main contribution to the $\tau$ integral comes from the vicinity of $\tau=0$ and $\tau=1$. According to Eqs. (2.28), (2.40), and (2.41), these values of $\tau$ correspond to $K_-=0$ ($K_z=K_0$) and $K_+=0$ ($K_z=-K_0$), respectively. Using formulas (A1) and (A4) of Appendix A, the $n$th order approximation to Eq. (3.3) is

$$W^{el}_{1,2}=\frac{\Gamma^2(1+\alpha\bar{A})}{2\Gamma(2\alpha\bar{A})}\left[\frac{(2\nu_E-Q_E{}^2)^2}{A_EA_{E'}}\right]^{\alpha\bar{A}}e^{\alpha\hat{F}(r)}\times V_H{}^2(Q_E{}^2)\frac{2M}{2\nu_E-Q_E{}^2}\sum_{i=1}^{n}C_{in}F_{1,2}(\tau_{in}),\tag{3.21}$$

where

$$F_{1,2}=\frac{\hat{W}_{1,2}(Q^2)V_p{}^2(Q^2)}{(1-z\tau)^{\alpha\bar{A}}(Q^2/Q_E{}^2)^2}\tag{3.22}$$

and $V_p{}^2$ takes into account the contribution of vacuum polarization as defined in Eq. (3.12). In Eq. (3.21), $\tau_{in}$ is related to the $i$th root $x_{in}$ of the Gegenbauer polynomial $P_n^{\alpha\bar{A}-1/2}(x)$ of degree $n$, through $x_{in}=2\tau_{in}-1$, while $C_{in}$ is proportional to the weight of the $x_{in}$ root. As shown in Appendix A, there are always, for $n\geq2$, two roots $\tau_{1n}=O(\alpha\bar{A})$ and $\tau_{nn}=1-O(\alpha\bar{A})$ with weight of order 1, while the other roots have weights of order $\alpha A$. Thus the leading contributions to Eq. (3.21) are from the $\tau_{1n}$ and $\tau_{nn}$ roots. We have given the analytical expressions of $\tau_{in}$ and $C_{in}$ up to $n=5$ in Appendix A and used these expressions in our first programs. We found a very rapid convergence with $n$. This is illustrated in Table I which we discuss after we derive the expressions for the hard factors.

### C. The hard factors

Let us first consider, along with Eq. (3.21), the elastic contribution to the structure functions one obtains in lowest-order perturbation theory in the soft-peaking approximation. The relative probability for one soft photon of momentum $k$ is covariantly given by[11]

$$\alpha I_1(k)=\alpha\bar{A}\int_{\bar{\lambda}}^{\infty}\frac{d\sigma}{\sigma}\delta^4(k-\sigma l)+\alpha\bar{A}\int_{\bar{\lambda}}^{\infty}\frac{d\sigma'}{\sigma'}\delta^4(k-\sigma' l').\tag{3.23}$$

Here, $\bar{A}$ comes from the angular integration over the direction of the emitted photon and $\bar{\lambda}$ is a quantity proportional to the photon mass regulator $\lambda$, whose exact expression is irrelevant for our discussion. $l$ and $l'$ are the positive-energy lightlike four-vectors defined in Eq. (2.9). The corresponding one-photon elastic contribution to the cross section is [compare Eq. (2.1)]

$$\frac{d\sigma^{1\gamma,el}}{d\Omega'dE'}=F_0\int\frac{d^4k}{(Q^2/Q_E{}^2)^2}\left[W_{2\,NR}+\frac{2(Q_E{}^2-2m^2)}{4EE'-Q_E{}^2}W_{1\,NR}\right]\times\alpha I_1(k),\tag{3.24}$$

where we have used Eq. (2.14) in lowest-order perturbation theory. Of course, $\nu$ and $Q^2$ are still

TABLE I. Elastic contribution to the $e$-$p$ $\nu W_2$ function in the peaking approximation for one-photon emission ($1\gamma$) and for multiphoton emission using Gegenbauer's polynomials of increasing degrees ($n=2$–$5$). $E=20$ GeV, $\theta=5°$, $E'_{\rm el}=18.499$.

| $E'$ (GeV) | $Q_E^2$ (GeV$^2$) | $x_E$ | $10^2\nu_E W_2^{\rm el}$ $1\gamma$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $10^3 M W_1^{\rm el}$ $1\gamma$ | $n=5$ |
|---|---|---|---|---|---|---|---|---|---|
| 18.48 | 2.813 | 0.986 | 3.378 | 2.235 38 | 2.235 38 | 2.235 38 | 2.235 38 | 64.66 | 42.79 |
| 18.46 | 2.810 | 0.972 | 1.695 | 1.178 82 | 1.178 82 | 1.178 82 | 1.178 82 | 32.00 | 22.26 |
| 18.40 | 2.800 | 0.933 | 0.7056 | 0.523 977 | 0.523 977 | 0.523 977 | 0.523 977 | 12.80 | 9.50 |
| 18.20 | 2.770 | 0.820 | 0.2726 | 0.218 862 | 0.218 862 | 0.218 862 | 0.218 862 | 4.36 | 3.50 |
| 18.00 | 2.740 | 0.730 | 0.1881 | 0.156 707 | 0.156 607 | 0.156 607 | 0.156 607 | 2.69 | 2.24 |
| 17.80 | 2.709 | 0.656 | 0.1533 | 0.130 759 | 0.130 760 | 0.130 760 | 0.130 760 | 1.98 | 1.68 |
| 17.50 | 2.663 | 0.568 | 0.1295 | 0.113 202 | 0.113 203 | 0.113 203 | 0.113 203 | 1.45 | 1.27 |
| 17.40 | 2.648 | 0.543 | 0.1250 | 0.110 021 | 0.110 022 | 0.110 022 | 0.110 022 | 1.34 | 1.18 |
| 17.30 | 2.633 | 0.520 | 0.1259 | 0.107 652 | 0.107 653 | 0.107 653 | 0.107 653 | 1.25 | 1.11 |
| 17.10 | 2.603 | 0.478 | 0.1171 | 0.104 726 | 0.104 729 | 0.104 729 | 0.104 729 | 1.11 | 0.99 |
| 16.90 | 2.572 | 0.442 | 0.1147 | 0.103 566 | 0.103 571 | 0.103 571 | 0.103 571 | 1.00 | 0.91 |
| 16.70 | 2.541 | 0.410 | 0.1140 | 0.103 705 | 0.103 712 | 0.103 712 | 0.103 712 | 0.93 | 0.84 |
| 16.50 | 2.511 | 0.382 | 0.1145 | 0.104 874 | 0.104 884 | 0.104 884 | 0.104 884 | 0.87 | 0.80 |
| 16.00 | 2.454 | 0.324 | 0.1201 | 0.111 406 | 0.111 432 | 0.111 432 | 0.111 432 | 0.77 | 0.72 |
| 15.50 | 2.359 | 0.279 | 0.1307 | 0.122 446 | 0.122 505 | 0.122 506 | 0.122 506 | 0.72 | 0.68 |
| 14.00 | 2.131 | 0.189 | 0.1946 | 0.185 313 | 0.185 722 | 0.185 728 | 0.185 728 | 0.72 | 0.69 |
| 12.50 | 1.903 | 0.135 | 0.3328 | 0.318 640 | 0.325 565 | 0.320 616 | 0.320 617 | 0.87 | 0.83 |
| 10.00 | 1.522 | 0.081 | 0.9824 | 0.938 396 | 0.955 583 | 0.956 511 | 0.956 554 | 1.46 | 1.44 |
| 7.50 | 1.442 | 0.049 | 3.4210 | 3.227 31 | 3.352 88 | 3.364 27 | 3.365 27 | 2.88 | 2.86 |
| 5.00 | 0.761 | 0.027 | 13.988 | 12.824 0 | 13.733 1 | 13.894 9 | 13.915 3 | 5.57 | 5.64 |

given by Eqs. (2.25) and (2.26) with $K\to k$. Using Eq. (3.23), the $k$ integral in Eq. (3.24) is trivially effected. We can cast the result in the form of Eq. (2.22) (with $C_1$ and $C_2$ set equal to zero) where, using Eqs. (3.1) and (3.2), the radiatively corrected structure functions are given by

$$W_{1,2}=\alpha\bar{A}M\int_{\bar\lambda}^{\infty}\frac{d\sigma}{\sigma}\,\delta\!\left(\nu-\frac{Q^2}{2}\right)\left(\frac{Q_E^{\,2}}{Q^2}\right)^2\hat{W}_{1,2}(Q^2)\bigg|_{k=\sigma l}+\alpha\bar{A}M\int_{\bar\lambda}^{\infty}\frac{d\sigma'}{\sigma'}\,\delta\!\left(\nu-\frac{Q^2}{2}\right)\left(\frac{Q_E^{\,2}}{Q^2}\right)^2\hat{W}_{1,2}(Q^2)\bigg|_{k=\sigma' l'}. \tag{3.25}$$

