Gravitational waves from rotating and precessing rigid bodies. II. General solutions and computationally useful formulas

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A rigid, freely precessing Newtonian body emits gravitational radiation. In this paper I review the classical-mechanics results for free precession which are needed in order to calculate the weak-field slow-motion quadrupole-moment gravitational waves. Within that formalism, I give algorithms for computing the exact gravitational power radiated and waveforms produced by arbitrary rigid-body freely precessing sources. I also present the dominant terms in series expansions of the waveforms for the case of an almost-spherical object precessing with a small wobble angle. These series expansions, which retain the precise frequency dependence of the waves, may be useful for gravitational astronomers when freely precessing sources begin to be observed.

I. INTRODUCTION

In this paper I analyze the quadrupole gravitational radiation emitted by a freely precessing, rigid, Newtonian body. An earlier work[1] (hereinafter referred to as paper I) presented the solutions for axisymmetric objects and, in the small-wobble-angle limit, an approximate solution for nonaxisymmetric bodies. Paper I also discussed some astrophysical applications of those calculations to neutron stars as sources of gravitational waves. Here, I give algorithms for computing the exact results for the gravitational power radiated and waveforms produced by an arbitrary rigid Newtonian object, rotating free of external torques, in the standard quadrupole-moment formalism. I also give computationally useful formulas for the interesting case of an almost-spherical object precessing with a small wobble angle. These series expansions retain the precise frequency dependence of the waves—an important point for observers who may have to integrate over long times in order to see a signal. The results are compared with the simpler, approximate waveforms of paper I. Since that paper discussed at length the application of these calculations to astrophysical systems, only a few remarks on that topic are included here.

Section II of this paper reviews some of the classical Newtonian-mechanics results for free precession, defines the coordinate system and terminology used herein, and presents formulas useful for calculations of the power radiated in gravitational waves by a rotating rigid body. That section also gives the dominant terms in the gravitational luminosity for an object with small wobble angle, small oblateness, and small nonaxisymmetry, and interprets those terms. Section III reviews more of the classical free-precession results, and uses them to derive formulas for the gravitational waveforms $h_+(t)$ and $h_\times(t)$. That section also presents explicitly the dominant terms, with their exact frequency dependences, for the same astrophysically relevant limit as in Sec. II. The waveforms are interpreted and compared with the approximate results of paper I. Figures 1 and 2 show the exact results for h_+ and $h_\times$ as calculated according to the algorithm discussed in Sec. III C, in two specific cases, for a variety of observer inclinations relative to the precessing body. Finally, Sec. IV summarizes the conclusions of this paper.

II. POWER RADIATED IN GRAVITATIONAL WAVES

A. Review of classical free-precession results and specification of coordinate system

Throughout this paper, I shall use the physical conventions of Landau and Lifshitz[2] in describing rigid-body motions, and the mathematical notation of Abramowitz and Stegun[3] for elliptic functions and integrals. Much of the material necessarily repeated here in the course of specifying the problem is taken directly from Ref. 2. I work in units where $G=c=1$.

A rigid, Newtonian object in flat space has its inertial properties completely specified by its mass and by a symmetric tensor $\overleftrightarrow{I}$ with components $I_{ij} \equiv \int \rho(\delta_{ij}r^2 - x_i x_j)d^3x$. In some noninertial coordinate system called the "body frame" $\overleftrightarrow{I}$ is diagonalized, with diagonal components I_1, I_2, I_3, and the center of mass of the object is stationary at the origin. Choose the body-frame unit basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ to form a right-handed coordinate system such that $I_1 < I_2 < I_3$. (If any two of the principal moments of inertia are equal, the analysis in paper I applies.) I shall use Latin subscripts for components of tensors evaluated in the inertial-space reference frame, and Greek subscripts in the body frame. When specific components are referred to explicitly, the letters x, y, and z are used in the inertial frame and the digits 1, 2, and 3 in the body frame.

The components of a tensor (such as $\overleftrightarrow{I}$) in the body frame and in the inertial frame are related by the "rotation matrix" $R_{j\mu} \equiv \vec{e}_j \cdot \vec{e}_\mu$. At any moment, the body frame's instantaneous angular velocity may be described by a vector $\vec{\Omega}$. The total angular momentum of the body is $\vec{J} = \overleftrightarrow{I} \cdot \vec{\Omega}$, a constant (if gravitational radiation-reaction torques are ignored). Choose the coordinate system of the inertial frame so that $\vec{J} = J\vec{e}_z$.

The orientation of the body frame relative to the inertial system is described by three Euler angles: θ is the angle between $\vec{e}_z$ and $\vec{e}_3$, φ is the longitude of the ascending node (that is, the angle between $\vec{e}_x$ and the line of nodes formed by the intersection of the $\vec{e}_x$-$\vec{e}_y$ plane and the $\vec{e}_1$-$\vec{e}_2$ plane), and ψ is the angle in the $\vec{e}_1$-$\vec{e}_2$ plane between the line of nodes and $\vec{e}_1$. (See Sec. 35 of Ref. 2 for illustrations and comments.)

Choose the origin of time and the orientation of $\vec{e}_x$ and $\vec{e}_y$ such that at $t = 0$, θ is at its maximum value, $\psi = \pi/2$, and $\varphi = 0$; that is, $\vec{e}_2$ lies in the $\vec{e}_x$-$\vec{e}_y$ plane and $\vec{e}_1$ and $\vec{e}_3$ lie in the $\vec{e}_y$-$\vec{e}_z$ plane. (This completes the specification of the two coordinate systems, and results in formulas which agree with the conventions of paper I and Ref. 2.)

