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Bianchi type-I cosmologies and spinor fields

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Using Hamiltonian techniques, we derive the general solution to Einstein-Dirac equations (with a cosmological constant and a mass term) when the metric and the spinor field are both invariant under an Abelian three-parameter group of transformations acting transitively on spacelike hypersurfaces.

I. INTRODUCTION

In recent years, a lot of attention has been paid to massless spinor fields in curved spacetimes. Besides general theorems about the coupled Einstein-Weyl equations,[1] many explicit solutions have been constructed and studied (let us mention the plane-wave solution of Brill and Cohen,[2] among others[3]).

However, to our knowledge, the combined Einstein-Dirac equations with a mass term have only been solved in a few cases.[4,5] The purpose of this paper is to present a new class of solutions which correspond to homogeneous anisotropic universes of type I (according to the classification given by Bianchi) filled with a massive spinor field.

We use Hamiltonian techniques, particularly suited to spacetimes possessing spatial isometries. This enables us to derive easily a great number of conserved quantities, even when a cosmological term is included (Secs. II and III). We then carry out the integration of the equations (Secs. IV and V). We also discuss the introduction of torsion (Sec. VI).

Let us describe briefly some of the qualitative features of our solutions. Most of them represent anisotropically expanding universes with an initial singularity (for some values of the integration constants, the expansion is isotropic and the solutions reduce to the cosmologies studied by Isham and Nelson[4]). The behavior in time of the average rate of expansion—that is, the dependence of the "volume" $\sqrt{g}$ of these universes on the proper time t—is the same as in type-I models filled with a pressure-free perfect fluid (discussed for example in the book by Ryan and Shepley[6]). As shown below, this rather intriguing property is a consequence of the strict conservation law obeyed by the energy of the spinor field measured in a comoving frame. The expansion presents, however, different features from the perfect-fluid case due to the spin of the Dirac field. For instance, as already noticed by Belinsky and Khalatnikov for type-I models filled with neutrinos,[7] the principal directions of expansion, referred to the usual synchronous invariant coordinate systems,[6] vary with time. Accordingly, one cannot assume that the spatial metric $g_{ab}(t)$ is diagonal in these coordinate systems.

Besides its mathematical interest, we believe that the present study is of some physical relevance since the universe could have been highly anisotropic during its early stages. Although we describe the spin-$\frac{1}{2}$ particles with the help of a complex spinor field, the solutions given below still satisfy Einstein-Dirac equations when the spinor field components are regarded as "classical anticommuting numbers."

II. HAMILTONIAN FOR TYPE-I EINSTEIN-DIRAC MODELS

We are considering gravitational and spinor fields which are invariant under an Abelian three-parameter group of transformations G_3 acting transitively on spacelike hypersurfaces.[8] For simplicity, we use invariant orthonormal frames $h_{(\alpha)}{}^{\beta}$ ($\alpha, \beta, \ldots = 0, 1, 2, 3$; indices in parentheses are tetrad indices) chosen so that their timelike leg $h_{(0)}{}^{\beta}$ points along the (positive) normal to the hypersurfaces of transitivity. In suitable coordinate systems, the fields are functions of one coordinate only:

$$ds^2 = -N^2(x^0)(dx^0)^2 + g_{ab}(x^0)dx^a dx^b\,, \tag{2.1}$$

$$\begin{aligned} &h_{(0)}{}^0 = N^{-1}(x^0),\quad h_{(0)}{}^k = 0,\quad h_{(k)}{}^0 = 0\,,\\ &h_{(k)}{}^a(x^0)h_{(l)}{}^b(x^0)g_{ab}(x^0) = \eta_{(kl)} = \mathrm{diag}(+,+,+)\,, \end{aligned} \tag{2.2}$$

$$\psi = \psi(x^0),\quad \bar{\psi} \equiv i\psi^{\dagger}\gamma_{(0)} = \bar{\psi}(x^0) \tag{2.3}$$

($a, b, \ldots = 1, 2, 3$). The intrinsic geometry of the surfaces of transitivity $x^0 = \text{const}$ is thus flat, but their extrinsic curvature

$$K_{ab} = -\frac{1}{2N}g_{ab,0},\quad g_{ab,0} \equiv \frac{dg_{ab}}{dx^0} \tag{2.4}$$

does not vanish in general.

The lapse function $N(x^0)$ in (2.1) is arbitrary since the coordinate systems in which (2.1) holds are determined up to the coordinate transformations

$$x'^0 = f(x^0), \quad x'^a = \Lambda^a{}_b x^b , \tag{2.5}$$

where $f(x^0)$ is an arbitrary function of x^0 (such that $f_{,0} \neq 0$) and $\Lambda^a{}_b$ is an arbitrary 3 by 3 constant matrix (such that $\det\Lambda \neq 0$). If desired, one can make $N=1$ by a suitable choice for $f(x^0)$ in (2.5). The corresponding time coordinate, denoted by t, is called the proper time.