The $\delta$ functions in the last equation make the $\sigma$ or $\sigma'$ integrations also trivial. The relevant kinematics may be read from Eqs. (2.40)–(2.45) with $w^2=M^2$ and $\sigma'=K_-=0$ for $k=\sigma l$ and $\sigma=K_+=0$ for $k=\sigma' l'$. The values of $\sigma$, $\sigma'$, and $Q^2$ at the $\delta$-function peaks are thus

$$\sigma_p=\frac{2\nu_E-Q_E^{\,2}}{A_E}, \tag{3.26}$$

$$t_p\equiv Q_p^{\,2}=Q_E^{\,2}(1-\sigma_p)=\frac{2M\tilde{E}'Q_E^{\,2}}{A_E}, \tag{3.27}$$

$$\sigma'_{p'}=\frac{2\nu_E-Q_E^{\,2}}{A_{E'}}, \tag{3.28}$$

and

$$t_{p'}\equiv Q_{p'}^{\,2}=Q_E^{\,2}(1+\sigma'_{p'})=\frac{2M\tilde{E}Q_E^{\,2}}{A_{E'}}. \tag{3.29}$$

The final form of Eq. (3.25) is therefore

$W^{1\gamma,\rm el}$(soft-peaking)

$$=\alpha\bar{A}\frac{2M}{2\nu_E-Q_E^{\,2}}\left[\left(\frac{Q_E^{\,2}}{Q_p^{\,2}}\right)^2\hat{W}(Q_p^{\,2})+\left(\frac{Q_E^{\,2}}{Q_{p'}^{\,2}}\right)^2\hat{W}(Q_{p'}^{\,2})\right]. \tag{3.30}$$

We note that Eq. (3.30) may be obtained also from Eq. (3.21) in the limit $\alpha\bar{A}\to 0$. The peaking approximation to the exact one-photon-emission formula has been derived by many authors.[12-15] In our notation, the Bjorken[12] result can be expressed as

$$\alpha I_1(k)\to\alpha\bar{A}\int_{\bar\lambda}^{\infty}\frac{d\sigma}{\sigma}(1-\sigma+\tfrac{1}{2}\sigma^2)\delta^4(k-\sigma l)+\alpha\bar{A}\int_{\bar\lambda}^{\infty}\frac{d\sigma'}{\sigma'}(1+\sigma'+\tfrac{1}{2}\sigma'^2)\delta^4(k-\sigma' l'). \tag{3.31}$$

Consequently, Eq. (3.30) is modified to

$W^{1\gamma,\rm el}$(peaking)

$$=\frac{2M\alpha\bar{A}}{2\nu_E-Q_E^{\,2}}\left[\left(\frac{Q_E^{\,2}}{Q_p^{\,2}}\right)^2\hat{W}(Q_p^{\,2})H_p+\left(\frac{Q_E^{\,2}}{Q_{p'}^{\,2}}\right)^2\hat{W}(Q_{p'}^{\,2})H_{p'}(1+\sigma'_{p'})^2\right], \tag{3.32}$$

where $H_p$ and $H_{p'}$ are hard factors for the $p$ and $p'$ peaks,

$$H_p = 1 - \sigma_p + \frac{\sigma_p^2}{2} \tag{3.33}$$

and

$$H_{p'} = 1 - \frac{\sigma'_{p'}}{1+\sigma'_{p'}} + \frac{1}{2}\left(\frac{\sigma'_{p'}}{1+\sigma'_{p'}}\right)^2 . \tag{3.34}$$

It is easy enough to generalize the hard factors so that they can be used to correct for hard-photon effects in the nonperturbative formulas. For that purpose, we note that since $\sigma = K_+/Q_E$ and $\sigma' = K_-/Q_E$ reduce, respectively, to $\sigma_p$ and $\sigma'_{p'}$ at the one-photon-emission level, we can replace $\sigma_p$ by $\sigma$ and $\sigma'_{p'}$ by $\sigma'$ in the $H$ factors. Furthermore, comparison between Eqs. (3.3), (3.21), and (3.32) shows that we can, at least to lowest-order perturbation theory, introduce a hard factor *multiplicatively* in the nonperturbative formulas through

$$y_-^2 H = y_-^2 H_+ H_- . \tag{3.35}$$

Here,

$$1 - \sigma + \tfrac{1}{2}\sigma^2 = \tfrac{1}{2}(1 + x_+^2) , \tag{3.36}$$

$$1 - \frac{\sigma'}{1+\sigma'} + \frac{1}{2}\left(\frac{\sigma'}{1+\sigma'}\right)^2 = \frac{1}{2}(1 + x_-^2) , \tag{3.37}$$

and $y_-$, $x_+$, and $x_-$ have been defined in Eqs. (2.42) and (2.43).

Thus, in practical applications, we modify Eq. (3.22) to

$$F = \frac{\hat{W}(Q^2) V_p^2 y_-^2 H}{(1 - z\tau)^{\alpha\bar{A}} (Q^2/Q_E^2)^2} = \frac{\hat{W}(Q^2) V_p^2 H}{(1 - z\tau)^{\alpha\bar{A}} x_+^2} , \tag{3.38}$$

where the second form uses Eq. (2.47).

In Table I, we illustrate our results for $\nu_E W_2^{el}$ and $MW_1^{el}$ in electron-proton scattering in the peaking approximation for one-photon and multiphoton emission. The columns labeled $1\gamma$ correspond to one-photon emission and are obtained from Eq. (3.32). For $\nu_E W_2^{el}$, we present also the results of Eqs. (3.21) and (3.38) for $n$ increasing from 2 to 5. We see that for $x_E \gtrsim 0.1$, $n=2$ is already an excellent approximation to $\nu_E W_2^{el}$. The same fast convergence with $n$ is found for $W^{el}$ and for the cross section. For very inelastic scattering (small $x_E$ and large $Q_E^2$) one may need the roots and weights of Gegenbauer polynomials of degree higher than five, owing to the strong variation with $\tau$ of the integrand of Eq. (3.3). In this case, our present program finds the roots and weights numerically with good precision up to $n \simeq 25$. We must note, however, that slow convergence with $n$ in the very inelastic region is the signal for hard bremsstrahlung which we discuss later on.

In Table II we present the results for the elastic contribution to the $ep$ cross section. Column $1\gamma$ gives the results of the peaking approximation for one-photon emission using Eqs. (2.22) and (3.32) while the column labeled $n=5$ gives the results for multiphoton emission using Eqs. (2.22), (3.21) with $n=5$, and (3.38). Let us remark that the one-photon and multiphoton contributions to the cross section are quite different near the elastic peak and become comparable in the very inelastic region. The discrepancy between the one-photon peaking approximation and the exact result in the very inelastic region is not due to a failure of the peaking approximation but rather to a new physical contribution which comes into play, the hard bremsstrahlung. The exact one-photon cross sec-

TABLE II. Elastic contributions to $ep$ cross section $d\sigma/d\Omega' dE'$ in units of $10^{33}$ cm$^2$/GeV sr. $E=20$ GeV, $\theta=5°$, $E'_{el}=18.4995$, $F_0=3574.9\times10^{-33}$ cm$^2$. HCS and HCS′ are the values of the hard cross section given in Eq. (3.44) for $t_s=(t_{\min}t_p)^{1/2}$ and $t_s$ at the position of the minimum between the $t$ and $p$ peaks, respectively.

| $E'$ (GeV) | $Q_E^2$ (GeV$^2$) | $x_E$ | $1\gamma$ | Exact | HCS | HCS′ | $n=5$ |
|---|---|---|---|---|---|---|---|
| 18.495 | 2.815 | 0.9967 | 345.78 | 344.32 | 0.030 | 0.030 | 206.29 |
| 18.480 | 2.813 | 0.9861 | 79.92 | 79.58 | 0.079 | 0.079 | 52.89 |
| 18.46 | 2.810 | 0.9723 | 39.57 | 39.41 | 0.046 | 0.046 | 27.53 |
| 18.40 | 2.801 | 0.9328 | 15.86 | 15.80 | 0.028 | 0.028 | 11.78 |
| 17.50 | 2.664 | 0.5678 | 1.862 | 1.886 | 0.015 | 0.018 | 1.628 |
| 16.50 | 2.511 | 0.3824 | 1.176 | 1.258 | 0.049 | 0.070 | 1.077 |
| 14.50 | 2.207 | 0.2138 | 1.092 | 1.370 | 0.160 | 0.267 | 1.037 |
| 12.50 | 1.903 | 0.1352 | 1.593 | 2.237 | 0.346 | 0.654 | 1.534 |
| 10.00 | 1.522 | 0.081 | 3.523 | 5.083 | 0.789 | 1.679 | 3.430 |
| 7.50 | 1.141 | 0.049 | 9.805 | 13.437 | 1.831 | 4.203 | 9.645 |
| 5.00 | 0.761 | 0.027 | 33.38 | 42.732 | 5.094 | 12.07 | 33.21 |
| 2.50 | 0.381 | 0.011 | 169.60 | 207.7 | 24.58 | 58.57 | 170.1 |
| 1.50 | 0.228 | 0.0065 | 475.51 | 577.2 | 73.55 | 174.9 | 473.9 |

tion is discussed next, while the hard bremsstrahlung accompanied by soft photons will be treated in a forthcoming paper.