If the components of $\vec{\Omega}$ in the body frame are denoted by Ω_1, Ω_2, and Ω_3, then the body has rotational energy $E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$ and angular momentum $J \equiv |\vec{J}| = (I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2)^{1/2}$. Now, for specificity, make one additional assumption about the precession: Assume that $J^2 > 2EI_2$. This is equivalent to assuming that, in the body frame, the apparent precessional motion of $\vec{J}$ is a closed curve around the $\vec{e}_3$ axis. (If $J^2 = 2EI_2$, the motion of $\vec{J}$ is along a curve passing through the $\vec{e}_2$ axis and the solutions for the gravitational radiation may be obtained as a limit of the equations given below. If $J^2 < 2EI_2$, the motion of $\vec{J}$ is along a closed curve around the $\vec{e}_1$ axis, and by consistently interchanging the indices 1 and 3 below, the correct solutions appear.)

The components of $\vec{\Omega}$ in the body frame are simple elliptic functions of time. Define the initial-value constants $a \equiv \Omega_1(t=0)$ and $b \equiv \Omega_3(t=0)$, and the dimensionless time variable τ according to the equation

$$\tau = bt\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]^{1/2}. \tag{1}$$

Then

$$\Omega_1 = a\,\mathrm{cn}\tau\,,$$
$$\Omega_2 = a\left[\frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\right]^{1/2}\mathrm{sn}\tau\,, \tag{2}$$
$$\Omega_3 = b\,\mathrm{dn}\tau\,.$$

The parameter m of the elliptic functions in Eqs. (2) is

$$m = \frac{(I_2 - I_1)I_1 a^2}{(I_3 - I_2)I_3 b^2}. \tag{3}$$

As $m \to 0$, $\mathrm{sn}\tau \to \sin\tau$, $\mathrm{cn}\tau \to \cos\tau$, $\mathrm{dn}\tau \to 1$, and the solutions reduce to the symmetric-object solutions of paper I. The elliptic functions are periodic in their argument τ, with period $4K$ where $K(m)$ is the "complete elliptic integral of the first kind" defined and tabulated in Ref. 3.

B. Derivation of equations useful for the quadrupole-moment formalism calculation

The quadrupole-moment formalism[4] says that the total energy radiated per unit time in gravitational waves is

$$\tfrac{1}{5}\langle \dddot{I\!\!\!-}_{jk}\dddot{I\!\!\!-}_{jk}\rangle \text{ where } I\!\!\!-_{jk} \equiv \int \rho(x_j x_k - \tfrac{1}{3}\delta_{jk} r^2)d^3x$$

and

$$\dddot{I\!\!\!-}_{jk} \equiv \frac{d^3}{dt^3} I\!\!\!-_{jk} = -\frac{d^3}{dt^3} I_{jk}.$$

The angular brackets denote a time average over a few periods.

The solution for the body's precessional motion is much simpler in the body frame than in the inertial frame, so it is profitable to work in the body frame as much as possible. In evaluating the total power radiated in gravitational waves, in fact, one can work entirely in the body frame, and I shall do so.

Since $I_{jk} = R_{j\mu}R_{k\nu}I_{\mu\nu}$ and the body-frame $I_{\mu\nu}$ is constant, simple differentiation with respect to time gives

$$\dddot{I}_{jk} = I_{\mu\nu}(\dddot{R}_{j\mu}R_{k\nu} + 3\ddot{R}_{j\mu}\dot{R}_{k\nu} + 3\dot{R}_{j\mu}\ddot{R}_{k\nu} + R_{j\mu}\dddot{R}_{k\nu}).$$

The derivatives of the rotation matrices are

$$\dot{R}_{j\mu} = \epsilon_{jkl}\Omega_k R_{l\mu} = \epsilon_{\nu\mu\gamma}\Omega_\nu R_{j\gamma}\,,$$
$$\ddot{R}_{j\mu} = \epsilon_{\nu\mu\gamma}\dot{\Omega}_\nu R_{j\gamma} + \Omega_\mu\Omega_\nu R_{j\nu} - |\vec{\Omega}|^2 R_{j\mu}\,, \tag{4}$$
$$\dddot{R}_{j\mu} = [\epsilon_{\mu\gamma\beta}(\ddot{\Omega}_\beta - |\vec{\Omega}|^2\Omega_\beta) + 2\Omega_\mu\dot{\Omega}_\gamma + \dot{\Omega}_\mu\Omega_\gamma]R_{j\gamma}$$
$$-3\Omega_\beta\dot{\Omega}_\beta R_{j\mu}\,.$$

