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Twisted scalar and spinor strings in Minkowski spacetime

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The construction of twisted scalar and spinor field configurations in Minkowski spacetime is described. Twisted field configurations are normally associated with a nonsimply connected space; however, it is shown that it is also possible to construct such configurations in a simply connected space. For each choice of a line in three-dimensional space there exists an inequivalent twisted scalar and spinor field configuration, referred to as strings. The vacuum expectation value of the energy-momentum tensor of such a quantized twisted field is calculated for the case of both scalar strings and spinor strings. It is found to be singular along the chosen line. The energy density off of this line is positive for scalar strings and negative for spinor strings. However, it is shown that the total energy of both types of strings is zero. Nonrelativistic quantum mechanics of these twisted field configurations is discussed, and the eigenfunctions and energy levels of a twisted particle in a Coulomb field are calculated.

I. INTRODUCTION

It has recently been pointed out by Isham[1,2] that in a spacetime which is not simply connected, new varieties of scalar and spinor fields (twisted fields) may be defined. For example, in a spacetime with topology $S^1\times R^3$ where the S^1 direction is spacelike, there exist two types of both scalar fields and spinor fields. In addition to the usual, untwisted fields one may have a twisted scalar field and a twisted spinor field in this spacetime. The twisted scalar field satisfies the same Klein-Gordon equation as the untwisted field but is antiperiodic rather than periodic in the S^1 direction. The local physics of both field configurations is identical, but their global behavior is not. The twisted spinor field is associated with a different choice of the spin connection; however, in the case of $S^1\times R^3$ it is possible to replace the twisted spin connection by antiperiodicity conditions.

The twisted field configurations are physically inequivalent to the untwisted configurations. For example, the Casimir energies of the quantized scalar and spinor field in $S^1\times R^3$ are different according to whether the field is twisted or untwisted.[3] The one-loop photon vacuum polarization in this spacetime is also different according to whether the photons are coupled to twisted spinors or to untwisted spinors.[4]

Even in a simply connected spacetime, such as Minkowski space, it is possible to construct twisted fields provided that suitable boundary conditions are satisfied. The purpose of this paper is to discuss such twisted fields in Minkowski space. It will be shown that an infinite number of inequivalent scalar and spinor field configurations may be constructed in any spacetime. For reasons which will become apparent, these configurations will be referred to as strings.

II. TWISTED SCALAR STRINGS

Consider a given axis in three-dimensional Euclidean space and establish a cylindrical polar coordinate system about that axis. The Minkowski-space metric becomes

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2\,. \tag{2.1}$$

The scalar wave equation

$$\Box\phi = 0 \tag{2.2}$$

may be separated in these coordinates and possesses regular solutions of the form

$$F \propto J_{|\mu|}(Kr)e^{i(\mu\theta + kz - \omega t)}\,, \tag{2.3}$$

where $K=(\omega^2-k^2)^{1/2}$. The twisted scalars are obtained by requiring that ϕ be antiperiodic in θ or that μ be a half integer. These solutions all vanish on the $r=0$ axis. Thus we may understand how it is possible for a twisted field configuration to exist in Minkowski spacetime; this axis could be removed from the space without affecting the twisted scalars, but the space would then be nonsimply connected.

There is an important difference between the twisted and the untwisted scalar configurations. In the case of untwisted scalars, which are obtained from Eq. (2.3) by letting μ be an integer, the sets of mode functions associated with two different choices of the axis are related by a linear transformation. This is not true for twisted scalars. Because a complete set of twisted scalar modes which are antiperiodic about a given axis vanish on that axis, it is not possible to use them to expand functions which are antiperiodic about a second axis but nonzero on the first axis. Thus each choice of axis gives rise to an inequivalent twisted scalar field configuration.

A given configuration may be quantized by the usual canonical procedure: Let $\tilde{\phi}$ be an antiperiodic Hermitian operator solution of Eq. (2.2) and $\tilde{\pi} = \dot{\tilde{\phi}}$ be its conjugate momentum. Impose the equal-time commutation relations

$$[\tilde{\phi}(\vec{x}, t), \tilde{\pi}(\vec{x}', t)] = i\delta(x - x'). \tag{2.4}$$

Then

$$\tilde{\phi} = \sum_{\lambda}(a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*), \tag{2.5}$$

where $\{F_\lambda\}$ are a complete set of antiperiodic functions of the form of Eq. (2.3) and where

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}. \tag{2.6}$$

A Fock space of states may now be defined. The vacuum state so obtained is not rotation or translation invariant. Each possible axis produces an inequivalent field theory and hence a distinct vacuum state; the selection of any particular vacuum results in the spontaneous breaking of translation and rotation invariance.

It will be shown below that even the vacuum state of the twisted field theory is associated with a nonzero density of energy in space concentrated about the axis of antiperiodicity. For this reason the twisted field configuration is called a "string" (the state will here be assumed to be the vacuum, but one could investigate "excited strings" associated with other state vectors). These strings should not be confused with those introduced in other contexts, such as the Dirac string in the theory of magnetic monopoles.[5] In this paper it will be assumed that the strings are straight and exist in a Minkowski spacetime. Both requirements could, however, be relaxed. Twistedness is a topological property, so it is possible to construct strings of arbitrary shape and in a general background spacetime.