### D. Exact one-photon contribution to the elastic peak

The exact, *one-photon*-emission contribution to the lepton-proton cross section has been computed by Mo and Tsai.[13] This is the standard reference for radiative corrections in $ep$ and $\mu p$ scattering, currently used by experimentalists to extract the nonradiative structure functions from the data. In Appendix B, we give a brief summary of the derivation of the original Mo and Tsai formula, the needed results being quoted in Eqs. (B26) and (B27). In the general case, this equation involves a double integral, one over the squared momentum transfer $t$, the other over the squared mass $w^2$ produced at the nucleon vertex. Since, as we shall see, the importance of the bremsstrahlung cross section for small $x_E$ arises from the vicinity of small $t$, we shall limit ourselves for the rest of this paper to the elastic contribution in which the smallest values of $t$ may be reached.

Using Eqs. (3.1) and (3.2) into Eq. (B18), the latter equation reads

$$\frac{d\sigma_{\text{ex}}^{1\gamma,\text{el}}}{d\Omega' dE'} = \frac{\alpha^3 p'}{4\pi u p} \int_{t_{\min}}^{t_{\max}} \frac{dt}{t^2} [t_1 \hat{W}_1(t) + t_2 \hat{W}_2(t)] , \quad (3.39)$$

where $t_1$ and $t_2$ are components of the lepton tensor integrated over the azimuthal angle with $\vec{u} = \vec{p} - \vec{p}'$ as polar axis, given in Eqs. (B26) and (B27) with $w^2 = M^2$. From Eq. (B19) we have

$$t_{\max}^{\min} = \frac{Q_E{}^2 + 2(E-E')(E-E' \mp u)}{1 + \dfrac{E - E' \mp u}{M}} . \quad (3.40)$$

In the very inelastic region, the variable $u$ becomes

$$\begin{aligned} u &= [(E-E')^2 + Q_E{}^2]^{1/2} \\ &= (E-E') \left(1 + \frac{Q_E{}^2}{2(E-E')^2} - \frac{Q_E{}^4}{8(E-E')^4} + \cdots \right) \end{aligned} \quad (3.41)$$

and the minimum value of $t$ simplifies to

$$t_{\min} \simeq \frac{Q_E{}^4}{4(E-E')^2} = M^2 x_E{}^2 . \quad (3.42)$$

Comparison of Eqs. (3.42) and (3.27) shows that $t_{\min}$ could be smaller than $t_p = Q_p{}^2$ by two orders of magnitude or more. For example, in $e+p$ scattering, for $E = 20$ GeV, $E' = 10$ GeV, and $\theta = 5°$, we get $Q_E{}^2 = 1.52$ GeV$^2$, $t_{\min} = 5.8 \times 10^{-3}$ GeV$^2$, and $t_p = 7.9 \times 10^{-1}$ GeV$^2$. For $t$ reaching $t_{\min}$, the exchanged photon propagator leads to a third peak (besides the $p$ and $p'$ peaks already discussed) in the integrand of Eq. (3.39). The discrepancy between the exact and peaking one-photon cross sections has its origin in the existence of this third peak, ignored in the peaking approximation, which is called the $t$ peak. Figure 4 shows the graph of the integrand of Eq. (3.39),

$$\mathcal{F}(t) = \frac{t_1 \hat{W}_1(t) + t_2 \hat{W}_2(t)}{t^2} , \quad (3.43)$$

for $e+p$ scattering with $E = 20$ GeV, $\theta = 5°$, and $E' = 5$ GeV. We shall define the hard bremsstrahlung cross section as the integral of $\mathcal{F}(t)$ under the $t$ peak,

$$\frac{d\sigma^{1\gamma H}}{d\Omega' dE'} = \frac{\alpha^3 p'}{4\pi u p} \int_{t_{\min}}^{t_s} \frac{dt}{t^2} [t_1 \hat{W}_1(t) + t_2 \hat{W}_2(t)] , \quad (3.44)$$

where $t_s$ is a value of $t$ which separates the $t$ and $p$ peaks. In Table II, we list the hard cross section given in Eq. (3.44), in the columns labeled HCS and HCS' corresponding to $t_s = (t_{\min} t_p)^{1/2}$ and $t_s$ at the position of the minimum between the $t$ and $p$ peaks, respectively. The column labeled "exact" is the result of Eq. (3.39). Under the elastic peak ($x_E \gtrsim 0.9$, say), the hard bremsstrahlung cross section is negligible and the peaking ($1\gamma$) and the exact cross sections almost coincide. Thus in this domain of $x_E$, we may be confident in the peaking approximation and we can use the column $n = 5$ as the elastic contribution to the cross section.

For smaller values of $x_E$, the difference between the exact and ($1\gamma$) peaking approximation becomes sizable. To solve this difficulty, Mo and Tsai[13] decided to take into account the exact one-photon cross section for the elastic contribution and to modify the peaking approximation for the inelastic one. The problem then is how to take properly into account multiphoton effects. Since our ap-

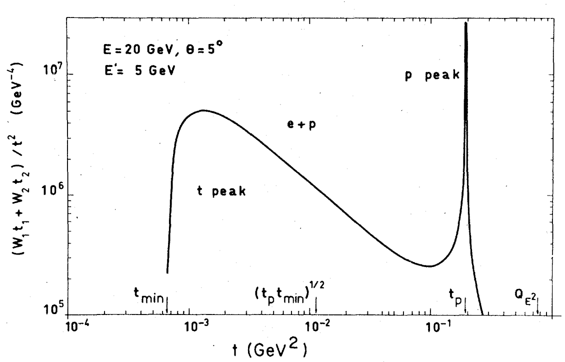


FIG. 4. Differential hard bremsstrahlung cross section versus the squared momentum transfer emphasizing the importance of $t$ peak for very inelastic scattering. The small-$p'$ peak is not shown.

proach to the radiative corrections is nonperturbative (by this we mean that we include the effect of soft photons to all orders), we shall treat this problem differently. Following Grammer and Yennie's[16] ideas, the contribution of the $t$ peak, which clearly originates from the emission of one hard photon, appears as a higher-order term in a rearranged perturbation series. Physically, the rearranged series instructs us to include first the cross section with the emission of no hard photon and an arbitrary number of soft photons, then the contribution of one hard photon plus an arbitrary number of soft photons, and so on. To take into account the soft photons which accompany the hard emitted one, one includes a spectral weight function in Eq. (B1) and generalizes the discussion of Sec. II. However, we must note the following problems associated with such a program.

(i) We do not know a nonperturbative form of the $\gamma ee\gamma$ vertices whose infrared-divergent part must cancel the infrared-divergent factor in the spectral weight function $E_\lambda$ introduced in I. In fact, one can assume that $\hat{E}$, given in Eq. (2.3), is still the right weight function to use and which takes into account the effects of real and virtual soft photons.

(ii) There is probably some double counting in this approach. However, this is not expected to be important since the $t$ and $p$ peaks are well separated when the hard bremsstrahlung contribution needs to be taken into account.

(iii) Since we are concerned with the small-$t$ domain, one may wonder if multiphoton exchange between the lepton and proton vertices leads to important contributions. In fact, infrared arguments show that these exchanges are characterized by the exponent of the order $(\alpha/\pi)\ln E/E'$, much smaller than $\alpha\bar{A}$ since it does not involve the lepton mass.[17] Consequently, multiphoton exhanges are expected to be small.

(iv) Computation using a rearranged perturbation series show that soft-photon effects in the hard bremsstrahlung cross section are important. We present these results in a forthcoming paper.