Taking Eqs. (4) and plugging into the equation for $\dddot{I}_{jk}$ yields $\dddot{I}_{jk} = R_{j\mu}R_{k\nu}B_{\mu\nu}$ where the body-frame quantity $B_{\mu\nu}$ is

$$B_{\mu\nu} \equiv -6\Omega_\gamma\dot{\Omega}_\gamma I_{\mu\nu} + I_{\gamma\nu}[\epsilon_{\gamma\mu\epsilon}(\ddot{\Omega}_\epsilon - 4|\vec{\Omega}|^2\Omega_\epsilon) + 2\Omega_\gamma\dot{\Omega}_\mu + \dot{\Omega}_\gamma\Omega_\mu] + I_{\mu\gamma}[\epsilon_{\gamma\nu\epsilon}(\ddot{\Omega}_\epsilon - 4|\vec{\Omega}|^2\Omega_\epsilon) + 2\Omega_\gamma\dot{\Omega}_\nu + \dot{\Omega}_\gamma\Omega_\nu]$$
$$+ 3I_{\gamma\delta}\Omega_\epsilon[\Omega_\gamma(\epsilon_{\epsilon\delta\nu}\Omega_\mu + \epsilon_{\epsilon\delta\mu}\Omega_\nu) + \dot{\Omega}_\eta(\epsilon_{\epsilon\delta\nu}\epsilon_{\eta\gamma\mu} + \epsilon_{\epsilon\delta\mu}\epsilon_{\eta\gamma\nu})]. \quad (5)$$

The problem of calculating the total power radiated P thus reduces to the problem of evaluating $P = \frac{1}{5}\langle \dddot{I}_{jk}\dddot{I}_{jk}\rangle = \frac{1}{5}\langle B_{\mu\nu}B_{\mu\nu}\rangle$.

The terms of $B_{\mu\nu}$ are not really as complicated as they may appear to be when written in tensorial notation. Using the fact that $I_{\mu\nu}$ is diagonal in the body frame, one finds

$$B_{11} = 6(\Delta_2\Omega_2\dot{\Omega}_2 - \Delta_3\Omega_3\dot{\Omega}_3 - \Delta_1\Omega_1\Omega_2\Omega_3), \quad (6)$$

where

$$\Delta_1 \equiv I_2 - I_3, \quad \Delta_2 \equiv I_3 - I_1, \quad \Delta_3 \equiv I_1 - I_2. \quad (7)$$

The other diagonal components of $B_{\mu\nu}$ follow by cycling the indices $1 \to 2 \to 3 \to 1$. For the off-diagonal terms

$$B_{12} = \Delta_3(\ddot{\Omega}_3 - 4|\vec{\Omega}|^2\Omega_3) + \dot{\Omega}_1\Omega_2(2\Delta_1 - \Delta_2)$$
$$-\dot{\Omega}_2\Omega_1(2\Delta_2 - \Delta_1) - 3\Omega_1{}^2\Omega_3\Delta_2 - 3\Omega_2{}^2\Omega_3\Delta_1, \quad (8)$$

and the other components of $\vec{B}$ follow by cycling the indices and by symmetry $(B_{\mu\nu} = B_{\nu\mu})$.

Equation (5) is quite general and in fact can be used to calculate for any time-varying rotation rate $\vec{\Omega}(t)$ the inertial-frame time derivatives of any rank-2 tensor $\vec{I}$ which is constant in the body frame. For our special case, where $\vec{\Omega}$ is that of free precession and $\vec{I}$ is the inertia tensor, the equations of motion and their derivatives determine the derivatives of Ω_μ:

$$\dot{\Omega}_\mu = I^{-1}{}_{\mu\beta}\epsilon_{\beta\gamma\delta}I_{\gamma\epsilon}\Omega_\epsilon\Omega_\delta,$$
$$\ddot{\Omega}_\mu = I^{-1}{}_{\mu\beta}\Omega_\beta\Omega_\gamma I_{\gamma\delta}\Omega_\delta - |\vec{\Omega}|^2\Omega_\mu \quad (9)$$
$$+ I^{-1}{}_{\mu\beta}\epsilon_{\beta\gamma\delta}I_{\gamma\epsilon}\Omega_\epsilon I^{-1}{}_{\delta\eta}\epsilon_{\eta\kappa\sigma}I_{\kappa\chi}\Omega_\sigma\Omega_\chi.$$

Using these identities to remove the derivatives of $\vec{\Omega}$ from $B_{\mu\nu}$ gives

$$B_{11} = 6\Omega_1\Omega_2\Omega_3\left(-\Delta_1 + \frac{\Delta_2{}^2}{I_2} - \frac{\Delta_3{}^2}{I_3}\right),$$
$$B_{12} = -4\Omega_3|\vec{\Omega}|^2\Delta_3 + \Delta_1\Omega_2{}^2\Omega_3\left(\frac{\Delta_3{}^2}{I_1I_3} + \frac{2\Delta_1 - \Delta_2}{I_1} - 3\right) \quad (10)$$
$$+ \Delta_2\Omega_1{}^2\Omega_3\left(\frac{\Delta_3{}^2}{I_2I_3} - \frac{2\Delta_2 - \Delta_1}{I_2} - 3\right).$$

The other components follow by symmetry and by cyclically permuting subscripts.

