Group weight and vanishing graphs

P. Butera,* G. M. Cicuta,* and M. Enriotti†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 23 August 1979)

Various properties of the group weight of Feynman graphs in non-Abelian gauge theories are discussed. Infinitely many skeleton graphs with vanishing weight are exhibited for every compact Lie group. The $1/N^2$ dependence of the topological expansion is related to a $1/N^2$ expansion in some channels with the exchange of definite quantum numbers.

I. INTRODUCTION

In the perturbative analysis of gauge theories it is convenient to represent every Feynman graph G as the product of a weight factor W_G depending on the gauge group times a momentum integral. An efficient graphic method to compute W_G for most simple Lie groups has been described.[1] We make use of that method to discuss further properties of the weight factor.[2]

In Sec. II we show that the weight factors of a SU(N) gauge theory are polynomials in N^2 (apart possibly from a factor N) and we relate this property to the topological expansion of the scattering amplitude in the channels where definite quantum numbers are exchanged. In Sec. III we observe that, at first sight surprisingly, infinitely many skeleton graphs have vanishing weight in every non-Abelian gauge theory. This feature is typical of non-Abelian gauge theories and may be of help in the analysis of perturbation theory, although a rough estimate suggests that the number of nonvanishing graphs grows with the perturbative order much faster than the number of the vanishing ones.

In Sec. IV we derive the projection operators corresponding to exchanged states with definite quantum numbers in gluon-gluon elastic scattering in SU(N) gauge theory.

II. PLANARITY AND TOPOLOGICAL EXPANSION

In this section we discuss the dependence on N of the weight W_G of an arbitrary graph in the SU(N) gauge theory.[3-8]

At order v in the coupling constant g, the graph has v trilinear vertices (in the usual way the four-gluon vertex is replaced by couples of three-gluon vertices)[1] some of which being three-gluon vertices v_g, the others being quark-quark-gluon v_q, $v=v_g+v_q$. The Feynman graph has $p=p_g+p_q$ propagators (we do not count the external lines), that is, p_g gluon and p_q quark propagators. If the graph has n external lines, $n=3v-2p$, the group factor W_G is a tensor of rank n. The graphical method described by Cvitanović is an efficient way to express the generic tensor W_G as a linear combination of a complete set of independent tensors having the same rank, the basis tensors, which are associated to graphs without internal gluons.[1]

In the SU(N) gauge theory the evaluation of the group factor W_G for any graph only involves the following two steps, (a) to re-express the three-gluon vertices in terms of the fundamental representation (see Fig. 1),

$$if_{ijk}=2\,\mathrm{Tr}(T_iT_jT_k-T_kT_jT_i)\,; \qquad (2.1)$$

(b) to replace all internal gluon lines with gluon projection operators (see Fig. 2),

$$2(T_i)^a_b(T_i)^c_d=\delta^a_d\delta^c_b-\frac{1}{N}\delta^a_b\delta^c_d. \qquad (2.2)$$

W_G is then expressed as the sum of $2^{v_g+p_g}$ "double-line" graphs. As an example, Fig. 3(a) shows a graph at order g^{10} in perturbation theory, with six three-gluon vertices and nine internal gluon propagators. In Fig. 3(b) there is one of the 2^{15} double-line graphs obtained after steps (a) and (b). In the double-line graphs may appear index loops, i.e., fermion loops unconnected to the rest of the graph and to external lines, each contributing a factor N, which are called windows. There are also index paths called boundaries which are attached to the external lines. There are no boundaries for graphs where the external sources are all color singlets. Each boundary represents a tensor with rank equal to the number of external lines attached to the index path. For example, the double-line configuration in Fig. 3(b) has one window and one boundary, as shown in Fig. 3(c).