## IV. RADIATIVE CORRECTIONS TO CONTINUOUS SPECTRA

Using the formulas derived in Sec. II, we can now carry out the radiative correction for a continuous spectrum almost as simply as in the discrete (elastic) case. We discuss first, in general, how to compute the measured structure functions in terms of the nonradiative ones. We apply the resulting expressions to the radiative corrections of the $\Delta_{33}$ resonance and to the phenomenological $F_2(x)$ function derived by Feynman and Field.[18] From the formulas we obtain one can discuss the unfolding procedure,[13] that is, the way to get the nonradiative structure functions from the measured ones. Surprisingly enough, taking into account soft, multiphoton effects makes our results simpler than those of Mo and Tsai. Correspondingly, the unfolding procedure is simplified.

### A. General formula for a continuous spectrum

The discussion of radiative corrections to a continuous spectrum relies on Eq. (2.46) in which the factor $Hy_-^2$ must multiply $W_{1\,\mathrm{NR}}$ and $W_{2\,\mathrm{NR}}$. For orientation, we remind the reader that in the approximation where the lepton mass is neglected, $H_1 \overset{m}{\simeq} H_2 \overset{m}{\simeq} H = (H_+)(H_-)$, where $H_\pm = (1+x_\pm)^2/2$ and the variables $x_\pm$ and $y_-$ are given in Eqs. (2.42) and (2.43). After the elastic contribution to Eq. (2.46) has been subtracted, the lower limit in the $w^2$ integration is the nucleon-pion threshold: $M^2$ should be replaced by $\Sigma^2 = (M+m_\pi)^2$, where $m_\pi$ is the pion mass. In practice we shall let $\Sigma$ be any mass such that the contribution to the $W_i$ functions for $w^2 < \Sigma^2$ has been already subtracted.

For later convenience, we make the linear change of the $w$ integration variable

$$\xi = \frac{2w - (w_E + \Sigma)}{w_E - \Sigma}\,, \tag{4.1}$$

where the experimental variable $w_E$ is defined in (2.37), and Eq. (2.46) becomes

$$W_{1,2} = (\alpha\bar{A})^2 e^{\alpha\hat{F}(\tau)} V_H^{\,2} \left(\frac{4w_E^{\,2}(w_E-\Sigma)^2}{A_E A_{E'}}\right)^{\alpha\bar{A}} 2^{-\alpha A}$$
$$\times \int_0^1 \frac{d\tau}{[\tau(1-\tau)]^{(1-\alpha\bar{A})}} \int_{-1}^{+1} \frac{d\xi}{(1-\xi)^{1-\alpha A}} \; \frac{w/w_E}{[(w_E+w)/2w_E]^{1-\alpha A}} \; \frac{W_{1,2\mathrm{NR}} H_{1,2} V_p^{\,2}}{x_+^{\,2}[1-z(w^2)\tau]^{\alpha A}}\,. \tag{4.2}$$

Here, we have used Eq. (2.47) and introduced vacuum polarization to lowest order through the factor $V_p^{\,2}$ given in Eq. (3.12). Notice that $w^2$ and $Q^2$ are known explicit functions of $\tau$ and $\xi$.

We have already discussed, in Sec. III, the $\tau$ integration method using the roots of the Gegenbauer polynomials and the corresponding weights. The $\xi$ integration receives the most important contribution from the vicinity of $\xi = 1$, that is, $w = w_E$. Consequently, we use the Gauss-Jacobi integration method described briefly in Appendix A. For an approximation of degree $n$, the formula involves the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ with $\alpha=-1+\alpha A$ and $\beta=0$ and the corresponding weights $C_{\nu n}$. For $\alpha A$ small, there is always a root of the form $\xi=1-O(\alpha A/n)$, whose weight is of order one, the weights of the other roots being of order $\alpha A$. When the nonradiative structure functions are smooth enough, low values of $n$ give sufficient precision as we shall discuss later on with some examples. For simplicity, we have chosen in our numerical programs the degree $n'$ for the $\tau$ integration equal to $n+1$, where $n$ is the degree used for the $\xi$ integration. In general, for comparison we list the results for different values of $n$ from 1 to $n_{\max}\simeq 20$. (For higher values of $n_{\max}$, we cannot determine for the moment, with sufficient precision, the roots and weights.) Using the normalization and weights given in Appendix A, the approximation of degree $n$ to Eq. (4.2) is given by

$$W=e^{\alpha\hat{F}(r)}V_H{}^2\left[\frac{4w_E{}^2(w_E-\Sigma)^2}{A_EA_{E'}}\right]^{\alpha\bar{A}}\Gamma^{-1}(1+\alpha A)\Gamma^2(1+\alpha\bar{A})\tfrac{1}{4}\sum_{\nu=1}^{n}\sum_{\mu=1}^{n'}C_{\mu n'}(-1+\alpha\bar{A},\ -1+\alpha\bar{A})C_{\nu n}(-1+\alpha A,0)\mathcal{G}_{\nu n,\mu n'}\,. \tag{4.3}$$

Here, $W$ is $W_1$ or $W_2$, $\mathcal{G}$ is the corresponding integrand in Eq. (4.2),

$$\mathcal{G}=\frac{W_{\rm NR}HV_P{}^2}{x_+{}^2[1-z(w^2)\tau]^{\alpha\bar{A}}}\,\frac{w/w_E}{[(w+w_E)/2w_E]^{1-\alpha A}}\,, \tag{4.4}$$

$H$ is the hard factor defined in Eqs. (3.35)–(3.37), and $W_{\rm NR}$ the nonradiative structure function. The indices which affect the integrand remind us that it has to be evaluated at the roots of the Jacobi polynomials $P^{(-1+\alpha A,0)}(\xi)$ and $P_{n'}^{(-1+\alpha\bar{A},-1+\alpha\bar{A})}(2\tau-1)$,

$$\xi\rightarrow\xi_{\nu n},\quad \tau\rightarrow\tau_{\mu n'}\,. \tag{4.5}$$

Let us note that, according to Eq. (4.1), $w^2\rightarrow w_{\nu n}{}^2$ while, since $Q^2$ is a function of both $\tau$ and $\xi$ [see Eqs. (2.40)–(2.45) and (2.47)], $Q^2\rightarrow Q_{\nu n,\mu m'}{}^2$. Equations (4.3) and (4.4) are the basic formulas for radiative corrections of a continuous spectrum, the corresponding formulas for the elastic contribution being Eqs. (3.2) and (3.38). As an example of how Eq. (4.3) is used, we give in Table III the values of the weights $C_{\nu n}$ and $C_{\mu m'}$ for $n=8$ and $n'=9$ and also the relevant values of $w$ and $Q^2$. From this table, we see that the terms in Eq. (4.3) corresponding to ($\nu=8$, $\mu=1$) or ($\nu=8$, $\mu=9$), reminiscent of the $p$ and $p'$ peaks, have each a weight of 1.83 $\times$0.955 out of a total weight of 2 $\times$2. The next dominant terms will have $\nu=8$ or $\mu=1$ or $\mu=9$ with only one large weight. The remaining terms involve the product of two small weights. It is illuminating to draw, even schematically, the ($\nu$, $Q^2$) points with their corresponding weights, on the analog of Fig. 1, for the actual kinematics. The dominant points are located near the lines $K_+=0$, $K_-=0$. Among these points, two of them, located in the vicinity of the ($\nu_E$, $Q_E{}^2$) point will have the largest weights. Of course, one can get large contributions from nondominant points when it happens that the integrand of Eq. (4.4) is large for these points. This is the case, for example, for the elastic tail and to a lesser extent, for a resonance tail. Then, one must take into account these contributions separately since the Gauss-Jacobi integration method breaks down when the behavior of the integrand is irregular as a function of $w$ or $\xi$.