In order to evaluate the actual power radiated in gravitational waves, it is necessary to know the average values over a cycle of $\mathrm{sn}^2\tau$, $\mathrm{sn}^4\tau$, $\mathrm{sn}^6\tau$, etc. These can be expressed in terms of the complete elliptic integrals of the first and second kinds, $K(m)$ and $E(m)$ (see Ref. 3). The results of time averaging over a cycle are[5]

$$\langle \mathrm{sn}^2\tau\rangle = \frac{K-E}{mK} = \frac{1}{2} + \frac{m}{16} + \frac{m^2}{32} + \frac{41m^3}{2048} + \cdots,$$
$$\langle \mathrm{sn}^4\tau\rangle = \langle \mathrm{sn}^2\tau\rangle\left[\frac{2(1+m)}{3m}\right] - \frac{1}{3m}$$
$$= \frac{3}{8} + \frac{m}{16} + \frac{35m^2}{1024} + \cdots, \quad (11)$$
$$\langle \mathrm{sn}^6\tau\rangle = \langle \mathrm{sn}^4\tau\rangle\left[\frac{4(1+m)}{5m}\right] - \langle \mathrm{sn}^2\tau\rangle\left(\frac{3}{5m}\right)$$
$$= \frac{5}{16} + \frac{15m}{256} + \cdots.$$

The identities $\mathrm{cn}^2\tau = 1 - \mathrm{sn}^2\tau$ and $\mathrm{dn}^2\tau = 1 - m\,\mathrm{sn}^2\tau$ which relate other elliptic functions to $\mathrm{sn}\tau$ enable all of the other averages to be calculated from the above ones. From these averages, the exact power output in gravitational radiation is straightforward to write out.

C. Exact quadrupole-moment gravitational luminosity

The total power P radiated in gravitational waves depends on the parameters I_1, I_2, and I_3 (principal moments of inertia of the rigid body), and a and b (initial values of the components of the body's angular velocity along the $\vec{e}_1$ and $\vec{e}_3$ body axes).

To compute the total gravitational luminosity for any choice of these parameters, one can proceed as follows: (1) evaluate the elliptic-function parameter m from Eq. (3); (2) evaluate the averages over a cycle $\langle \mathrm{sn}^6\tau\rangle$, $\langle \mathrm{sn}^2\tau\,\mathrm{cn}^4\tau\rangle$, $\langle \mathrm{sn}^2\tau\,\mathrm{cn}^2\tau\,\mathrm{dn}^2\tau\rangle$, etc. of the various combinations of even powers of $\mathrm{sn}\tau$, $\mathrm{cn}\tau$, and $\mathrm{dn}\tau$ with exponents adding up to 6, using Eqs. (11) and the elliptic function identities which follow them; (3) evaluate the averages $\langle B_{\mu\nu}{}^2\rangle$ for μ,ν running 1 through 3, using Eqs. (2), (7), (10), and the averages calculated in step (2); (4) add up the results of step (3) and divide by 5 to get $P \equiv \frac{1}{5}\langle B_{\mu\nu}B_{\mu\nu}\rangle$, the quadrupole-moment formalism result for the luminosity in gravitational waves.

D. Series expansions for small wobble angle, small oblateness, and near-axisymmetry

Because the gravitational power radiated P must be invariant under a reversal of the direction of rotation $(\vec{\Omega} \to -\vec{\Omega})$, P contains only even powers of Ω_1, Ω_2, and Ω_3. Define coefficients F_μ, G, and $H_{\mu\nu}$ for each of the types of terms in $\vec{\Omega}$ by

$$P \equiv \tfrac{1}{5}\left(F_\mu\langle\Omega_\mu{}^6\rangle + G\langle\Omega_1{}^2\Omega_2{}^2\Omega_3{}^2\rangle + H_{\mu\nu}\langle\Omega_\mu{}^4\Omega_\nu{}^2\rangle_{\mu\neq\nu}\right). \quad (12)$$

One can expand F_μ, G, and $H_{\mu\nu}$ for the interesting case of small oblateness, where the differences between the principal moments of inertia are small compared to the principal moments themselves. The results are simple; through order $\Delta_\mu{}^2$:

$$F_\mu = 32\Delta_\mu{}^2,$$

$$G = 100(\Delta_1{}^2 + \Delta_2{}^2 + \Delta_3{}^2) + 84(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3),$$

$$H_{\mu\nu} = 2(13\Delta_\mu + \Delta_\nu)(5\Delta_\mu + \Delta_\nu) \text{ for } \mu \neq \nu, \quad (13)$$

$$H_{11} = H_{22} = H_{33} \equiv 0.$$

The equation $F_\mu = 32\Delta_\mu{}^2$ is in fact exact to all orders in Δ_μ. The F_μ terms in P, which are proportional to a sixth power of a single body-frame angular velocity, are precisely $(32/5)(I_2-I_3)^2\langle\Omega_1{}^6\rangle$, $(32/5)(I_3-I_1)^2\langle\Omega_2{}^6\rangle$, and $(32/5)(I_1-I_2)^2\langle\Omega_3{}^6\rangle$. These are familiar from the case of rotation about a principal axis, where there are no other terms.

The expression for P in terms of F_μ, G, and $H_{\mu\nu}$ still contains unevaluated averages of angular velocities. In the astrophysically relevant case of small wobble angle, small oblateness, and near-axisymmetry those averages can be conveniently expanded. Small wobble angle means that the ratio of the body-frame angular velocities $\Omega_1(0)/\Omega_3(0) = a/b \ll 1$. Small oblateness implies that $(I_3-I_1)/I_3 \ll 1$ (since $I_1<I_2<I_3$, there is no need to mention I_2 here). Near-axisymmetry causes $(I_2-I_1)/(I_3-I_1) \ll 1$; that is, the equatorial moments of inertia are close to each other compared to their difference from the polar moment. If equal weights are given to all three of these small parameters, the power radiated by a freely precessing rigid body can be expanded to give, at lowest order,

$$P \approx \tfrac{32}{5} b^6(I_2-I_1)^2 + \tfrac{2}{5}a^2b^4(I_3-I_{1:2})^2, \quad (14)$$

where $I_{1:2}$ is some average of I_1 and I_2, the precise nature of which is irrelevant to this order.