Basic notions are those of planarity and degree of nonplanarity. It is very convenient to "complete" the graph G by adding one more vertex, called P_∞, where all external lines of G are incident (see Fig. 4). One can now "draw" the completed graph on a sphere with h handles so that

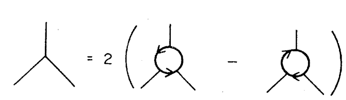

FIG. 1. Graphical representation of Eq. (2.1).

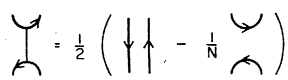

FIG. 2. Graphical representation of Eq. (2.2).

lines intersect only at vertices (embedding). The minimum h_m for which the embedding is possible is a characterization of the degree of nonplanarity of the graph. The graph is planar if and only if $h_m = 0$.[9] The graph in Fig. 3(a) has $h_m = 1$, that is, it may be embedded on a torus. The completed graph embedded on a sphere with h_m handles may be regarded as a polyhedron whose edges are the lines of the graph. Then the Euler formula relating the number of vertices V, of edges P, and faces F holds,

$$V - P + F = 2 - 2h_m, \tag{2.3}$$

which in terms of the original graph is

$$(v+1) - (p+n) + (f+n-1) = 2 - 2h_m, \tag{2.4}$$

where $f \equiv F - n + 1$.

The multitude of double-line graphs originating from a Feynman graph has a different number of faces and handles but the same number of

$$f + 2h = p - v + 2 = \frac{v}{2} - \frac{n}{2} + 2. \tag{2.5}$$

Indeed the number of faces ranges between the

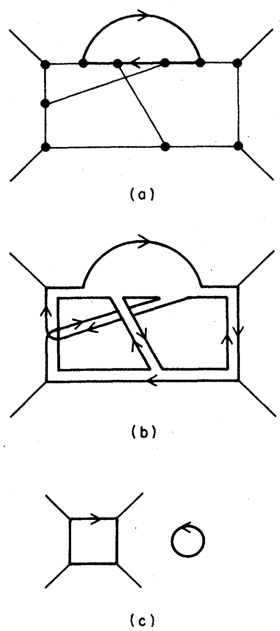

FIG. 3. (a) A graph at order g^{10} in perturbation theory. (b) One of the 2^{15} double-line contributions to the group weight of the graph in (a). (c) The same contribution as in (b), now exhibiting boundaries and windows.

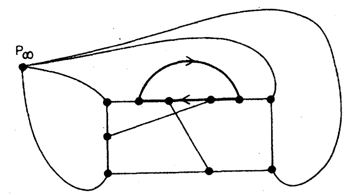

FIG. 4. The graph of Fig. 3(a) is completed here by adding one more vertex P_∞ in order to exhibit its degree of nonplanarity.

maximum $f_M = 2 - v + p - 2h_m$ and the minimum $f_m = 0$ (if f_M is even) or $f_m = 1$ (if f_M is odd).

The quark loops in the original graph do not contribute to the number of faces of the double-line graph. Then

$$f - q = b + w, \tag{2.6}$$

where q, b, and w represent the number of quark loops, boundaries, and windows.

The group-theoretic weight W_G can then be expressed

$$W_G = g^v \sum N^w c_m T^{(m)},$$

where $T^{(m)}$ is one of the basis tensors, the summation extends over all the double-line graphs, and the coefficient c_m counts the multiplicity of the configuration with w windows and the factors of 2 and $(-1/N)$ arising from steps (a) and (b).

By use of (2.5) and (2.6) the power N^w can be rewritten,

$$\begin{aligned} W_G &= g^v \sum c_m N^{f-q-b} T^{(m)} \\ &= (g\sqrt{N})^v N^{-q} N^{1-n/2} \sum c_m N^{1-b} N^{-2h} T^{(m)}. \end{aligned} \tag{2.7}$$

The coefficient c_w may contain a dependence on N only for those double-line configurations which arise from the singlet subtraction term in step (b):

$$2(T_i)^a_b (T_i)^c_d \rightarrow -\frac{1}{N} \delta^a_b \delta^c_d.$$

For the first such replacement, the double-line configuration has $v-2$ vertices and $p-3$ propagators, then its invariant $f + 2h$ is a positive integer with different parity from the original graph or any configuration, where instead the replacement

$$2(T_i)^a_b (T_i)^c_d \rightarrow \delta^a_d \delta^c_b \tag{2.8}$$

has been made.