TABLE III. Values of $C_{\nu 8}$, $C_{\mu 9}$, $w$, and $Q^2$ for $\mu+p$, $E=219$ GeV. $w_E=16.94$, $Q_E{}^2=4.85$ GeV$^2$, $\omega=60$. The third through tenth columns give values of $Q^2$ in GeV$^2$ for the indicated $\mu$ and $\nu$.

| | $w$ | 1.43 | 2.89 | 5.28 | 8.24 | 11.32 | 14.06 | 16.03 | 16.934 |
|---|---|---|---|---|---|---|---|---|---|
| | $C_{\nu 8}$ | 0.003 | 0.007 | 0.011 | 0.017 | 0.025 | 0.038 | 0.072 | 1.829 |
| $C_{\mu 9}$ | $\mu \backslash \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.9555 | 1 | 1.44 | 1.51 | 1.75 | 2.23 | 2.95 | 3.78 | 4.49 | 4.848 |
| 0.020 | 2 | 1.79 | 1.87 | 2.10 | 2.57 | 3.24 | 3.97 | 4.56 | 4.848 |
| 0.012 | 3 | 2.71 | 2.78 | 3.00 | 3.40 | 3.93 | 4.41 | 4.73 | 4.849 |
| 0.009 | 4 | 4.34 | 4.39 | 4.53 | 4.77 | 5.00 | 5.06 | 4.96 | 4.851 |
| 0.009 | 5 | 6.71 | 6.71 | 6.71 | 6.65 | 6.39 | 5.87 | 5.24 | 4.853 |
| 0.009 | 6 | 9.61 | 9.54 | 9.32 | 8.83 | 7.94 | 6.73 | 5.52 | 4.855 |
| 0.012 | 7 | 12.6 | 12.4 | 11.9 | 11.0 | 9.42 | 7.51 | 5.77 | 4.856 |
| 0.020 | 8 | 14.9 | 14.7 | 14.0 | 12.6 | 10.5 | 8.08 | 5.94 | 4.857 |
| 0.955 | 9 | 16.02 | 15.77 | 15.01 | 13.44 | 11.07 | 8.35 | 6.03 | 4.858 |

### B. Radiative corrections for the $\Delta_{33}$ resonance

We have applied the preceding formalism to the radiative corrections of the $\Delta_{33}$ resonance cross section using the nonradiative structure functions given by Mo and Tsai.[13] In our notations, these functions are

$$W_{1\mathrm{NR}}=\frac{\vec{q}^{\,2}}{Q^2}W_{2\mathrm{NR}}=F_{BW}(w^2)\vec{q}^{\,2}2M^2C_3^{\,2}(Q^2)\frac{(E_i^*+M)}{3M}. \tag{4.6}$$

Here $F_{\mathrm{BW}}$ is the Breit-Wigner resonance function

$$F_{\mathrm{BW}}(w^2)=\frac{\Gamma M_{33}}{\pi[(w^2-M_{33}{}^2)^2+\Gamma^2 M_{33}{}^2]}, \tag{4.7}$$

where $M_{33}=1.236$ GeV is the mass, $\Gamma$ the width of the $\Delta_{33}$ resonance, $\vec{q}^{\,2}$ is the squared three-momentum of the exchanged photon in the laboratory frame, covariantly given by

$$\vec{q}^{\,2}=\left(\frac{w^2+Q^2-M^2}{2M}\right)^2+Q^2, \tag{4.8}$$

and $E_i^*$ is the incident proton energy in the $\Delta_{33}$ rest frame

$$E_i^*=\frac{w^2+M^2+Q^2}{2w}.$$

The squared $\Delta_{33}$ form factor and the width $\Gamma$ are chosen according to Mo and Tsai[13]:

$$M^2C_3{}^2(Q^2)=(2.05)^2\exp(-6.3Q)(1+9.0Q), \tag{4.9}$$

$$\Gamma=\frac{(0.1293\ \mathrm{GeV})[0.85(p^*/m_\pi)]^3}{1+[0.85(p^*/m_\pi)]^2}, \tag{4.10}$$

where

$$p^{*2}=\left(\frac{w^2-M^2+m_\pi{}^2}{2w}\right)^2-m_\pi{}^2. \tag{4.11}$$

In Fig. 5, we give two examples of our results for the $e+p\to e+\Delta_{33}$ cross section, including the elastic tail from multiphoton emission discussed in Sec. III. The energies and angles are the same as those chosen by Mo and Tsai so that the reader may compare our Fig. 5 with Fig. 2 of Ref. 13. Unfortunately, our *nonradiative* cross-section curves, although using the Mo-Tsai parametrization, are already in disagreement with the corresponding curves of these authors. Since Figs. 20 and 7 of Ref. 13 seem to be inconsistent (compare the cross sections at $E'=17.5$ GeV), we cannot push the comparison further.

Let us note that, using Eq. (4.3) in our program, we have printed for every point the results for $n=1,\ldots,12$. When the point is in the central resonance region, convergence is achieved already for $n=3$ or 4. For points belonging to the tail of the resonance, convergence becomes slow. In this case, it is advisable to choose for $\Sigma$ a value between $M_{33}$ and $w_E$ and then use Eq. (4.2) to compute separately the contribution under the resonance.

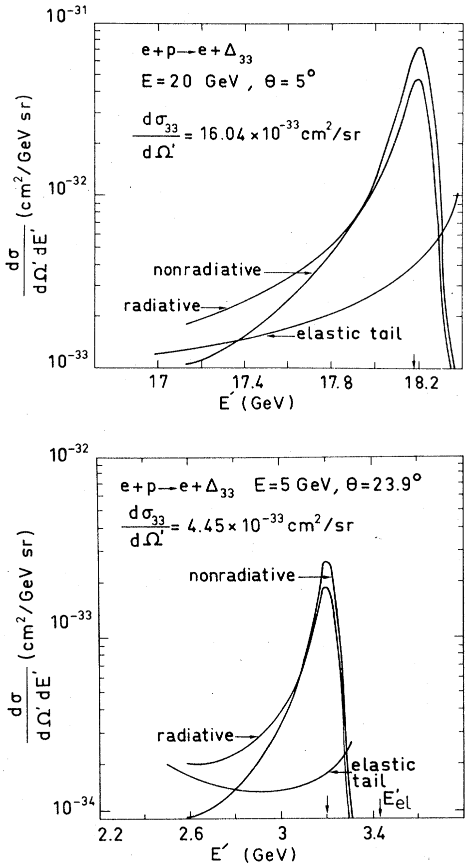


FIG. 5. Radiative corrections for the $\Delta_{33}$ resonance. The elastic radiative tail is also shown for comparison.

### C. Radiative corrections to $F_2$

Recently, two experimental groups extracted from their data the structure function $F_{2\mathrm{NR}}=\nu W_{2\mathrm{NR}}$ at energies above 200 GeV. In the first experiment,[19] the incident muon energy is $E=219$ GeV and the largest $\omega=1/x_E$ is $10^3$. In the second experiment,[20] the incident muon energy is $E=270$ GeV and the smallest $x_E$ is 0.04. As usual, radiative corrections were applied to the data before extracting the structure functions. Unfortunately, the magnitude of these corrections is not indicated. Since we do not have at our disposal the raw data, we cannot compare our radiative corrections with those used in these papers. However, we have computed the radiatively corrected structure function $F_2=\nu_E W_2$ using as input the scale-invariant

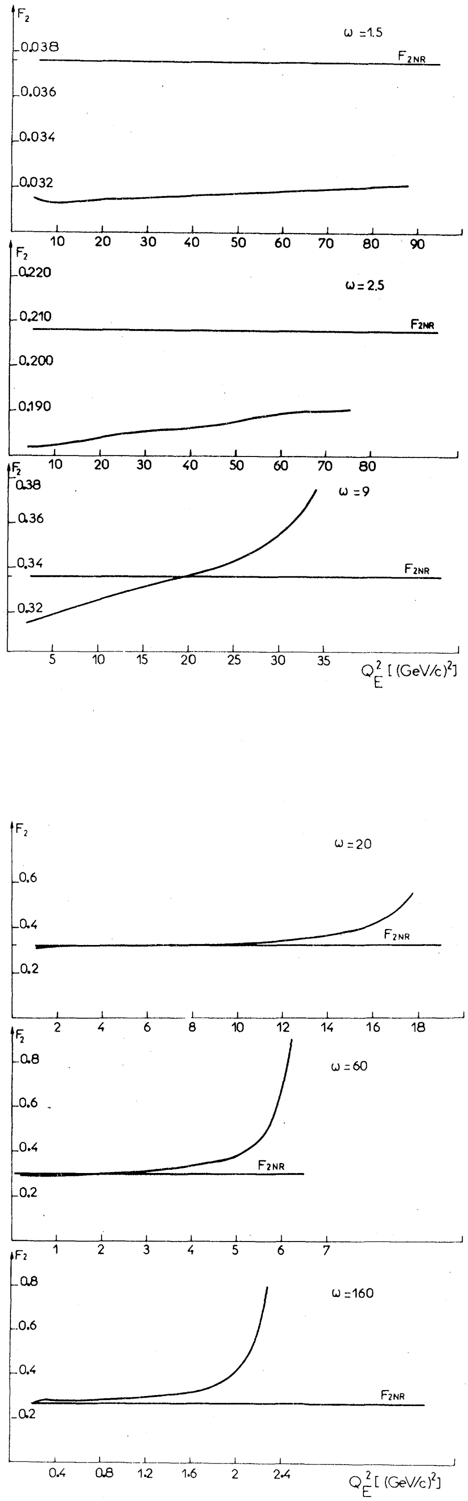


FIG. 6. Radiative corrections for $\mu+p$ scattering at $E=219$ GeV for various $\omega$ values. The horizontal lines represent the Feynman scale-invariant, nonradiative structure function.