The quantization prescription which has been adopted here treats the twisted scalar field as a distinct quantum field to be quantized separately from the untwisted field. Avis and Isham[6] have proposed combining twisted and untwisted fields into a single field theory by taking the generating functional to be a linear combination of those for untwisted and twisted fields. In such a theory, the twisted and untwisted configurations are identified with different sectors of the same theory rather than distinct quantum fields. In the present context this proposal would require either selecting a subset of all the possible twisted configurations or else summing over an infinite number of twisted generating functionals. Although this is an interesting possibility, it will not be pursued further here.

Let us now turn to the calculation of the energy-momentum tensor for a scalar string. The two-point function is

$$\tilde{G}(x, x') = \langle 0 | \tilde{\phi}(x)\tilde{\phi}(x') | 0 \rangle, \tag{2.7}$$

where $\tilde{\phi}$ is a twisted scalar field antiperiodic about a given axis and $|0\rangle$ is the associated vacuum state. In Appendix A this quantity is calculated explicitly and shown to be

$$\begin{aligned}\tilde{G}(x, x') &= (2\pi^2)^{-1}(rr')^{1/2}\cos\tfrac{1}{2}\Delta\theta[r^2 + r'^2 + 2rr' + (\Delta z)^2 - (\Delta t)^2]^{-1/2} \\ &\quad\times[r^2 + r'^2 - 2rr'\cos\Delta\theta + (\Delta z)^2 - (\Delta t)^2]^{-1},\end{aligned} \tag{2.8}$$

where $\Delta\theta = \theta - \theta'$, etc. The two-point function for the untwisted scalar field is

$$\begin{aligned}G(x, x') &= -\frac{1}{4\pi^2\sigma} \\ &= (4\pi^2)^{-1}[r^2 + r'^2 - 2rr'\cos\Delta\theta + (\Delta z)^2 - (\Delta t)^2]^{-1},\end{aligned} \tag{2.9}$$

where σ is the square of the geodesic distance between x and x'. Define the renormalized Green's function by

$$G_R(x, x') = \tilde{G}(x, x') - G(x, x'). \tag{2.10}$$

In the coincidence limit this quantity is finite except at $r = 0$:

$$G_R(x, x) = -\frac{1}{32\pi^2 r^2}. \tag{2.11}$$

The energy-momentum tensor for a twisted scalar field is of the same form as that for an untwisted scalar field. In flat spacetime it is

$$\begin{aligned}T_{\mu\nu} &= (1 - 2\xi)\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}(1 - 4\xi)g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} \\ &\quad - 2\xi\phi\,\phi_{;\mu\nu},\end{aligned} \tag{2.12}$$

where ξ is an arbitrary parameter which takes the value zero for minimal coupling and $\frac{1}{6}$ for conformal coupling. The finite vacuum expectation value of this tensor may be defined as

$$\begin{aligned}\langle T_{\mu\nu}\rangle &= \lim_{x \to x'}[(1 - 2\xi)\partial_\mu\partial_{\nu'} - \tfrac{1}{2}(1 - 4\xi)g_{\mu\nu}\partial_\rho\partial^{\rho'} \\ &\quad - 2\xi\nabla_\mu\partial_\nu]\tilde{G}_R(x, x'),\end{aligned} \tag{2.13}$$

where ∇_μ is the covariant derivative at x in the metric Eq. (2.1), ∂_μ the ordinary derivative at x, and $\partial_{\mu'}$ that at x'. After some calculation, it is found that the nonzero components of $\langle T_{\mu\nu}\rangle$ are

$$\begin{aligned}\langle T_{tt}\rangle &= -\langle T_{zz}\rangle = \frac{32\xi - 5}{256\pi^2 r^4}, \\ \langle T_{rr}\rangle &= -\tfrac{1}{3}r^{-2}\langle T_{\theta\theta}\rangle = \frac{16\xi - 3}{256\pi^2 r^4}.\end{aligned} \tag{2.14}$$

The energy-momentum tensor is singular on the $r=0$ axis. Similar behavior often occurs in the expectation value of the energy-momentum tensor of a quantized field which is subject to boundary conditions on an arbitrary surface. If we define the total energy of the string as

$$E=\int \langle T_{tt}\rangle d^3x, \tag{2.15}$$

then the energy is infinite unless $\xi=\frac{5}{32}$.

However, an alternative definition of total energy is obtained by first defining the Hamiltonian operator

$$H=\int T_{tt}d^3x \tag{2.16}$$

and then imposing a regularization-subtraction procedure. This does not in general yield E; the two operations do not commute. This noncommutativity occurs in the case of the Casimir effect for a minimally coupled scalar field[7] and in the case of a quantized electromagnetic field in the presence of curved perfectly conducting boundaries.[8] One might expect the same to be true in the present case. Indeed, the energy per unit length obtained by ζ-function regularization[9,10] vanishes.