Therefore $(1/N)$ times the value of the double-line configuration, where the singlet subtraction term has been used, has the same parity of the power of N as the graph with the replacement (2.8) everywhere and both can be written in the form (2.7) with coefficients c_m now independent of N. The same argument holds for multiple use of the singlet subtraction terms and Eq. (2.7) holds for the general graph, with coefficient c_m independent of N.

It is then clear that if one is interested in processes with fixed number of boundaries, for instance, processes with color-singlet sources only ($b=0$), then Eq. (2.7) arranges the contribution of each graph in decreasing powers of N^2. Or one may sum over the perturbation series and take the limit $N\to\infty$ with $g^2N=\gamma^2$ fixed and one would obtain that the contribution to amplitudes with fixed number of boundaries and quark loops are arranged in decreasing powers of N^2 and are associated with increasing degree of nonplanarity.[4-8]

A simple remark may shorten the computation of W_G. Step (b) [Eq. (2.2)] may be substituted by the simpler replacement (2.8) when (i) the gluon connects two three-gluon vertices, or (ii) the gluon connects one three-gluon vertex with one quark-quark-gluon vertex.

Proofs are straightforward:

$$\begin{aligned} f_{jik}f_{klm} &= 4\,\mathrm{Tr}(T_jT_iT_k - T_kT_iT_j)\mathrm{Tr}(T_kT_lT_m - T_mT_lT_k) \\ &= 2\,\mathrm{Tr}(T_jT_iT_lT_m - T_jT_iT_mT_l \\ &\quad - T_iT_jT_lT_m + T_iT_jT_mT_l). \end{aligned} \tag{2.9}$$

For point (ii) we have

$$\begin{aligned} i(T_k)^b_a f_{kij} &= 2i(T_k)^b_a \mathrm{Tr}(T_kT_iT_j - T_jT_iT_k) \\ &= i[T_i, T_j]^b_a. \end{aligned} \tag{2.10}$$

Therefore, in all Feynman graphs where there are no gluons which directly connect quark lines, the number of double-line graphs originating from a simple graph is reduced to 2^{v_g}. Of course this happens in a pure SU(N) gauge theory (without quarks). Then one finds that for two-point functions and three-point functions, where there is just one basic tensor (respectively δ_{ab} and f_{abc}), the group weight W_G of the generic Feynman graph is

$$W_G = \delta_{ab}(Ng^2)^s \sum_{P=0}^{[s/2]} c_P(N^2)^{-P} \text{ at order } g^{2s}, \tag{2.11}$$

$$W_G = f_{abc}\,g(Ng^2)^s \sum_{P=0}^{[s/2]} c_P(N^2)^{-P} \text{ at order } g^{2s+1}, \tag{2.12}$$

where the leading coefficient c_0 is different from zero if and only if the graph is planar.[10]

As is shown in Sec. IV, for the four-point function one has six basic tensors, three of which (A,B,C) have one boundary and three (D,E,F) have two boundaries. At order g^{2s+2} one finds

$$\begin{aligned} W_G = g^2(g^2N)^s \Bigg\{ & A\sum_0^{[s/2]} a_P(N^2)^{-P} + B\sum_0^{[s/2]} b_P(N^2)^{-P} + C\sum_0^{[s/2]} c_P(N^2)^{-P} \\ & + \frac{1}{N}\left[D\sum_0^{[s/2]} d_P(N^2)^{-P} + E\sum_0^{[s/2]} e_P(N^2)^{-P} + F\sum_0^{[s/2]} f_P(N^2)^{-P}\right]\Bigg\}. \end{aligned} \tag{2.13}$$

Higher n-point functions have weights W_G expressed in the same form after one has taken care of the N factors associated with the number of boundaries of the basic tensors.