This simple result for the gravitational luminosity is also quite reasonable. The first term, $(32/5)b^6(I_2-I_1)^2$, is the standard result for a rigid body freely rotating about its principal axis I_3 at angular velocity b. The second term is the small-wobble-angle limit of the energy radiated by a freely rotating axisymmetric rigid body,[1] with equatorial moments of inertia $I_1 = I_2$.

III. GRAVITATIONAL WAVEFORMS FROM FREE PRECESSION

A. Further review of classical free precession results

The calculation of the waveforms radiated by a precessing object is both simpler and more complex than the calculation of the total power radiated by that body. It is simpler in that only two time derivatives occur, instead of three, and that only terms linear in $\vec{I}$ occur, instead of terms quadratic. It is more complex in that the Euler angles of the body appear explicitly. It is also complicated somewhat by the appearance of one more parameter, the observer's inclination angle i relative to the invariant $\vec{J}$ direction.

The components Ω_1, Ω_2, and Ω_3 of $\vec{\Omega}$ in the body frame are periodic in time, with period

$$T \equiv \frac{4K}{b}\left[\frac{I_1I_2}{(I_3-I_2)(I_3-I_1)}\right]^{1/2} \quad (15)$$

[see Eqs. (1)–(3)].

The Euler angles θ and ψ are also periodic, with period $T/2$:

$$\cos\theta = \frac{I_3 b}{J}\,\mathrm{dn}\tau,$$
$$\tan\psi = \left[\frac{I_1(I_3-I_2)}{I_2(I_3-I_1)}\right]^{1/2}\frac{\mathrm{cn}\tau}{\mathrm{sn}\tau}. \quad (16)$$

Here and throughout I use the notation and initial-value choices of Sec. II A and of Ref. 2, wherein the classical free-precession results which I quote are derived. Note that if the oblateness of the body is small, the period T is very long. As $I_1 \to I_2$ and the object approaches axisymmetry, $m \to 0$, $\Omega_3(t) \to$ constant, and $T \to 2\pi I_1/[\Omega_3(I_3-I_1)]$, the usual free-precession period of a symmetric body. Note also that for precession around the $\vec{e}_3$ axis, $\dot\psi < 0$.

The Euler angle φ, unfortunately, is complicated; if it is written as a sum $\varphi \equiv \varphi_1 + \varphi_2$, then the function φ_1 can be expressed by

$$\exp[2i\varphi_1(t)] = \frac{\vartheta_4\left(\dfrac{2\pi t}{T} - i\pi\alpha\right)}{\vartheta_4\left(\dfrac{2\pi t}{T} + i\pi\alpha\right)}, \quad (17)$$

where α is a solution of $\mathrm{sn}(2i\alpha K) = iI_3b/(I_1a)$ and ϑ_4 is a theta function in the notation of Ref. 3. (Because of the common periodicity of the elliptic functions and the ϑ functions, all solutions α are equivalent.) If $K'(m) \equiv K(1-m)$ and $q \equiv \exp(-\pi K'/K)$, then a useful series expansion of φ_1 can be written

$$\varphi_1(t) = \sum_{n=1}^{\infty}\frac{-2q^n}{n(1-q^{2n})}\sin\left(\frac{4n\pi t}{T}\right)\sinh(2n\pi\alpha). \quad (18)$$

The function $\varphi_1(t)$ is periodic in t with period $T/2$. The other part of φ is a linear function of time, $\varphi_2(t)=2\pi t/T'$, where

$$\frac{2\pi}{T'}\equiv\frac{J}{I_1}-\frac{2i}{T}\frac{\vartheta_4'(i\pi\alpha)}{\vartheta_4(i\pi\alpha)}$$
$$=\frac{J}{I_1}+\frac{2b}{K}\left[\frac{(I_3-I_2)(I_3-I_1)}{I_1I_2}\right]^{1/2}$$
$$\times\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sinh(2n\pi\alpha). \qquad (19)$$

Thus, $\cos(\varphi_2(t))$ has a period T' not in general commensurate with T, and so the body's motion typically is nonperiodic. The period $T'\to 2\pi I_1/J$ as the body becomes axisymmetric.

B. Derivation of equations for the quadrupole-moment waveform calculation

The general expression for the waveforms radiated is a simple one: In the transverse-traceless gauge of Ref. 4, the dimensionless gravitational-wave amplitudes are

$$h_+\equiv h^{TT}_{\hat v\hat v}=-h^{TT}_{\hat w\hat w}=\frac{-1}{r}(\ddot I_{\hat v\hat v}-\ddot I_{\hat w\hat w}),$$
$$h_\times\equiv h^{TT}_{\hat v\hat w}=\frac{-2}{r}\ddot I_{\hat v\hat w}. \qquad (20)$$

In these equations, r is the distance from the observer to the source of the radiation, and $\hat v$ and $\hat w$ are unit vectors transverse to the waves' direction of propagation. Specifically, for a source at the origin of the inertial frame and a distant observer in the $\vec e_y$-$\vec e_z$ plane at colatitude i from the $\vec e_z$ axis, the vectors $\hat v$ and $\hat w$ may be defined as $\hat v\equiv\vec e_y\cos i-\vec e_z\sin i$ and $\hat w=-\vec e_x$. Such an observer would, in the usual astronomical convention, define the body's inclination to be angle i.