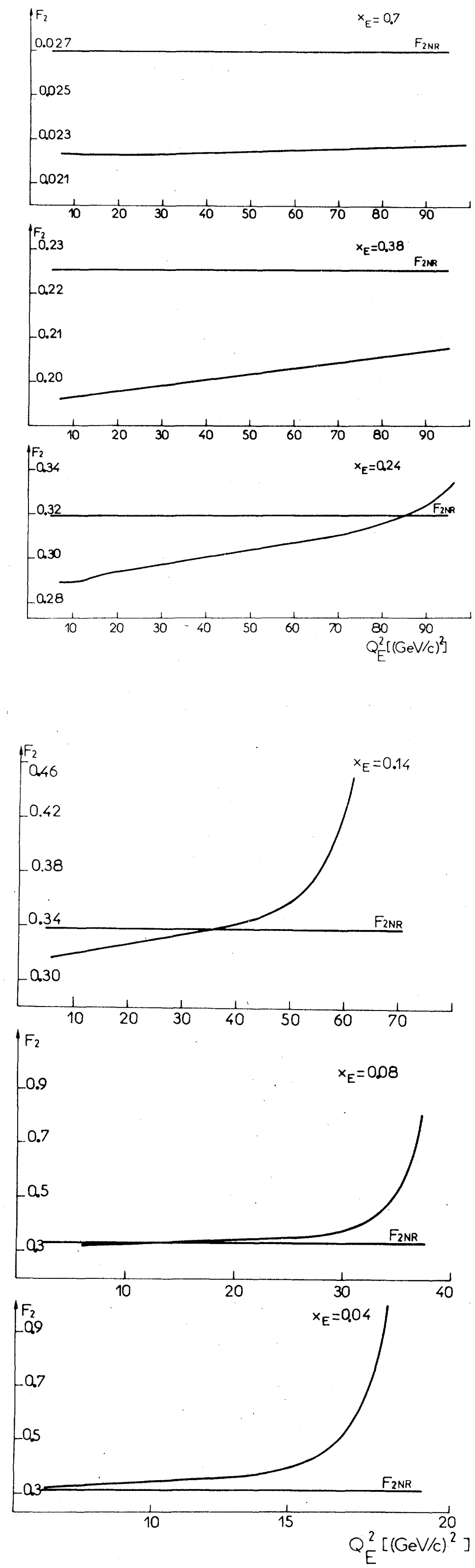


FIG. 7. Radiative corrections for $\mu+p$ scattering at $E=270$ GeV for various $x_E$ values.

structure function $F_{2\mathrm{NR}}$ of Fields and Feynman,[18]

$$F_{2\mathrm{NR}}(x)=x[\tfrac{4}{9}(u+\bar{u})+\tfrac{1}{9}(d+\bar{d})+\tfrac{1}{9}(s+\bar{s})]\,, \qquad (4.12)$$

where

$$\begin{aligned}xu(x)=0.17(1-x)^3[1&+41.88x+1412.38x^2+1276.42x^3\\&+x^{1/2}(9.09-389.89x-2080.05x^2\\&-258.07x^3)]\,,\\ x\bar{u}(x)=0.17(1-x)^{10}&\,,\\ xd(x)=0.17(1-x)^4[1&-14.11x-139.39x^2+126.87x^3\\&+x^{1/2}(5.48+112.51x-37.93x^2\\&-46.49x^3)]\,,\end{aligned} \qquad (4.13)$$

$$x\bar{d}(x)=0.17(1-x)^7\,,$$

$$xs(x)=x\bar{s}(x)=0.1(1-x)^8\,.$$

The results of using Eqs. (4.12) and (4.13) into (4.3) and (4.4) are illustrated, for the two experiments, in Figs. 6 and 7. These radiative corrections include neither the radiative tail from the elastic peak nor the hard bremsstrahlung which must be computed separately. These figures show in particular the steep rise of $F_2$ for small fixed $x_E$ and increasing $Q_E{}^2$, that is, in the very inelastic region.

Our discussion of the unfolding procedure,[13] that is, the procedure for extracting the nonradiative structure functions from the measured cross section, will be very sketchy. The reason is that experimentalists found it more practical[21] to adjust a parametrized form of the nonradiative structure functions, on which radiative corrections are applied, until agreement with the data is reached. In this case, Eqs. (4.3) and (4.4) need not be inverted, although inversion is possible similarly to the procedure described in Ref. 13.

Two other important parts of radiative corrections, the straggling and ionization in the target, are not considered in this paper since these subjects are very clearly exposed in Ref. 13 and can be incorporated in our formulas.

To conclude, we emphasize that the nonperturbative approach described in this paper, which takes into account mainly collinear photon emission, is expected to be the dominant contribution to the internal bremsstrahlung, except in the very inelastic region. In that region, the $t$ peak is important and hard bremsstrahlung must be included as a separate effect. We intend to discuss this subject from a nonperturbative point of view in a forthcoming paper.

## ACKNOWLEDGMENTS

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## APPENDIX A: GAUSS-JACOBI INTEGRATION METHOD

Our formulas for radiative corrections involve integrals of the form

$$I(f;\alpha,\beta)=\frac{2^{-\alpha-\beta}\,\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{+1}dx(1-x)^{\alpha}(1+x)^{\beta}f(x)\,, \qquad (\mathrm{A}1)$$

where $f$ is assumed to be a smooth function on the integration interval. The factors introduced in the definition of $I$ make the final formulas simpler since the normalization of $I$ is

$$I(1;\alpha,\beta)=2\,. \qquad (\mathrm{A}2)$$

Equation (A2) results from the definition of Euler's $B$ function. In practice, we shall need these integrals for $\alpha=\beta=-1+\alpha\bar{A}$ (elastic tail) and for $\alpha=-1+\alpha A$, $\beta=0$ (for a continuous spectrum). The Gauss-Jacobi integration method[22] is suited for the computation of these integrals.

Let $x_{\nu n}$ be the zeros of $P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree $n$. These polynomials are orthogonal on the $[-1,1]$ interval with the weight function $(1-x)^{\alpha}(1+x)^{\beta}$. In the Gauss method of mechanical quadrature, one shows that if $f(x)$ is approximated by an osculating polynomial of degree $2n-1$ through the points $x_{\nu n}$, the following formula holds:

$$I(f;\alpha,\beta)=\sum_{\nu=1}^{n}C_{\nu n}(\alpha,\beta)f(x_{\nu n})\,. \qquad (\mathrm{A}3)$$

Here, $C_{\nu n}$ is the weight attached to the $x_{\nu n}$ root which is proportional to $\lambda_{\nu n}$, the Christoffel or cote number for the Jacobi polynomial. Taking into account the normalization of Eq. (A1) and the known expression[22] of $\lambda_{\nu n}$ we get

$$C_{\nu n}(\alpha,\beta)=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\beta+1)(\beta+2)\cdots(\beta+n-1)(1-x_{\nu n}{}^2)(\alpha+\beta+2n)^2}{2(\alpha+n)(\beta+n)n!\,(\alpha+\beta+2)(\alpha+\beta+3)\cdots(\alpha+\beta+n)[P_{n-1}^{(\alpha,\beta)}(x_{\nu n})]^2}\,. \qquad (\mathrm{A}4)$$

When the function $f$ is smooth enough, the use of a low-degree Jacobi polynomial in Eq. (A3) is sufficient to get very good precision in the mechanical quadrature. Since the reader may like to use our formulas without being involved in a computer program, we shall give the analytical expressions of the roots and weights for the lowest values of $n$ and for the two cases of interest which we have computed by hand.