As one can see from (4.3)–(4.8) in Sec. IV, the first four projection operators do not mix basis tensors with different boundaries or they mix them with the proper pure factor N. Therefore the gluon-gluon elastic scattering amplitude in those channels will be a polynomial in N^2, apart from an overall normalization independent of the order in perturbation theory, while the scattering amplitude in the channels associated with the last two projectors loses the simpler dependence on N^2.

III. VANISHING GRAPHS

It is easy to show that in non-Abelian gauge theories there are infinitely many skeleton graphs with vanishing weight. They are identified in an obvious way by only using the antisymmetry property of the three-gluon vertex so that the results of this section hold for any compact Lie group. It is convenient to restrict ourselves first to a pure gauge non-Abelian theory. From the graphical rules[1] it is obvious that a graph containing a subgraph with vanishing weight will also have vanishing weight. Furthermore, since there is a single independent tensor of rank two and a

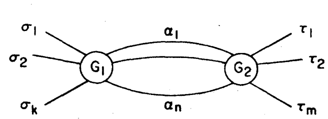

FIG. 5. A generic graph G whose weight is considered as a convolution of the weights of the subgraphs G_1 and G_2.

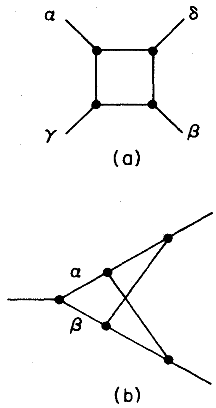

FIG. 6. (a) The lowest-order graph with four external lines that has a symmetry plane through the external lines (γ, δ). (b) The corresponding vanishing graph obtained by convolution of the graph in part (a), with a three-gluon vertex $f_{\tau\alpha\beta}$.

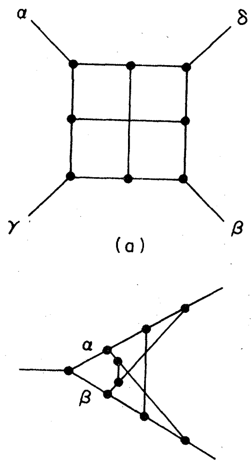

FIG. 8. See caption for Fig. 6.

single one of rank three, we may restrict ourselves to skeleton graphs with vanishing weight. In fact, each such skeleton will produce vanishing graphs if arbitrary self-energy or vertex insertions are made. While it may be difficult to give necessary and sufficient conditions for the vanishing of the weight of a skeleton graph in a general non-Abelian theory, our remarks select a large class of vanishing graphs.

Let us consider the weight $T^G_{\sigma_1\cdots\sigma_k\tau_1\cdots\tau_m}$ of a graph G (see Fig. 5). It may be obtained by partial saturation of the weights of the subgraphs G_1 and G_2

$$T^G_{\sigma_1\cdots\sigma_k\tau_1\cdots\tau_m}=T^{G_1}_{\sigma_1\cdots\sigma_k\alpha_1\cdots\alpha_n}T^{G_2}_{\alpha_1\cdots\alpha_n\tau_1\cdots\tau_m}.$$

A simple sufficient condition for the vanishing of T^G is that T^{G_1} and T^{G_2} are respectively symmetric and antisymmetric in two corresponding saturated α indices. In particular, a vanishing weight is obtained for any three-gluon diagram, which is the product of a three-gluon vertex times a four-gluon tensor symmetric in the two saturated indices. Because of the nature of the three-gluon vertex every planar, or nonplanar, four-leg graph with a plane of symmetry through two of the external lines is associated with a tensor W_G symmetric in the couple of indices (say α, β) of the external gluons not lying in the symmetry plane. The lowest-order[11] examples of such symmetric skeleton graphs are shown in Figs. 6(a) – 9(a).