As in Sec. II, it is advantageous to work as much as possible in the body frame. Using the relation $\ddot I_{jk}=I_{\mu\nu}(\ddot R_{j\mu}R_{k\nu}+2\dot R_{j\mu}\dot R_{k\nu}+R_{j\mu}\ddot R_{k\nu})$, and substituting the results for $\dot R_{j\mu}$ and $\ddot R_{j\mu}$ from Sec. II B, Eqs. (4), I obtain

$$\ddot I_{jk}=R_{j\mu}R_{k\nu}A_{\mu\nu},$$

where

$$A_{\mu\nu}\equiv-2|\vec\Omega|^2I_{\mu\nu}+(\epsilon_{\delta\gamma\mu}\dot\Omega_\delta+\Omega_\gamma\Omega_\mu)I_{\gamma\nu}$$
$$+(\epsilon_{\delta\gamma\nu}\dot\Omega_\delta+\Omega_\gamma\Omega_\nu)I_{\mu\gamma}+2\epsilon_{\delta\gamma\mu}\epsilon_{\eta\chi\nu}\Omega_\delta\Omega_\eta I_{\gamma\chi} \qquad (21)$$

is defined completely in terms of body-frame quantities. Combining Eqs. (20) and (21) with the definitions of $\hat v$, $\hat w$, and inclination i, I obtain

$$h_+=\frac{-1}{r}[(\cos i R_{y\mu}-\sin i R_{z\mu})$$
$$\times(\cos i R_{y\nu}-\sin i R_{z\nu})-R_{x\mu}R_{x\nu}]A_{\mu\nu}, \qquad (22)$$
$$h_\times=\frac{2}{r}(\cos i R_{y\mu}-\sin i R_{z\mu})R_{x\nu}A_{\mu\nu},$$

where the explicit components of $A_{\mu\nu}$ are

$$A_{11}=2(\Delta_2\Omega_2{}^2-\Delta_3\Omega_3{}^2),$$
$$A_{12}=(\Delta_1-\Delta_2)\Omega_1\Omega_2+\Delta_3\dot\Omega_3 \qquad (23)$$
$$=(\Delta_1-\Delta_2+\Delta_3{}^2/I_3)\Omega_1\Omega_2,$$

and symmetry and cyclic index permutation give the rest.

The components of the rotation matrix $R_{j\mu}$ in terms of the Euler angles θ, φ, and ψ are reproduced here for convenient reference. They are

$$R=j\downarrow\overset{\mu\longrightarrow}{\begin{pmatrix}\cos\psi\cos\varphi-\cos\theta\sin\psi\sin\varphi & -\sin\psi\cos\varphi-\cos\theta\cos\psi\sin\varphi & \sin\theta\sin\varphi\\ \cos\psi\sin\varphi+\cos\theta\sin\psi\cos\varphi & -\sin\psi\sin\varphi+\cos\theta\cos\psi\cos\varphi & -\sin\theta\cos\varphi\\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta\end{pmatrix}}. \qquad (24)$$

C. Exact quadrupole-moment gravitational waveforms

The gravitational wave amplitudes radiated by a freely precessing, Newtonian rigid body depend on the parameters I_1, I_2, and I_3 (principal moments of inertia of the body), a and b (initial values of the components of the body's angular velocity along the $\vec e_1$ and $\vec e_3$ body axes), i (inclination angle of the observer relative to the invariant $\vec J$ direction of the body), and time t.

To compute the gravitational waveforms h_+ and $h_\times$ for any choice of these parameters, one can proceed as follows: (1) Evaluate the elliptic function parameter m from Eq. (3); (2) evaluate the constant α defined by $\mathrm{sn}(2i\alpha K(m))=iI_3b/(I_1a)$ [following Eq. (17)]; (3) evaluate the time parameter τ using Eq. (1), the angular velocities Ω_1, Ω_2, and Ω_3 at time τ using Eqs. (2), and the Euler angles θ, φ, and ψ using Eqs. (15)–(19); (4) evaluate the components of $A_{\mu\nu}$ and $R_{j\mu}$ using Eqs. (23) and (24); (5) plug the results of the preceding evaluations into Eqs. (22) to compute $h_+(t)$ and $h_\times(t)$. This algorithm was used to calculate the waveforms shown in Figs. 1 and 2, which are discussed in the following subsection.

D. Series expansions for small wobble angle, small oblateness, and small nonaxisymmetry

While arbitrarily accurate values for h_+ and $h_\times$ may be computed using the algorithm described above, for many purposes it may be more useful to have available the first terms of a series expansion of the gravitational waveforms. In making these expansions one must be careful not to lose the correct, exact frequency dependence of the waves. Experiments to detect nearly monochromatic gravitational radiation often need to integrate for long times in order to build up an observable signal. Hence small errors in the calculated power spectrum are dangerous. There also may exist several closely spaced frequency components in the radiation, which will be confused and confounded by a series expansion that fails to preserve the correct frequency spectrum.