### 1. The Gegenbauer case, $\alpha = \beta$

When $\alpha=\beta$, the Jacobi polynomials are proportional to the Gegenbauer polynomials $P_n^\lambda(x)$ with $\lambda=\alpha+\frac{1}{2}$. Since these polynomials have definite parity, we have computed "by hand" the $x_{\nu n}$ and $C_{\nu n}$ up to $n=5$ and checked the results by taking $f=1,x,x^2\cdots$. Defining

$$\alpha=-1+\bar{\alpha} \tag{A5}$$

we get

$n=2$:

$$x_{12}=-(1+2\bar{\alpha})^{-1/2}=-x_{22}\,,\qquad C_{12}=C_{22}=1; \tag{A6}$$

$n=3$:

$$x_{13}=-(1+2\bar{\alpha}/3)^{-1/2}=-x_{33},\quad x_{23}=0\,,\qquad C_{13}=C_{33}=\frac{1+2\bar{\alpha}/3}{1+2\bar{\alpha}}\,,\quad C_{23}=\frac{8\bar{\alpha}}{3(1+2\bar{\alpha})}; \tag{A7}$$

$n=4$:

$$x_{\substack{1\\2}4}=-\left(\frac{3\pm2[(1+\bar{\alpha})/(1+2\bar{\alpha}/3)]^{1/2}}{5+2\bar{\alpha}}\right)^{1/2}=-x_{\substack{4\\3}4}\,,\qquad C_{\substack{1\\2}4}=\frac{\frac{1}{2}+\bar{\alpha}\pm(\frac{1}{2}-\bar{\alpha})[(1+2\bar{\alpha}/3)/(1+\bar{\alpha})]^{1/2}}{1+2\bar{\alpha}}=C_{\substack{4\\3}4}; \tag{A8}$$

$n=5$:

$$\begin{aligned} x_{\substack{1\\2}5}&=-\left(\frac{5\pm2[(1+\bar{\alpha})/(1+2\bar{\alpha}/5)]^{1/2}}{7+2\bar{\alpha}}\right)^{1/2}=-x_{\substack{5\\4}5},\quad x_{35}=0\,,\\ C_{\substack{1\\2}5}&=\frac{[1+(2\bar{\alpha}/5)]\{[1+(14\bar{\alpha}/9)]\pm[1+(4\bar{\alpha}/9)](1-2\bar{\alpha})/(1+\bar{\alpha})[(1+\bar{\alpha})/(1+2\bar{\alpha}/5)]^{1/2}\}}{2(1+2\bar{\alpha})(1+2\bar{\alpha}/3)}=C_{\substack{5\\4}5}\,,\\ C_{35}&=\frac{64}{45}\,\frac{\bar{\alpha}(1+\bar{\alpha})}{(1+2\bar{\alpha})(1+2\bar{\alpha}/3)}\,. \end{aligned} \tag{A9}$$

It is important to note that for small $\bar{\alpha}$ (the value of $\bar{\alpha}$ will be in fact $\alpha A$), for a degree $n\geq2$, there are always two roots in the vicinity of $\pm1$ with weight of order 1 whereas the other roots have weight of order $\bar{\alpha}$. With the change of variable $x=2\tau-1$, Eq. (A1) reads

$$I(f;\ \alpha,\alpha)=2\Gamma(2\bar{\alpha})\Gamma^{-2}(\bar{\alpha})\int_0^1 d\tau[\tau(1-\tau)]^{\bar{\alpha}-1}f(x(\tau))\,. \tag{A10}$$

### 2. The Jacobi case with $\beta = 0$

Here, we have computed the roots and weights for $n=1$ and $n=2$ only. We have used a program for higher values of $n$. In many cases, $n=2$ gives already precise results. Defining

$$z=\frac{x-1}{2} \tag{A11}$$

the results are

$n=1$:

$$z_{11}=-\frac{\alpha+1}{\alpha+2},\quad C_{11}=2\,, \tag{A12}$$

$n=2$:

$$z_{\substack{1\\2}2}=\frac{-(\alpha+2)(\alpha+3)\pm[2(\alpha+2)(\alpha+3)]^{1/2}}{(\alpha+3)(\alpha+4)}\,,\qquad C_{\substack{1\\2}2}=\frac{(\alpha+2)[2(\alpha+2)]^{1/2}\mp\alpha(\alpha+3)^{1/2}}{(\alpha+2)[2(\alpha+2)]^{1/2}}\,. \tag{A13}$$

We emphasize that when $\alpha\to-1+\alpha A$ with $\alpha A$ small, there is always for $n\geq1$ one root in the vicinity of $x=1$ with weight of order 2, the other roots having weights of order $\alpha A$.

## APPENDIX B: A REMINDER OF THE EXACT ONE-PHOTON-EMISSION FORMULA

For the reader's convenience, we give in this appendix a brief summary of the Mo and Tsai[13] formula. To lowest-order perturbation theory, the exact one-photon-emission contribution to the cross section is given by

$$\frac{d\sigma_{\text{ex}}^{1\gamma}}{d\Omega' dE'}=\frac{\alpha^2p'}{p}\,\frac{\alpha}{2\pi^2}\int\frac{d^3k}{2\omega(q^2)^2}\,W_{\text{NR}}^{\mu\nu}T_{\mu\nu}(p,p',k)\,. \tag{B1}$$

Here, $W^{\mu\nu}$ is the proton structure tensor defined in Eq. (2.13) and $T^{\mu\nu}$ is the lepton tensor describing one-photon emission from the lepton vertex (not necessarily soft), whose normalization is such that

$$\lim_{k\to 0} T^{\mu\nu}(p,p',k)=(-j^\rho j_\rho)X^{\mu\nu}\big|_{\alpha=0}\,, \tag{B2}$$

where the classical current $j^\rho$ is given by $j^\rho = p'^\rho/k\cdot p' - p^\rho/k\cdot p$ and thus

$$-j^2=-j^\rho j_\rho=\frac{2p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p)^2}-\frac{m^2}{(k\cdot p')^2}\,. \tag{B3}$$

From Eq. (2.12) with $\alpha=0$ we get

$$X^{\mu\nu}\big|_{\alpha=0}=2(m^2-p\cdot p')g^{\mu\nu}+2(p^\mu p'^\nu+p^\nu p'^\mu)\,. \tag{B4}$$

The explicit expression of $T^{\mu\nu}$ is easily obtained and reads

$$T^{\mu\nu}=\tfrac{1}{2}\sum_{(\epsilon)}\mathrm{Tr}\left\{(\not p'+m)\left[\frac{\not\epsilon(\not p'+\not k+m)\gamma^\mu}{2k\cdot p'}-\frac{\gamma^\mu(\not p-\not k+m)\not\epsilon}{2k\cdot p}\right](\not p+m)\left[\frac{\gamma^\nu(\not p'+\not k+m)\not\epsilon}{2k\cdot p'}-\frac{\not\epsilon(\not p-\not k+m)\gamma^\nu}{2k\cdot p}\right]\right\}, \tag{B5}$$

where $\epsilon$ is the photon polarization vector. Computing the trace and the sum over photon polarizations, the result can be written as

$$\begin{aligned}T^{\mu\nu}=\;&2(-j^2)[(m^2-p\cdot p')g^{\mu\nu}+(p^\mu p'^\nu+p^\nu p'^\mu)]\\&-2g^{\mu\nu}\left[(2p\cdot p'+m^2)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+m^2\left(\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}\right)+\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right]\\&+2(p^\mu p'^\nu+p^\nu p'^\mu)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+\frac{4p'^\mu p'^\nu}{k\cdot p'}-\frac{4p^\mu p^\nu}{k\cdot p}\\&-2\left(\frac{p'^\mu k^\nu+p'^\nu k^\mu}{k\cdot p}\right)\left(\frac{p\cdot p'}{k\cdot p'}-\frac{m^2}{k\cdot p}-1\right)+2\left(\frac{p^\mu k^\nu+p^\nu k^\mu}{k\cdot p'}\right)\left(\frac{p\cdot p'}{k\cdot p}-\frac{m^2}{k\cdot p'}+1\right)-\frac{4m^2k^\mu k^\nu}{(k\cdot p)(k\cdot p')}\,.\end{aligned} \tag{B6}$$

In this equation, the term involving the classical current represents the soft part of the lepton tensor. We have checked that Eq. (B6) satisfies the identities

$$q^\mu T_{\mu\nu}=q^\nu T_{\mu\nu}=0\,, \tag{B7}$$

which result from gauge invariance. We noted in I that $q_\mu X^{\mu\nu}=q_\nu X^{\mu\nu}=0$ only in the limit $K\to 0$. Using the complete expression of $T^{\mu\nu}$, gauge invariance is maintained in the domain where the emitted photons are not necessarily soft, at least at the one-emitted-photon level.