To show this, consider a twisted scalar field in a cylinder of radius R and length L; that is, we require that $\tilde{\phi}(r=R)=0$ and $\tilde{\phi}(z+L)=\tilde{\phi}(z)$ in addition to $\tilde{\phi}(\theta+2\pi)=-\tilde{\phi}(\theta)$. The resulting eigenfrequencies are

$$\omega_{nl}=(\xi_{nl}{}^2R^{-2}+4\pi^2m^2L^{-2})^{1/2}, \tag{2.17}$$

where ξ_{nl} is the nth zero of $J_{|l+1/2|}(x)$ and where $n=1,2,\ldots;\ l=0,\pm1,\pm2,\ldots;$ and $m=0,\pm1,\pm2,\ldots.$ The vacuum expectation value of H is

$$\langle H\rangle=\tfrac{1}{2}\sum_{nlm}\omega_{nlm}. \tag{2.18}$$

Define the ζ-function-regularized energy as

$$\langle H\rangle_s=\tfrac{1}{2}\sum_{nlm}\omega_{nlm}{}^{-2s}=\tfrac{1}{2}R^{-1}Z(s), \tag{2.19}$$

where $Z(s)$ is the ζ-function of the operator $-\nabla^2$ in acting on twisted scalars in a cylinder of unit radius and length L/R. Explicitly,

$$Z(s)=\sum_j \lambda_j{}^{-s}, \tag{2.20}$$

where $\lambda_j=\lambda_{nlm}=\xi_{nl}{}^2+4\pi m^2R^2L^{-2}$ are the eigenvalues of $-\nabla^2$ subject to the above boundary conditions.

Let

$$Z(s)=\frac{1}{\Gamma(s)}\int_0^\infty Y(t)t^{s-1}dt. \tag{2.21}$$

The function $Y(t)$ may be expanded as[10,11]

$$Y(t)=\sum_{n=0}(B_n+C_n)t^{(n-3)/2} \tag{2.22}$$

in the case of a three-dimensional space. The ζ function $Z(s)$ has a simple pole at $s=-\frac{1}{2}$ with residue

$$-\tfrac{1}{2}\pi^{-1/2}(B_4+C_4), \tag{2.23}$$

so we may write

$$\langle H\rangle_{s}\ {}_{s\to -1/2}\!\!\sim -\tfrac{1}{4}R^{-1}\pi^{-1/2}\left(\frac{B_4+C_4}{s+\frac{1}{2}}+\text{finite term}\right). \tag{2.24}$$

The coefficient B_4 vanishes in the present case[12] because the curvature of the space is zero. The coefficient C_4 is unknown, but depends upon the details of the boundary (in this case the cylinder). If $C_4\neq0$, then the pole term in Eq. (2.24) must be subtracted. The details of how this is performed, whether by the introduction of a surface counterterm in the action or by some other procedure, and of whether some of the finite term is removed as well, are irrelevant to our present purposes. The result must be of the form

$$\langle H\rangle_{\text{ren}}=R^{-1}f(L/R). \tag{2.25}$$

Thus if $R\to\infty$ with L/R fixed, then the renormalized energy must vanish:

$$\langle H\rangle_{\text{ren}}\to0 \text{ as } R\to\infty. \tag{2.26}$$

Hence we conclude that the energy per unit length of the string is zero.

How is this result to be reconciled with the previous result that $\langle T_{tt}\rangle>0$ if $r\neq0$? The simplest resolution is to assign a singular negative energy to the line $r=0$ which is such that the integral of the energy density over all space vanishes. The energy density may be modeled by considering the choice

$$\langle T_{tt}\rangle=\frac{32\xi-5}{256\pi^2}\times\begin{cases} r^{-4}, & r>\epsilon\\ -3\epsilon^{-3}, & r<\epsilon\end{cases} \tag{2.27}$$

in the limit of small ϵ. Critchley, Dowker, and Kennedy[13] have recently proposed the introduction of such singular surface energies in order to reconcile the discrepancy between $\int\langle T_{tt}\rangle d^3x$ and $\langle H\rangle$ in the presence of boundaries. This interpretation leads us to conclude that the twisted scalar string is a configuration of zero total energy but nonzero energy density.

III. TWISTED SPINOR STRINGS

In analogy with the twisted scalar strings, it is possible to construct twisted spinor field configurations in Minkowski spacetime which are inequivalent to the usual Poincaré-invariant field theory. Let us first recall the tetrad formalism for the description of spinors in a general metric. The Dirac equation for spinors of mass m is

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0\,, \tag{3.1}$$

where the spinor covariant derivative is

$$\nabla_\mu = \partial_\mu - \Gamma_\mu \tag{3.2}$$

and Γ_μ is the spinor connection. The γ matrices satisfy the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\,. \tag{3.3}$$

The connection satisfies

$$\partial_\mu \gamma_\alpha - \Gamma^\rho_{\alpha\mu}\gamma_\rho = \Gamma_\mu \gamma_\alpha - \gamma_\alpha \Gamma_\mu\,, \tag{3.4}$$

where $\Gamma^\rho_{\alpha\mu}$ are the Christoffel symbols for the metric $g_{\mu\nu}$. Let e^μ_a be a field of tetrad vectors which satisfy

$$e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab} \text{ and } e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu}\,, \tag{3.5}$$

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. Greek indices are raised by $g^{\mu\nu}$ and Latin indices by η^{ab}. If $\hat{\gamma}^a$ satisfy $\{\hat{\gamma}^a, \hat{\gamma}^b\} = 2\eta^{ab}$, then the matrices defined by

$$\gamma^\mu = e^\mu_a \hat{\gamma}^a \tag{3.6}$$

satisfy Eq. (3.3). The spinor connection in the absence of an electromagnetic field may be expressed as

$$\Gamma_\mu = \tfrac{1}{4} g_{\mu\nu}(e^\mu_a e^a_{\rho,\alpha} - \Gamma^\mu_{\rho\alpha}) s^{\nu\rho}\,, \tag{3.7}$$

where $s^{\nu\rho} = \frac{1}{2}[\gamma^\nu, \gamma^\rho]$ and the γ matrices are those of Eq. (3.6).