By convolution with the bare three-gluon vertex $f_{\tau\alpha\beta}$ (or equivalently with any three-gluon Green's function $\Gamma_{\tau\alpha\beta}$) one obtains a vanishing graph [see

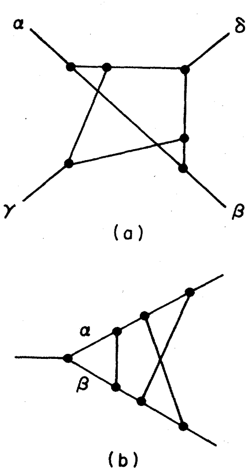

FIG. 7. See caption for Fig. 6.

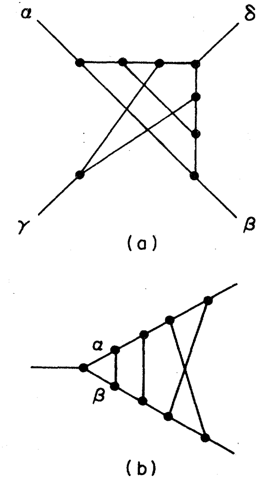

FIG. 9. See caption for Fig. 6.

Figs. 6(b) – 9(b)].

This procedure suggests a very rough estimate of the number of such symmetric four-leg skeletons. At large order n in the coupling constant, the number of symmetric skeletons with only the two external vertices lying in the symmetry plane is roughly $x(\frac{1}{2}n)$, where $x(m) \sim m!$ is the number of skeletons (which in this level of estimate are as many as the generic graphs)[12] at order m. Therefore the ratio R of the vanishing skeletons versus the nonvanishing ones for the three-point function would be, at order n,

$$R \sim (\tfrac{1}{2}n)!/n!.$$

This estimate neglects the facts that (a) there are symmetric skeletons with vertices on the symmetry plane, (b) not all symmetric four-point skeletons lead to three-point skeletons [see for example Fig. 7(b) or 9(b)], and (c) depending on the specific gauge group of the theory, there are nonsymmetric four-point skeletons having a symmetric tensor.[13] While (a) and (c) would increase the estimated ratio R, (b) would decrease it. It seems, however, that none of these points can substantially change the very rough previous estimate.

We can now consider a non-Abelian gauge theory with a multiplet of fermions transforming as the fundamental representation of the group. Again one may look for four-point gluon graphs with a symmetry plane, as in Fig. 10(a). When convoluted with the three-gluon vertex, they originate vanishing graphs, as in Fig. 10(b). Since the reflection around the symmetry plane must also

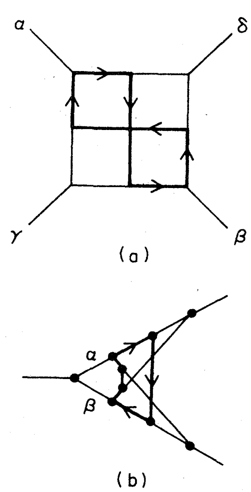

FIG. 10. (a) A graph with four external gluons and one fermion loop, which has a symmetry plane through the lines (γ, δ). (b) A vanishing graph obtained by convolution of the graph in (a) with the three-gluon vertex.

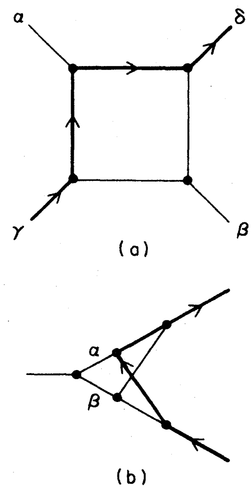

FIG. 11. (a) A graph which does not have a symmetry plane through the lines (γ, δ) but whose weight is a tensor symmetric in the indices (α, β). (b) A vanishing graph obtained by convolution of the graph in (a) with the three-gluon vertex.

preserve the direction in the fermion path, one expects that only special ways of replacing a gluon path with a fermion path in the vanishing gluon graphs will still give a vanishing graph.