Taking into account Eqs. (2.13) and (B6) we can write Eq. (B1) in the form

$$\frac{d\sigma^{1\gamma}_{\mathrm{ex}}}{d\Omega' dE'}=\frac{\alpha^3 p'}{2\pi^2 p}\int\frac{d^3k}{2\omega}\,\frac{W_{1\mathrm{NR}}T_1+W_{2\mathrm{NR}}T_2}{(q^2)^2}\,, \tag{B8}$$

where

$$\begin{aligned}T_1&\equiv -g^{\mu\nu}T_{\mu\nu}\\&=-4(j^2)(p\cdot p'-2m^2)+8p\cdot p'\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+4\left(\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right)+4m^2\left(\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}+\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)\end{aligned} \tag{B9}$$

and

$$\begin{aligned}T_2&\equiv\frac{P^\mu P^\nu T_{\mu\nu}}{M^2}\\&=-2j^2\left(\frac{2(P\cdot p)(P\cdot p')}{M^2}-p\cdot p'+m^2\right)-2\left[(2p\cdot p'+m^2)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+m^2\left(\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}\right)+\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right]\\&\quad-4\frac{(P\cdot p)(P\cdot p')}{M^2}\left(\frac{1}{k\cdot p}-\frac{1}{k\cdot p'}\right)-\frac{4(P\cdot p)^2}{M^2k\cdot p}+\frac{4(P\cdot p')^2}{M^2k\cdot p'}-\frac{4(P\cdot p')(P\cdot k)}{M^2}\left(\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p)^2}-\frac{1}{k\cdot p}\right)\\&\quad+\frac{4(P\cdot p)(P\cdot k)}{M^2}\left(\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p')^2}+\frac{1}{k\cdot p'}\right)-\frac{4m^2(P\cdot k)^2}{M^2(k\cdot p)(k\cdot p')}\,.\end{aligned} \tag{B10}$$

Equation (B8) can be written in a more manageable form, suited for numerical computations. Let us introduce in this equation as integration variables, the squared momentum transfer

$$t \equiv Q^2 = -(p - p' - k)^2 = Q_E{}^2 + 2\omega_L(E - E' - u\cos\theta_{kL}) \tag{B11}$$

and the hadronic invariant mass

$$w^2 = (P+q)^2 = -t + M^2 + 2M(E - E' - \omega_L)\,, \tag{B12}$$

where $\vec{u} = \vec{p} - \vec{p}'$ is a three-vector taken as the $z$ axis in the laboratory frame, $\omega_L$ is the laboratory energy, and $\theta_{kL}$ is the angle with respect to $\vec{u}$, of the momentum of emitted photon. In terms of the usual laboratory variables, $u = |\vec{u}|$ is given by

$$u = [(E - E')^2 + Q_E{}^2]^{1/2}\,. \tag{B13}$$

Using Eqs. (B11) and (B12) and also

$$\frac{1}{k\cdot p} - \frac{1}{k\cdot p'} = \frac{k\cdot(p' - p)}{k\cdot p k\cdot p'} = \frac{Q_E{}^2 - t}{2k\cdot p k\cdot p'}\,, \tag{B14}$$

one notes in Eq. (B10) an important cancellation of the large terms, those proportional to $p\cdot p'(E^2 + E'^2)/(k\cdot p)(k\cdot p')$. After some elementary algebra, Eqs. (B9) and (B10) become

$$T_1 = 2m^2(2m^2 - t)\left(\frac{1}{(k\cdot p)^2} + \frac{1}{(k\cdot p')^2}\right) + 8 + \frac{4[(t/2 + m^2)^2 + p\cdot p'(p\cdot p' - 4m^2)]}{k\cdot p k\cdot p'} \tag{B15}$$

and

$$\begin{aligned} T_2 = &-\frac{2m^2}{(k\cdot p)^2}\left[2E'\left(E' + \frac{t + w^2 - M^2}{2M}\right) - \frac{t}{2}\right] - \frac{2m^2}{(k\cdot p')^2}\left[2E\left(E - \frac{t + w^2 - M^2}{2M}\right) - \frac{t}{2}\right] - 4 \\ &+ \frac{2}{k\cdot p k\cdot p'}\left\{-\frac{Q_E{}^4}{4} - \frac{p\cdot p'(E - E')(w^2 - M^2)}{M} + t\left[E^2 + E'^2 - \frac{t}{4} - \frac{p\cdot p'(E - E')}{M}\right] + m^2(2E^2 + 2E'^2 - t - 2\omega_L{}^2)\right\} \\ &- \frac{2(w^2 - M^2 + t)}{M}\left(\frac{E}{k\cdot p} + \frac{E'}{k\cdot p'}\right). \end{aligned} \tag{B16}$$

Equations (B15) and (B16) reproduce the results of Mo and Tsai.

When $t$ and $w^2$ are taken as integration variables in Eq. (B8), the nonradiative structure functions do not depend on the azimuthal angle $\varphi_k$ of the emitted photon. Upon introducing

$$t_i = \frac{1}{2\pi}\int_0^{2\pi} T_i d\varphi_k\,, \quad i = 1, 2 \tag{B17}$$

and computing the Jacobian for the $(\omega_L, \cos\theta_{kL}) \to (w^2, t)$ transformation, Eq. (B8) reads

$$\frac{d\sigma_{\text{ex}}^{1\gamma}}{d\Omega' dE'} = \frac{\alpha^3 p'}{8\pi M p u} \times \int_{M^2}^{w_E{}^2} dw^2 \int_{t_{\min}}^{t_{\max}} \frac{dt}{t^2}(t_1 W_{1\text{NR}} + t_2 W_{2\text{NR}})\,, \tag{B18}$$

where

$$t_{\max}^{\min} = \frac{Q_E{}^2 + 2(E - E' \mp u)[E - E' - (w^2 - M^2)/2M]}{1 + (E - E' \mp u)/M} \tag{B19}$$

and $w_E{}^2$ is given in Eq. (2.37).

To compute the functions $t_i$ associated with the lepton vertex, we define

$$a = E - p\cos\theta_p\cos\theta_{kL}\,, \quad a' = E' - p'\cos\theta_{p'}\cos\theta_{kL}\,, \tag{B20}$$

and

$$b = -p\sin\theta_{kL}\sin\theta_p = b' = -p'\sin\theta_{kL}\sin\theta_{p'}\,. \tag{B21}$$

Here, $\theta_p$ and $\theta_{p'}$ are the angles formed by $p$ and $p'$, respectively, with the $\vec{u}$ vector (in the laboratory frame). From simple kinematics one gets

$$\cos\theta_p = \frac{p - p'\cos\theta}{u}\,, \quad \cos\theta_{p'} = \frac{p\cos\theta - p'}{u} \tag{B22}$$

and

$$\sin\theta_p = p'\sin\theta/u\,, \quad \sin\theta_{p'} = p\sin\theta/u\,. \tag{B23}$$

In practical applications, we use the second form of the formula

$$\int_0^{2\pi} \frac{d\varphi_k}{(\hat{k}\cdot p)(\hat{k}\cdot p')} = \frac{2\pi}{a-a'}\left(\frac{1}{S'}-\frac{1}{S}\right) = \frac{2\pi(a+a')}{SS'(S+S')}, \tag{B24}$$

where

$$S=(a^2-b^2)^{1/2} \text{ and } S'=(a'^2-b^2)^{1/2}, \tag{B25}$$

which avoids the spurious $a=a'$ singularity. From Eq. (B17) we get

$$t_1 = \frac{2m^2(2m^2-t)}{\omega_L^2}\left(\frac{a}{S^3}+\frac{a'}{S'^3}\right) + \frac{4(a+a')[(m^2+t/2)^2+p\cdot p'(p\cdot p'-4m^2)]}{\omega_L^2 SS'(S+S')} + 8 \tag{B26}$$

and

$$t_2 = -\frac{2m^2 a}{\omega_L^2 S^3}\left[2E'\left(E'+\frac{t+w^2-M^2}{2M}\right)-\frac{t}{2}\right] - \frac{2m^2 a'}{\omega_L^2 S'^3}\left[2E\left(E-\frac{t+w^2-M^2}{2M}\right)-\frac{t}{2}\right]-4$$
$$+\frac{2(a+a')}{\omega_L^2 SS'(S+S')}\left[-\frac{Q_E^4}{4}-\frac{p\cdot p'(E-E')(w^2-M^2)}{M}+t\left(E^2+E'^2-\frac{t}{4}-p\cdot p'\frac{(E-E')}{M}\right)+m^2(2E^2+2E'^2-t-2\omega_L^2)\right]$$
$$-\frac{2(w^2-M^2+t)}{M\omega_L}\left(\frac{E}{S}+\frac{E'}{S'}\right). \tag{B27}$$

The width of the $p$ and $p'$ peaks appear if we make more explicit Eq. (B25):

$$S = p\left[\left(\cos\theta_{kL}-\frac{E}{p}\cos\theta_p\right)^2+\frac{m^2p'^2\sin^2\theta}{p^2u^2}\right]^{1/2}, \tag{B28}$$

$$S' = p'\left[\left(\cos\theta_{kL}-\frac{E'}{p'}\cos\theta_{p'}\right)^2+\frac{m^2p^2\sin^2\theta}{p'^2u^2}\right]^{1/2}, \tag{B29}$$

where, according to Eqs. (B11) and (B12),

$$\cos\theta_{kL} = \frac{E-E'}{u} - \frac{t-Q_E^2}{2u\left(E-E'-\dfrac{t+w^2-M^2}{2M}\right)}. \tag{B30}$$

In the main text we discuss in more detail the exact, elastic one-photon contribution.


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