Different choices of the tetrad field yield different spinor connections. Normally, two such choices may be related by a similarity transformation, but this is not always the case.[6] The existence of multiple spinor structures is manifested in the existence of inequivalent spinor connections.

An example of a nonsimply connected spacetime is $S^1 \times R^3$, flat space with periodicity in one spatial direction. The usual untwisted spinor field in this spacetime is obtained by choosing the vectors e^μ_a at different points to be parallel to one another. However, another possible tetrad field is obtained by requiring the vectors to undergo a 2π rotation about the axis of periodicity as one moves in the direction of periodicity. This choice leads to an inequivalent spinor field configuration, the twisted spinor field. That this configuration is physically inequivalent to the untwisted spinor field is evidenced by the fact that the two configurations give rise to different Casimir energies and different one-loop photon vacuum polarizations.

If the metric of Minkowski spacetime is given by Eq. (2.1), a suitable choice of the tetrad field is

$$\begin{aligned} &e^\mu_0 = \delta^\mu_0\,, \quad e^\mu_3 = \delta^\mu_3\,, \\ &e^\mu_1 = \delta^\mu_1 \cos\theta - \delta^\mu_2 r^{-1} \sin\theta\,, \\ &e^\mu_2 = \delta^\mu_1 \sin\theta + \delta^\mu_2 r^{-1} \cos\theta\,. \end{aligned} \tag{3.8}$$

The connection calculated from Eq. (3.7) vanishes: $\Gamma_\mu = 0$. This tetrad field is in fact the parallel one associated with a rectangular coordinate system and leads to the usual Poincaré-invariant spinor field theory. An alternative choice of the tetrad field is

$$\begin{aligned} &\tilde{e}^\mu_a = \delta^\mu_a\,, \quad a \neq 2 \\ &\tilde{e}^\mu_2 = r^{-1}\delta^\mu_2\,, \end{aligned} \tag{3.9}$$

which are tangent to the coordinate lines of the cylindrical coordinate system. Let

$$\tilde{\gamma}^\mu = \tilde{e}^\mu_a \hat{\gamma}^a\,. \tag{3.10}$$

Then the spinor connection becomes

$$\tilde{\Gamma}_\mu = \tfrac{1}{2} r \delta^2_\mu \tilde{\gamma}^2 \tilde{\gamma}^1\,. \tag{3.11}$$

It is possible to remove the connection by a similarity transformation. If $\tilde{\psi}$ is a solution of

$$i\tilde{\gamma}^\mu \tilde{\nabla}_\mu \psi - m\tilde{\psi} = 0\,, \tag{3.12}$$

then

$$\psi = S^{-1}\tilde{\psi} \tag{3.13}$$

is a solution of

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0\,, \tag{3.14}$$

where

$$\gamma^\mu = S^{-1}\tilde{\gamma}^\mu S \tag{3.15}$$

and

$$\nabla_\mu = S^{-1}\tilde{\nabla}_\mu S\,. \tag{3.16}$$

Let

$$S = e^{\theta \hat{\gamma}^2 \hat{\gamma}^1/2} = \cos\tfrac{1}{2}\theta + \hat{\gamma}^2\hat{\gamma}^1 \sin\tfrac{1}{2}\theta\,. \tag{3.17}$$

Then

$$\nabla_\mu = \partial_\mu \tag{3.18}$$

and γ^μ becomes the γ matrices defined by Eqs. (3.6) and (3.8). However, if $\tilde{\psi}$ periodic in θ, ψ must be antiperiodic:

$$\psi(\theta) = -\psi(\theta + 2\pi)\,. \tag{3.19}$$

Thus in this representation the twisted spinor field satisfies the same wave equation as the untwisted field but is distinguished by its antiperiodicity. Note that this antiperiodicity is not to be confused with the effect of a 2π rotation on a spinor. If $R(\theta)$ is the operator corresponding to a rotation by an angle θ, then $R(2\pi)\psi = -\psi$. This is, however, true for both twisted and untwisted spinors, whereas only the twisted spinors satisfy Eq. (3.19).

The untwisted spinors are single-valued solutions of Eq. (3.14) with $\nabla_\mu = \partial_\mu$:

$$\psi_u(\theta) = \psi_u(\theta + 2\pi)\,. \tag{3.20}$$

All possible choices of tetrad field and hence of the spinor connection may be reduced to one of these two possibilities. As in the scalar case, the antiperiodic spinors all vanish on the $r=0$ axis so the region of space where they are nonzero is not simply connected.