To make the expansions possible, in addition to demanding small elliptic function parameter m, it is also convenient to demand that the wobble angle be small and that the parameter

$$\delta \equiv 1-\left[\frac{I_1(I_3-I_2)}{I_2(I_3-I_1)}\right]^{1/2}$$

be small. This allows expansion of $\cos\psi$. The assumption of small δ is equivalent to the assumption that the body's nonaxisymmetry is not too large.

The resulting expansions of the cosines of the Euler angles are

$$\cos\theta=\frac{bI_3}{J}\left[1+\tfrac{1}{4}m(\cos 2\nu-1)+O(m^2)\right],$$

$$\cos\psi=\sin\nu\left[1+(\delta+\tfrac{1}{4}m)\cos^2\nu+O(\delta^2,m^2,m\delta)\right], \qquad (25)$$

$$\cos\varphi=\cos\frac{2\pi t}{T'}+\tfrac{1}{8}m\sinh(2\pi\alpha)\sin\frac{2\pi t}{T'}\sin 2\nu+O(m^2),$$

where $\nu\equiv\pi\tau/(2K)=2\pi t/T$.

One may now plug in and grind these explicit Euler angles through the equations for h_+ and $h_\times$. The results are simple and interesting for the astrophysically important case of small wobble angle, small oblateness, and near axisymmetry discussed in Sec. II:

$$h_+=\frac{-2}{r}(1+\cos^2 i)(I_2-I_1)\Omega^2\cos(2\Omega t)+\frac{\sin(2i)}{r}(I_3-I_{1:2})\left(\frac{aI_1}{bI_3}\right)\left(\frac{2\pi}{T'}\right)^2\cos\left(\frac{2\pi t}{T'}\right),$$

$$h_\times=\frac{-4}{r}\cos i(I_2-I_1)\Omega^2\sin(2\Omega t)+\frac{2}{r}\sin i(I_3-I_{1:2})\left(\frac{aI_1}{bI_3}\right)\left(\frac{2\pi}{T'}\right)^2\sin\left(\frac{2\pi t}{T'}\right), \qquad (26)$$

where $\Omega\equiv(2\pi/T')-(2\pi/T)$ and $I_{1:2}$ is an average of I_1 and I_2 (as before). These are the dominant terms in the radiation; corrections are of higher order in m, δ, $aI_1/(bI_3)$, $(I_3-I_1)/I_3$, and $(I_2-I_1)/(I_3-I_1)$. Equations (26) do, however, retain the exact frequency dependence of the dominant parts of the waves in the period T'. (The cost is that T' obeys a messy transcendental equation.) The results here agree with Eqs. (2) of paper I, where a simpler expansion was made which only gave the waves' approximate frequencies.

As was the case in Sec. II, the dominant components of h_+ and $h_\times$ [Eqs. (26)] have a simple physical interpretation. The waves at frequency 2Ω with strength independent (to this order in the expansion) of the wobble angle are from the differing moments of inertia I_1 and I_2. They are identical in strength, frequency, and angular distribution to the waves produced by a simple rigid rotor (a spinning dumbbell, for example). The waves at frequency $2\pi/T'$ are the small-wobble-angle limit of the waves produced by a freely precessing, axisymmetric ($I_1=I_2$) object [Eqs. (1) of paper I]. As in that case, the mean frequency of pulses seen from a spot fixed on the body's surface is not equal to the gravitational-wave frequency; the two differ by the precession frequency $2\pi/T$. As discussed in paper I, this frequency splitting may cause difficulties for some gravitational-wave detectors which rely on a high-Q system, mechanically synchronized with a pulsar's electromagnetic pulses, to integrate up an observable signal. On the other hand, if the frequency splitting can be observed, it will provide a direct measurement of a pulsar's oblateness. Other details of the gravitational waveforms give information about wobble angle, inclination, and nonaxisymmetry—information difficult or impossible to obtain by electromagnetic means. See paper I for a detailed discussion.

Figure 1 shows the computed waveforms h_+ and $h_\times$ for a freely precessing, nearly axisymmetric body ($I_1/I_3=0.99$, $I_2/I_3=0.991$) moving with a fairly small wobble angle ($a/b=0.1$). The exact solution as graphed agrees with the first terms in the series expansion [Eqs. (26)] to within the expected accuracy of $\sim 10\% \sim |a/b| \sim \delta$. The particular choice of initial conditions at $t=0$ used in this paper, and the location of the observer in the $\vec{e}_y$-$\vec{e}_z$ plane, produces the particular phase relationship between h_+ and $h_\times$ evident near $t=0$. At later times, the frequency splitting due to the (in this case slow) body-frame precession changes the relative phases of the two wave polarizations. The ~10% contributions from terms not retained in Eqs. (26) also cause slow (timescale T) amplitude variations of the waves; the variations are espe-