One may note, however, that even in some cases where the reflection around the symmetry plane does not preserve the direction of the loop, one may still produce a vanishing graph because of additional properties of weights depending on the gauge group. For instance, in the SU(N) case, it is easy to check that the tensor $T_{\alpha\gamma\beta\delta}$, which is the weight of the graph in Fig. 11(a), is a symmetric tensor in (α, β), although the graph has no symmetry plane through (γ, δ). Therefore the three-point graph in Fig. 11(b) has vanishing weight.[14]

We also remark that for any given graph with three external lines and vanishing weight one can obtain a vanishing "vacuum" graph by "completing" the former with one more coupling of the type f_{abc} or $(\lambda_a)^b_c$. Next by stereographic projection (as was mentioned in the definition of planarity in Sec. II) from another inequivalent vertex of the vacuum graph, one may obtain a new vanishing three-point graph. For instance, in this way one shows that the vanishing graph in Fig. 12 is related to that in Fig. 11(b).

We finally mention that the study of graphs with definite symmetry properties can be pursued in an algebraic way[15] through the study of the spectral properties of the adjacency matrix. It is amusing to notice that the two lowest-order vanishing graphs, already exhibited in Ref. 1, when completed by one more trigluon vertex, are just the

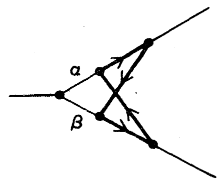

FIG. 12. This three-leg graph may be proved to vanish by first completing the vanishing graph of Fig. 11(b) and next by "stereographic projecting" from a three-gluon vertex.

first representatives of a peculiar class of graphs, sometimes called cages.[16] We checked that all five cages with trilinear vertices, exhibited in Ref. 16, have vanishing weight due to the mechanism previously described.

IV. BASIS TENSORS AND PROJECTORS

In order to discuss the cases where the $1/N$ expansion is actually a $1/N^2$ expansion (see Sec. II) and for completeness reasons, we write here the SU(N) tensor basis for processes with $r=4$ external gluons (no external quarks) and the linear combinations of the basis tensors which are associated to the exchange of definite quantum numbers. The set of all distinct traces over r T_i matrices form a natural tensor basis, but the tensors of rank r, $r \geq N$ so obtained are not linearly independent.[17] Actually in a pure SU(N)

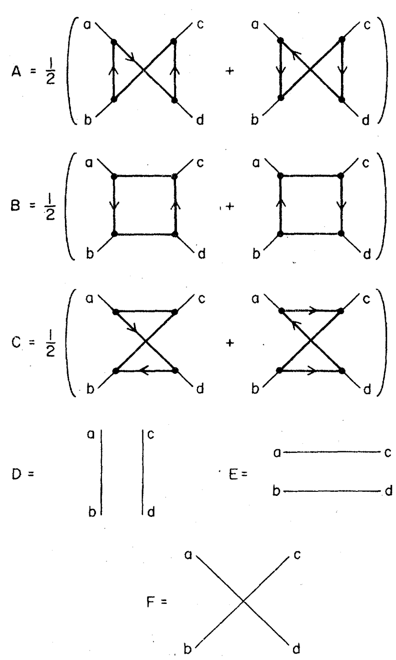

FIG. 13. Basis tensors for gluon-gluon scattering in SU(N).