The canonical quantization of the twisted spinor field is analogous to that of an untwisted spinor field. Let us work in a representation in which the twisted spinor field is a (single-valued) solution of Eq. (3.12). The associated Lagrangian density is

$$\mathcal{L} = i\bar{\tilde{\psi}}\tilde{\nabla}_\mu \tilde{\psi} - m\bar{\tilde{\psi}}\tilde{\psi}\,, \tag{3.21}$$

where $\bar{\tilde{\psi}} = \tilde{\psi}^\dagger \tilde{\gamma}^0$. Because $\tilde{\Gamma}_0 = 0$, the momentum conjugate to $\tilde{\psi}$ is

$$\tilde{\pi} = \frac{\delta \mathcal{L}}{\delta \tilde{\psi}_0} = i\bar{\tilde{\psi}}\,. \tag{3.22}$$

The equal-time anticommutation relations

$$\{\tilde{\psi}(\vec{x}, t),\ \tilde{\pi}(\vec{x}', t)\} = i\delta(\vec{x}, \vec{x}') \tag{3.23}$$

are imposed. Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be complete sets of positive- and negative-frequency solutions of Eq. (3.12) which satisfy

$$\int u^\dagger_{\lambda'} u_\lambda \sqrt{-g}\, d^3x = \int v^\dagger_{\lambda'} v_\lambda \sqrt{-g}\, d^3x = \delta_{\lambda'\lambda}. \tag{3.24}$$

Then we may write

$$\tilde{\psi} = \sum_\lambda (a_\lambda u_\lambda + b^\dagger_\lambda v_\lambda)\,, \tag{3.25}$$

where

$$\{a_\lambda, a^\dagger_{\lambda'}\} = \{b_\lambda, b^\dagger_{\lambda'}\} = \delta_{\lambda\lambda'}. \tag{3.26}$$

The energy-momentum tensor is

$$T_{\mu\nu} = \tfrac{1}{4} i[\bar{\tilde{\psi}}\tilde{\gamma}_\mu \tilde{\nabla}_\nu \tilde{\psi} + \bar{\tilde{\psi}}\tilde{\gamma}_\nu \tilde{\nabla}_\mu \tilde{\psi} - (\tilde{\nabla}_\mu \bar{\tilde{\psi}})\tilde{\gamma}_\nu \tilde{\psi} - (\tilde{\nabla}_\nu \bar{\tilde{\psi}})\tilde{\gamma}_\mu \tilde{\psi}]. \tag{3.27}$$

In particular,

$$T_{tt} = \tfrac{1}{2} i[\tilde{\psi}^\dagger(\partial_0 \tilde{\psi}) - (\partial_0 \tilde{\psi}^\dagger)\tilde{\psi}]. \tag{3.28}$$

The expectation value of $T_{\mu\nu}$ must be invariant under inversions of the t, θ, and z coordinates and is hence diagonal. On dimensional grounds it must be of the form

$$\langle T_{\mu\nu}\rangle = r^{-4}\mathrm{diag}(A, B, Cr^2, D) \tag{3.29}$$

in the limit that $m=0$. Because

$$\langle T^{\mu\nu}\rangle_{;\nu} = 0\,, \tag{3.30}$$

we have that

$$3B + C = 0\,. \tag{3.31}$$

The fact that the components of $\langle T_{\mu\nu}\rangle$ must be invariant under Lorentz boosts in the z direction implies that

$$D = -A\,. \tag{3.32}$$

Finally, if $m=0$ the expectation value is traceless,

$$\langle T^\mu_\mu\rangle = 0\,, \tag{3.33}$$

so that

$$A - B - C - D = 0\,. \tag{3.34}$$

Thus in this case $\langle T_{\mu\nu}\rangle$ is determined entirely by the constant A; in Appendix B it is shown that

$$A = -\frac{1}{128\pi} \tag{3.35}$$

for a twisted neutrino field. That is,

$$\langle T_{tt}\rangle = -\langle T_{rr}\rangle = \frac{1}{3r^2}\langle T_{\theta\theta}\rangle = -\langle T_{zz}\rangle = -\frac{1}{128\pi r^4}\,. \tag{3.36}$$

The energy-momentum tensor for a twisted, four-component spinor field is, in the massless limit, twice that given by Eq. (3.36). As in the scalar case, the energy density becomes singular at $r\to 0$. Here, however, it is negative. One may also calculate the total energy by first forming the Hamiltonian and then applying regularization, as was done in Sec. II for the scalar case. Again one finds that the total energy is zero, so the spinor string is interpreted as consisting of a negative energy density given by Eq. (3.36) and a compensating positive singular energy at $r=0$.

IV. NONRELATIVISTIC QUANTUM MECHANICS OF TWISTED FIELDS

It is well known that the Dirac and Klein-Gordon equations reduce in the nonrelativistic limit to the Schrödinger equation.[14] This is also true for the twisted spinors and scalars discussed in previous sections. A twisted spin-0 or spin-$\frac{1}{2}$ particle satisfies the usual Schrödinger equation

$$-\frac{1}{2m}\nabla^2\psi + V(\vec{x})\psi = i\frac{\partial\psi}{\partial t}\,, \tag{4.1}$$

where the wave function must be antiperiodic about a given axis:

$$\psi(\theta) = -\psi(\theta + 2\pi)\,. \tag{4.2}$$

[Equation (4.1) is written for the case that the vector potential $\vec{A}=0$, but twisted particles satisfy the same equation as untwisted particles in all cases.]

It is apparent that solutions of Eq. (4.1) which satisfy Eq. (4.2) will have very different properties from the familiar periodic solutions. As in the relativistic case, all such solutions vanish on the $r=0$ axis. Let us consider the "twisted hydrogen atom" problem. Take

$$V=Ze^2/\rho\,, \tag{4.3}$$

where

$$\rho=(r^2+z^2)^{1/2}. \tag{4.4}$$

This is just the Coulomb potential energy of a charge e in the field of a charge Ze located at the origin.