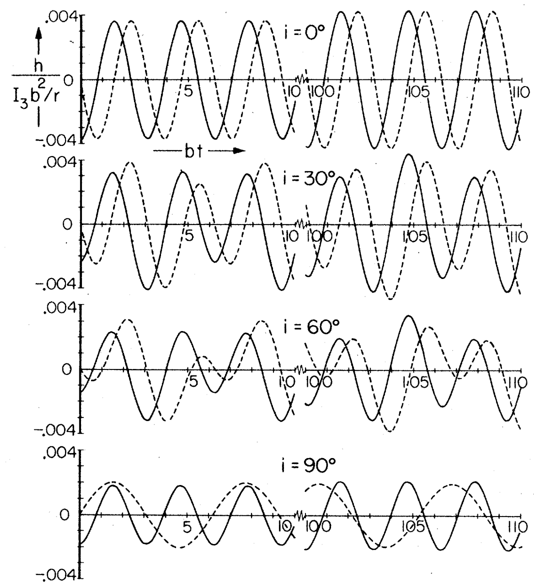

FIG. 1. Gravitational radiation waveforms h_+ (solid lines) and $h_\times$ (dashed lines) measurable by observers at inclinations $i = 0°$, 30°, 60°, and 90° relative to the $\vec{J}$ of the freely precessing, rigid, Newtonian source. The algorithm and equations of Sec. III were applied to a body with principal moments of inertia $I_1/I_3 = 0.99$ and $I_2/I_3 = 0.991$, rotating with initial values of its angular velocities $\Omega_1(0)/\Omega_3(0) \equiv a/b = 0.1$. For this case, solution of the equations of motion gave elliptic function parameter $m = 1.1 \times 10^{-3}$, $\alpha = 0.948\,93$, period $T' = 6.189\,06b$, and precession period $T = 656.19b$. The dimensionless units in terms of which h is plotted are

$$\frac{G}{c^4}\frac{I_3 b^2}{r} = 1.1 \times 10^{-21}\left(\frac{I_3}{10^{45}\,\mathrm{g\,cm^2}}\right)\left(\frac{b}{200\,\mathrm{rad\,sec^{-1}}}\right)^2 \times \left(\frac{1\,\mathrm{kpc}}{r}\right).$$

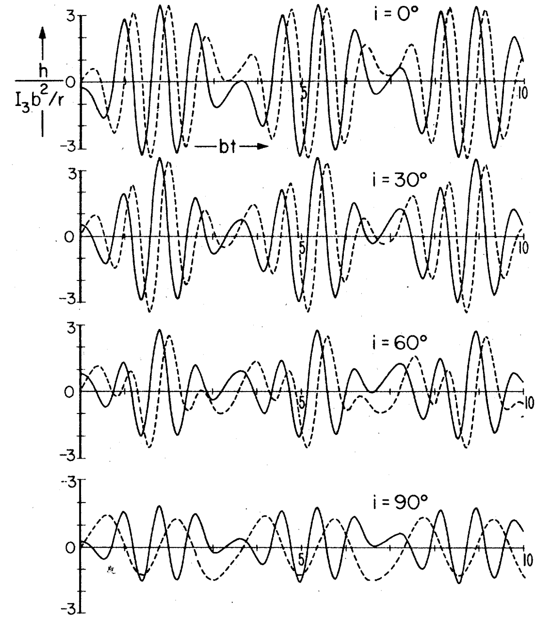

FIG. 2. Gravitational radiation waveforms h_+ (solid lines) and $h_\times$ (dashed lines) measurable by observers at inclinations $i = 0°$, 30°, 60°, and 90° relative to the $\vec{J}$ of the freely precessing, rigid, Newtonian source. In this case, $I_1/I_3 = \frac{1}{3}$, $I_2/I_3 = \frac{2}{3}$, $\Omega_1(0)/\Omega_3(0) \equiv a/b = 1$, $m = \frac{1}{3}$, $\alpha = 0.429\,786$, $T' = 1.790\,69b$, and $T = 6.935\,66b$. As in Fig. 1, the dimensionless units in terms of which h is plotted are GI_3b^2/rc^4.

cially visible at $i = 0$. The frequencies of the dominant Fourier components as calculated in Eqs. (26) are exact.

In Fig. 2, the waves emitted at various angles by a highly oblate ($I_1/I_3 = \frac{1}{3}$, $I_2/I_3 = \frac{2}{3}$) body precessing with a large wobble angle ($a/b = 1$) are shown. In this case the two timescales T and T' are of comparable magnitudes, and the waveforms at all inclinations i exhibit a wealth of information about their source.

IV. CONCLUSIONS AND OPEN QUESTIONS

The results given in Secs. II and III of this paper for the power and waveforms produced by a freely precessing, rigid, Newtonian body are simple applications of the quadrupole-moment formalism to one specific physical system. As discussed in paper I, these idealized calculations may be applicable to the astrophysically realistic case of a rapidly rotating neutron star. The sensitivities of gravitational experiments are improving at a rapid rate; it is conceivable that some precessing-body sources will be detectable within the next decade. The results presented here may then help save others some computational labor.

Papers I and II have only dealt with weak-field, slow-motion, small-stress sources (the standard Newtonian approximation to general relativity). Neutron stars have rather strong fields, since $GM/rc^2 \sim 0.2$ in typical models. I suspect, but have not proved, that the strong-field, slow-motion approximation to general relativity will give precisely the same waveform predictions as does the weak-field formalism, if the moment of inertia and quadrupole-moment tensors of the body are properly redefined. This topic might be worth further investigation. It might also be interesting to calculate more realistic models of precessing neutron stars, where the assumptions of infinite rigidity and zero external torques are relaxed. (Paper I, Sec. II, suggests but does not prove that such realistic models will typically not differ significantly from the models calculated here, except for having a longer precession time scale T.) Finally, more work on the interpretation of the gravitational waveforms might be valuable; paper I discussed how to deduce information about the source from the waves, but only for the cases of axisymmetric bodies and of small-wobble-angle precession for triaxial bodies.

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