gauge theory, by the rules (2.1) and (2.2), complete and independent bases are obtained by symmetrizing (or antisymmetrizing) traces of products of T_i matrices, which are graphically fermion loops of even (odd) length. By the Furry theorem, this is also true if the Lagrangian contains fermion fields.[18] Therefore, for $r=4$ one has six instead of nine basis tensors, i.e.,

$$\begin{aligned} A &= \tfrac{1}{2}[\mathrm{Tr}(T_aT_dT_cT_b)+\mathrm{Tr}(T_aT_bT_cT_d)],\\ B &= \tfrac{1}{2}[\mathrm{Tr}(T_aT_bT_dT_c)+\mathrm{Tr}(T_aT_cT_dT_b)],\\ C &= \tfrac{1}{2}[\mathrm{Tr}(T_aT_dT_bT_c)+\mathrm{Tr}(T_aT_cT_bT_d)],\\ D &= \delta_{ab}\delta_{cd},\quad E=\delta_{ac}\delta_{bd},\quad F=\delta_{ad}\delta_{bc}. \end{aligned} \tag{4.1}$$

They are shown in Fig. 13.

One may define a product of basis tensors as a convolution in the "vertical" channel (that is, $KL=H$ means $H_{acbd}=K_{actu}L_{tubd}$). Since the fermion loop is symmetrized, this product is commutative; D acts as the identity and one easily finds

$$\begin{aligned} A^2 &= \tfrac{1}{32}(D+F)-\frac{1}{4N}(B+C)+\frac{1}{16N^2}E,\\ AB &= AC=\frac{-1}{4N}(B+C)+\frac{1}{16N^2}E,\\ AF &= A,\quad BF=C,\quad CF=B,\\ BE &= CE=\tfrac{1}{4}(N-1/N)E,\\ AE &= \frac{-1}{4N}E,\quad FE=E,\quad F^2=D,\\ B^2 &= C^2=\tfrac{1}{8}\left[NB-\frac{2}{N}(B+C)+\left(\frac{1}{2N^2}+\frac{1}{4}\right)E\right],\\ BC &= \tfrac{1}{8}\left[NC-\frac{2}{N}(B+C)+\left(\frac{1}{2N^2}+\frac{1}{4}\right)E\right],\\ E^2 &= (N^2-1)E. \end{aligned} \tag{4.2}$$

The linear combinations of the basis tensors that are mutually orthogonal projection operators and that represent the exchange of a state with definite quantum numbers in the vertical channel are here labeled with the dimension of the irreducible representations in the decomposition of the product $(N^2-1)\otimes(N^2-1)$ and by the symmetry (or antisymmetry) property in the exchange of the indices a and c (or b and d)[19]:

$$P_{1,S}=\left(\frac{1}{N^2-1}\right)E \text{ (Pomeron channel)}, \tag{4.3}$$

$$P_{N^2-1,A}=\frac{4}{N}(B-C) \text{ (antisymm. adjoint channel)}, \tag{4.4}$$

$$P_{N^2-1,S}=\frac{4N}{N^2-4}\left(B+C-\frac{1}{2N}E\right)$$

(symm. adjoint channel), (4.5)

$$P_{(N^2-4)(N^2-1)/4+\overline{(N^2-4)(N^2-1)/4},A}=\frac{-4}{N}(B-C)+\tfrac{1}{2}(D-F)\,, \tag{4.6}$$

$$P_{N^2(N-1)(N+3)/4,S}=2A-\frac{2}{N+2}(B+C)+\tfrac{1}{4}(D+F)+\frac{1}{2(N+1)(N+2)}E\,, \tag{4.7}$$

$$P_{N^2(N-3)(N+1)/4,S}=-2A-\frac{2}{N-2}(B+C)+\tfrac{1}{4}(D+F)+\frac{1}{2(N-1)(N-2)}E\,, \tag{4.8}$$

for $N=3$ the representation $\frac{1}{4}[N^2(N-3)(N+1)]$ is not present and indeed the last projection operator vanishes because of the relation[20] [valid only in SU(3)]

$$8(A+B+C)=D+E+F\,.$$

In $N=2$, more relations exist (see, for instance, Ref. 1) and one is left with only three channels associated with $P_{1,S}, P_{3,A}, P_{5,S}$.