Equation (4.1) is separable in spherical polar coordinates if $V=V(\rho)$. However, it is not possible to obtain a normalizable set of eigenfunctions in these coordinates. The solutions of the angular equations are the spherical harmonics Y_l^μ where the antiperiodicity in the azimuthal angle requires that $\mu=\pm\frac{1}{2},\pm\frac{3}{2},\ \ldots$. These functions are singular on the axis $r=0$ and cannot be normalized. This is a consequence of the fact that Eq. (4.2) implicitly breaks the rotational invariance of the system. Hence one cannot construct a complete set of simultaneous eigenfunctions of the Hamiltonian and of the orbital angular momentum operator L^2.

It is, however, possible to obtain well-behaved solutions in parabolic coordinates. The separability of the Schrödinger equation in these coordinates for the hydrogen atom was demonstrated in the early days of quantum mechanics.[15-17] A textbook discussion of this problem is given, for example, by Schiff.[18] Let

$$\xi=\rho-z\,,\qquad \eta=\rho+z\,. \tag{4.5}$$

Then (ξ,η,θ) are the parabolic coordinates. The solutions of Eq. (4.1) subject to Eq. (4.2) are easily obtained by comparison with the periodic solutions; they are

$$\psi_{n_1 n_2 \mu}=Ne^{-\alpha(\xi+\eta)/2}(\xi\eta)^{|\mu|/2}L_{n_1}^{|\mu|}(\alpha\xi)L_{n_2}^{|\mu|}(\alpha\eta)\times e^{i\mu\theta}e^{-iEt}\,, \tag{4.6}$$

where N is a constant, n_1 and n_2 are non-negative integers, $\mu=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$, $L_n^{|\mu|}$ is a Laguerre polynomial,[19] and

$$\alpha=2\mu\,|E|\,. \tag{4.7}$$

The allowed energy eigenvalues are

$$E=E_n=-\frac{2mZ^2e^4}{(2n+1)^2}\,,\quad n=1,2,3,\ldots \tag{4.8}$$

where

$$n=n_1+n_2+|\mu|+\tfrac{1}{2}\,. \tag{4.9}$$

Compare this result with the usual hydrogen-atom energy eigenvalues

$$E_n'=-\frac{mZ^2e^4}{2n^2}\,,\quad n=1,2,3,\ldots\,. \tag{4.10}$$

The separations between the energy levels are different in the two cases, so the spectrum of the twisted hydrogen atom is quite distinct from that of the ordinary hydrogen atom.

V. DISCUSSION

We have seen how twisted scalar and spinor field configurations may be constructed in Minkowski spacetime. There are an infinite number of inequivalent configurations. The vacuum state of the quantum field theory of such a twisted field is associated with a singular distribution of energy (a string). The total energy of the string is nonetheless equal to zero. The one-particle states of the theory might be regarded as describing "twisted particles."

Although the mathematical construction of twisted field configurations is relatively straightforward, the physical interpretation is less clear. Does the singularity in the vacuum energy density of the twisted strings render the twisted field configurations physically unacceptable? Perhaps not. It is conceivable that the twisted field description is a good approximation at low energies but one which breaks down at high energies or short distances. If this were the case, the singular behavior might be avoided. It is not clear, however, what would determine the limits of validity of the twisted description.

Another unresolved question is whether tunneling between different configurations is possible. If this is possible, then ordinary electrons might under appropriate circumstances become "twisted electrons" and be described nonrelativistically by Eqs. (4.1) and (4.2). Such an occurrence should give rise to observable effects, as typified by the twisted hydrogen atom. Various stringlike objects have been proposed in recent years as models of hadrons.[20-24] Whether the strings discussed in this paper may be used to form the basis for a hadronic model is a subject for further investigation.

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APPENDIX A

Here the two-point function for a quantized twisted scalar field will be calculated. Let us confine the field to a finite quantization volume in order to obtain normalizable mode functions; this may be achieved by imposing the conditions $\phi(z) = \phi(z+L)$ and $\phi(r=R)=0$. Then the mode functions are

$$F_\lambda = F_{nlk} = N_{nl}\, J_{|l+1/2|}(K_{nl} r) e^{i[(l+1/2)\theta + kz - \omega t]}\,, \quad \text{(A1)}$$

where $K_{nl} = R^{-1}\xi_{nl}$ and ξ_{nl} is the nth zero of $J_{|l+1/2|}(x)$. The normalization constant is determined by the condition

$$\int |F_\lambda|^2 \sqrt{-g}\, d^3x = (2\omega)^{-1} \quad \text{(A2)}$$

to be

$$N_{nl} = \left[4\pi\omega L R \int_0^1 dt\, t J_{|l+1/2|}{}^2(\xi_{nl} t)\right]^{-1/2}. \quad \text{(A3)}$$

The two-point function is

$$G(x,x') = \langle 0|\tilde\phi(x)\tilde\phi(x')|0\rangle = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty}\sum_k F_{nlk}(x) F^*_{nlk}(x'). \quad \text{(A4)}$$

Take the limit that R and $L\to\infty$. Then

$$\sum_k \to L(2\pi)^{-1}\int_{-\infty}^{\infty} dk. \quad \text{(A5)}$$

As $n\to\infty$,

$$\xi_{nl} \sim \pi(n + \tfrac{1}{2}|l+\tfrac{1}{2}| - \tfrac{1}{4}), \quad \text{(A6)}$$

so that

$$\sum_n \to R\pi^{-1}\int_0^{\infty} dK. \quad \text{(A7)}$$

Furthermore, using Eq. (A3) and the asymptotic form

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi), \quad x\to\infty \quad \text{(A8)}$$

we find that

$$N_{nl} \sim \left(\frac{K_{nl}}{4\pi L R}\right)^{1/2}, \quad L, R\to\infty. \quad \text{(A9)}$$

Thus the two-point function becomes

$$\tilde G(x,x') = \frac{1}{8\pi^2}\sum_{l=-\infty}^{\infty}\int_0^\infty dK \int_{-\infty}^{\infty} dk\, K\omega^{-1} J_{|l+1/2|}(Kr) J_{|l+1/2|}(Kr') e^{i[(l+1/2)\Delta\theta + k\Delta z - \omega\Delta t]}\,, \quad \text{(A10)}$$

where $\Delta\theta = \theta - \theta'$, etc.

Let $\Delta t = 0$. Then the sum and integrals appearing in Eq. (A10) may be explicitly evaluated by use of the relations[25,26]

$$\int_{-\infty}^{\infty} dk (k^2 + K^2)^{1/2} e^{i\Delta z k} = 2K_0(K|\Delta z|)\,, \quad \text{(A11)}$$

where K_0 is a modified Bessel function,

$$\int_0^\infty dK\, K J_{|l+1/2|}(Kr) J_{|l+1/2|}(Kr') K_0(K|\Delta z|) = -i(2\pi)^{-1/2}(rr')^{-1}(u^2-1)^{-1/4} Q^{1/2}_{|l+1/2|-1/2}(u)\,, \quad \text{(A12)}$$

where Q^μ_ν is a Legendre function of the second kind,

$$u = (2rr')^{-1}[r^2 + r'^2 + (\Delta z)^2] \quad \text{(A13)}$$

and

$$Q^{1/2}_{\nu-1/2}(\cosh\alpha) = i\left(\frac{\pi}{2\sinh\alpha}\right)^{-1} e^{-\nu\alpha}. \quad \text{(A14)}$$

The result is

$$\tilde G(x,x')|_{t=t'} = (2\pi^2)^{-1}(rr')^{1/2} \times \cos\tfrac{1}{2}\Delta\theta [r^2 + r'^2 + 2rr' + (\Delta z)^2]^{-1/2} \times [r^2 + r'^2 - 2rr'\cos\Delta\theta + (\Delta z)^2]^{-1}. \quad \text{(A15)}$$

However, invariance under Lorentz boosts in the z direction requires that Δz and Δt appear in the combination $(\Delta z)^2 - (\Delta t)^2$. Thus two-point function for arbitrary x and x' is given by Eq. (2.8).

APPENDIX B

In this appendix $\langle T_{tt}\rangle$ for a massless twisted spinor string will be calculated. The Dirac equation for a twisted spinor, Eq. (2.12), with the connection of Eq. (3.11) may be written as

$$\left(\hat\gamma^0\partial_t + \hat\gamma^1\partial_r + r^{-1}\hat\gamma^2\partial_\theta + \hat\gamma^3\partial_z + \frac{1}{2r}\hat\gamma^1\right)\tilde\psi = 0 \quad \text{(B1)}$$

if $m=0$. Choose a representation in which

$$\hat\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}, \quad \hat\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}, \quad i=1,2,3 \quad \text{(B2)}$$

where the σ^i are the Pauli matrices and I is the 2×2 unit matrix. If we let

$$\tilde\psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}, \quad \text{(B3)}$$

where ϕ and χ are two-component spinors, then Eq. (B1) may be written as

$$\partial_t\phi - \sigma^1\left(\partial_r + \frac{1}{2r}\right)\phi - r^{-1}\sigma^2\partial_\theta\phi - \sigma^3\partial_z\phi = 0 \qquad \text{(B4)}$$

and

$$\partial_t\chi + \sigma^1\left(\partial_r + \frac{1}{2r}\right)\chi + r^{-1}\sigma^2\partial_\theta\chi + \sigma^3\partial_z\chi = 0\,. \qquad \text{(B5)}$$

Solutions of Eq. (B4) may be obtained in the form

$$\phi = \begin{pmatrix} F \\ G \end{pmatrix} e^{i(l\theta + kz - \omega t)} \qquad \text{(B6)}$$

if

$$\left(\frac{d}{dr} - \frac{l-\frac{1}{2}}{r}\right)F = -i(\omega - k)G \qquad \text{(B7)}$$

and

$$\left(\frac{d}{dr} + \frac{l+\frac{1}{2}}{r}\right)G = -i(\omega + k)F\,. \qquad \text{(B8)}$$

Combine Eqs. (B7) and (B8) to find

$$\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \left[1 - \frac{(l-\frac{1}{2})^2}{\rho^2}\right]F = 0\,, \qquad \text{(B9)}$$

where $\rho = kr$ and $K = (\omega^2 - k^2)^{1/2}$. Hence regular solutions are of the form

$$F = NJ_{|l-1/2|}(\rho)\,. \qquad \text{(B10)}$$

Because $\tilde{\psi}$ is required to be single-valued, l is an integer. From Eq. (B7) we obtain

$$G = -iNK(\omega - k)^{-1}J_{l+1/2}(Kr)\,, \qquad \text{(B11)}$$

if $l \geq 1$, and

$$G = iNK(\omega - k)^{-1}J_{-l-1/2}(Kr)\,, \qquad \text{(B12)}$$

if $l \leq 0$.