Note added. By using the Lie commutator for the quark representation, one easily shows that the tensor described by the graph in Fig. 11(a) is indeed symmetric in the indices (α,β) for every semisimple Lie group. Our point was only to show a simple example of a tensor, symmetric in a couple of indices, represented by a graph which does not exhibit that symmetry.

ACKNOWLEDGMENTS

We thank the SLAC Theory Group for their hospitality. This work was supported by the Department of Energy under Contract No. DE-AC03-76SF00515.

---

*On leave from Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy.

†On leave from Istituto di Fisica, Universita di Milano, Italy.

[1] P. Cvitanović, Phys. Rev. D 14, 1536 (1976).

[2] Part of the matter in Secs. II and III was briefly described by one of us, G. Cicuta, Phys. Rev. Lett. 43, 826 (1979).

[3] The subject of this section has already been discussed in the literature (Refs. 4–8) usually starting from a double-line Feynman graph, which would correspond in the approach of Cvitanović and our approach to various contributions to the same weight, and/or by restricting the study to the color-singlet external sources. Therefore, although this section overlaps with known literature, it may be useful. Furthermore, it does not seem that previous analyses obtained equations of the type of (2.13) or the conclusion of Sec. II.

[4] G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[5] G. Veneziano, Phys. Lett. 52B, 220 (1974); Nucl. Phys. B74, 365 (1974).

[6] M. Ciafaloni, G. Marchesini, and G. Veneziano, Nucl. Phys. B98, 472 (1975); B98, 493 (1975).

[7] G. P. Canning, Phys. Rev. D 12, 2505 (1975).

[8] Many papers on the topological expansion have appeared in the past few years. Many of them are quoted in G. Chew and C. Rosenzweig, Phys. Rep. 41C, 263 (1978) and E. Witten, Nucl. Phys. B160, 57 (1979).

[9] This definition of planarity appears in Sec. 4 of Nakanishi's book, *Graph Theory and Feynman Integrals* (Gordon and Breach, New York, 1971). It has the benefit of relating the notion of planarity of Feynman graphs to planarity in graph theory, then making possible the use of Kuratowski theorems. This definition also agrees with the common use in the literature about topological expansion although usually it is not written.

[10] All sums in Eqs. (2.11)–(2.13) extend up to $[\frac{1}{2}s]$, the entire part of $\frac{1}{2}s$.

[11] We recall that within the graphic method (Ref. 1) it is convenient to express four-gluon couplings in terms of trigluon couplings. Then the occurrence of vanishing graphs with some four-couplings results from the cancellation among different graphs with trilinear couplings only.

[12] P. Cvitanović, B. Lautrup, and R. Pearson, Phys. Rev. D 18, 1939 (1978).

[13] For instance, by looking at the basis tensor, Fig. 13, it is easy to see that every four-gluon graph whose weight is a tensor symmetric in one couple of indices will also be symmetric in the other couple even though the graph may not have such symmetry.

[14] This vanishing graph is a subgraph of a sixth-order graph which was found to have a vanishing weight in a study of the quark form factor by J. Carazzone, E. Poggio, and H. Quinn, Phys. Rev. D 11, 2286 (1975), their Figs. 12(b) and 13.

[15] N. Biggs, *Algebraic Graph Theory* (Cambridge University Press, Cambridge, 1974).

[16] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* (American Elsevier, New York, 1975), Appendix III.

[17] Tensor basis for all simple Lie groups are described in Ref. 1. The projection operators in SU(3) for gluon-gluon scattering appear in P. Yeung, Phys. Rev. D 13, 2306 (1976) with some misprints.

[18] We thank Michael Dine for a discussion on this point.

[19] We thank Franco Buccella for a discussion on this point.

[20] A. J. MacFarlane, A. Sudbery, and P. H. Weisz, Commun. Math. Phys. 11, 77 (1968).