A negative-helicity (neutrino) solution of Eq. (B1) is

$$u_\lambda = \begin{pmatrix} \phi_\lambda \\ 0 \end{pmatrix}, \qquad \text{(B13)}$$

where $\lambda = (nlk)$ and satisfies

$$(1 - i\gamma_5)u_\lambda = 0 \qquad \text{(B14)}$$

with $\gamma_5 = (-g)^{-1/2}\gamma_0\gamma_1\gamma_2\gamma_3$. We take the quantization volume to be as in Appendix A and impose the conditions

$$u_\lambda(z) = u_\lambda(z + L) \qquad \text{(B15)}$$

and

$$F_\lambda(r = R) = 0\,. \qquad \text{(B16)}$$

We could choose to replace Eq. (B16) by a condition upon G_λ without altering the final results. In analogy with the scalar case, we now have

$$K_{nl} = R^{-1}\xi'_{nl}\,, \qquad \text{(B17)}$$

where ξ'_{nl} denotes the nth zero of $J_{|l-1/2|}$. The normalization condition Eq. (3.24) implies

$$2\pi L\int_0^R dr\, r(|F_{nlk}|^2 + |G_{nlk}|^2) = 1\,. \qquad \text{(B18)}$$

In the limit that $R \to \infty$, we find that

$$N^2 = \frac{K(\omega - k)}{4\omega LR} \qquad \text{(B19)}$$

(for all values of l). Positive-helicity (antineutrino) solutions of Eq. (B1) are

$$v_\lambda = \begin{pmatrix} 0 \\ \chi_\lambda \end{pmatrix} \qquad \text{(B20)}$$

and satisfy

$$(1 + i\gamma_5)v_\lambda = 0\,. \qquad \text{(B21)}$$

The two-component spinors

$$\chi_\lambda = \begin{pmatrix} F_\lambda \\ G_\lambda \end{pmatrix} e^{i(l\theta + kz + \omega t)} \qquad \text{(B22)}$$

are solutions of Eq. (B5). If the normalization factor is chosen as before, the v_λ satisfy Eq. (3.24).

The formal vacuum expectation value of T_{tt} is obtained by substituting Eq. (3.25) into Eq. (3.28):

$$\langle 0|T_{tt}|0\rangle = -\sum_\lambda \omega_\lambda v_\lambda^\dagger v_\lambda\,. \qquad \text{(B23)}$$

Regularize this quantity by separating the points at which the field operators are evaluated. The regularized quantity may be written as

$$\begin{aligned}\langle T_{\mu\nu}(x,x')\rangle &= -\sum_\lambda \omega_\lambda v_\lambda^\dagger(x')v_\lambda(x) \\ &= -\frac{1}{8\pi^2}\int_0^\infty dK K\int_{-\infty}^\infty dk\, e^{i\Delta z k}\Bigg\{(\omega - k)\sum_{l=-\infty}^\infty J_{|l-1/2|}(Kr)J_{|l-1/2|}(Kr') \\ &\qquad + (\omega + k)\Bigg[\sum_{l=1}^\infty J_{l+1/2}(Kr)J_{l+1/2}(Kr') \\ &\qquad + \sum_{l=-\infty}^0 J_{-l-1/2}(Kr)J_{-l-1/2}(Kr')\Bigg]e^{i(l\Delta\theta - \omega\Delta t)}\Bigg\}\,, \end{aligned} \qquad \text{(B24)}$$

where the fact that

$$\sum_\lambda \to \frac{LR}{2\pi^2}\int_0^\infty dK\int_{-\infty}^\infty dK\int_{-\infty}^\infty dk\sum_{l=-\infty}^\infty \qquad \text{(B25)}$$

as $L, R \to \infty$, has been used.

Set $t = t'$ and evaluate the sums and integrals using Eqs. (A11), (A12), and (A14) and

$$\int_{-\infty}^{\infty} dk\omega\, e^{i\Delta z k} = -\frac{2K}{|\Delta z|} K_1(K\Delta z) = \frac{2}{\Delta z}\frac{d}{dz} K_0(K\Delta z)\,, \tag{B26}$$

where we take $\Delta z - z' > 0$. The result is

$$\langle T_{tt}(x,x')\rangle_{t=t'} = \tfrac{1}{4}\pi^{-1}(rr')^{-1}\{[r^2 + r'^2 + 2rr' + (\Delta z)^2]^{-1/2}[r^2 + r'^2 - 2rr'\cos\Delta\theta + (\Delta z)^2]^{-1} - 2[r^2 + r'^2 + 2rr' + (\Delta z)^2]^{1/2}[r^2 + r'^2 - 2rr'\cos\Delta\theta + (\Delta z)^2]^{-2}\}\,. \tag{B27}$$

Let $r = r'$ and $\theta = \theta'$. Then as $\Delta z \to 0$,

$$\langle T_{tt}(x,x')\rangle_{t=t',r=r',\theta=\theta'} \sim -\frac{1}{16\pi(\Delta z)^4} - \frac{1}{128\pi r^4} + O((\Delta z)^2)\,. \tag{B28}$$

The term proportional to $(\Delta z)^{-4}$ is the usual divergent Minkowski-space vacuum energy. The finite energy density for a string is obtained by subtracting this term:

$$\langle T_{tt}\rangle = -\frac{1}{128\pi r^4}\,. \tag{B29}$$

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