In the same way, the spacelike legs $h_{(k)a}$ are not completely determined by the relations (2.2). Indeed, one still has the freedom of performing arbitrary spatial rotations (depending only on x^0), the infinitesimal form of which reads

$$\delta h_{(a)}{}^l(x^0) = \omega_{(a}{}^{b)}(x^0) h_{(b)}{}^l(x^0) , \tag{2.6a}$$

$$\delta\psi(x^0) = \omega_{(ab)}(x^0) S^{(ab)}\psi(x^0), \quad S^{(ab)} = \tfrac{1}{8}[\gamma^{(a)}, \gamma^{(b)}] \tag{2.6b}$$

where $\omega_{(ab)} \equiv \eta_{(bc)}\omega_{(a}{}^{c)}$ are six arbitrary infinitesimal functions of x^0, antisymmetric in a, b (note that $\omega_{(ab)}$ is just short for $\omega_{(a)(b)}$). In view of this freedom, the three functions

$$\Omega_{(ab)} = \tfrac{1}{2}(h_{(a)c,0} h_{(b)}{}^c - h_{(b)c,0} h_{(a)}{}^c) \tag{2.7}$$

are completely arbitrary. It is sometimes convenient to impose the gauge conditions

$$\Omega_{(ab)} = 0 \tag{2.8a}$$

which are equivalent to

$$h_{(a)c,0} = \tfrac{1}{2} h_{(a)}{}^b g_{bc,0} . \tag{2.8b}$$

In this gauge, the triads $h_{(a)}{}^c$ are parallel propagated along the geodesics $x^a = \text{const}$. Note that the transformations (2.6) with constant $\omega_{(ab)}$ preserve the conditions (2.8).

It is straightforward to write Einstein-Dirac equations for the tetrads (2.2) and the spinor field (2.3). We assume some familiarity with the now classical work of Arnowitt, Deser, and Misner. The Dirac equations read[9]

$$\chi_{,0} = \Omega_{(ab)} S^{(ab)}\chi - Nm\gamma^{(0)}\chi , \tag{2.9}$$

where we have introduced the weighted spinors

$$\chi = g^{1/4}\psi, \quad g = \det g_{ab} \tag{2.10}$$

in order to get rid of the extrinsic curvature in the right-hand side of (2.9), while the Einstein equations become

$$\mathcal{H} \equiv g^{-1/2}(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2) + 2\lambda g^{1/2} - m\phi^\dagger\gamma^{(0)}\chi = 0 , \tag{2.11}$$

$$\pi^{ab}{}_{,0} + \frac{\partial\mathcal{H}}{\partial g_{ab}} - \tfrac{1}{2}K^a{}_c \mathcal{H}^{bc}_{1/2} - \tfrac{1}{2}K^b{}_c \mathcal{H}^{ac}_{1/2} = 0 . \tag{2.12}$$

In (2.11) and (2.12), we have defined, for reasons to be clarified soon,

$$\pi^{ab} = -(K^{ab} - Kg^{ab})g^{1/2}, \quad K = K^a{}_a, \quad \pi = \pi^a{}_a, \tag{2.13}$$

$$\phi^\dagger = \bar\psi\gamma^{(0)} g^{1/4} = i\chi^\dagger \tag{2.14}$$

(spatial indices are lowered and raised with g_{ab} and its inverse g^{ab}), and

$$\mathcal{H}^{ab}_{1/2} = h_{(l)}{}^a h_{(m)}{}^b \mathcal{H}^{(lm)}_{1/2}, \quad \mathcal{H}^{(lm)}_{1/2} = \phi^\dagger S^{(lm)}\chi . \tag{2.15}$$

When taking the derivatives of $\mathcal{H}$ with respect to the metric coefficients g_{ab} in (2.12), one keeps π^{ab}, χ, and $\phi^\dagger$ constant. The stress tensor of the spinor field [i.e., the last two terms in the left-hand side of the G_{ab} equations (2.12)] has two noticeable properties

$$T^{ab}g_{ab} = 0 \text{ and } T^{ab}K_{ab} = 0 \tag{2.16}$$

and cannot, in general, be diagonalized together with the metric g_{ab} and the extrinsic curvature K_{ab}. It identically vanishes in the exceptional case when the three eigenvalues of K_{ab} are equal (isotropy).

The set of equations (2.9), (2.11), and (2.12) (i.e., all the nontrivial Einstein-Dirac equations) can be deduced from a variational principle applied to the canonical action

$$S = \int (\pi^{(l)a} h_{(l)a,0} + \phi^\dagger\chi_{,0} - N\mathcal{H} - \Omega_{(ab)}\mathcal{H}^{(ab)}) dx^0, \tag{2.17}$$

in which the independent variables are the spatial components $h_{(l)a}$ of the tetrads, the spinor field χ, their conjugate momenta $\pi^{(l)a}$ and $\phi^\dagger$—thus $\phi^\dagger$ is not regarded as a function of χ in the action—and the Lagrange multipliers N, $\Omega_{(ab)}$. The quantities π^{ab} are treated as the following functions of the momenta $\pi^{(l)a}$:

$$\pi^{ab} = \tfrac{1}{4}(\pi^{(l)a} h_{(l)}{}^b + \pi^{(l)b} h_{(l)}{}^a) , \tag{2.18}$$

whereas the "super angular momentum" $\mathcal{H}^{(ab)} = -\mathcal{H}^{(ba)}$ is defined by

$$\mathcal{H}^{(ab)} = \mathcal{H}^{(ab)}_G + \mathcal{H}^{(ab)}_{1/2} , \tag{2.19}$$

$$\mathcal{H}^{(ab)}_G = \tfrac{1}{2}(\pi^{(a)k} h^{(b)}{}_k - \pi^{(b)k} h^{(a)}{}_k) . \tag{2.20}$$

We can of course introduce a Poisson bracket structure in the space of the dynamical variables

$$[h_{(l)a}, \pi^{(m)b}] = \delta^{(m)}_{(l)}\delta^b_a, \quad [\chi_A, (\phi^\dagger)^B] = \delta^B_A \tag{2.21}$$

(all others vanish) and rewrite the equations as

$$h_{(l)a,0} = [h_{(l)a}, H], \quad \psi_{A,0} = [\psi_A, H], \text{ etc.},$$

where the Hamiltonian is a linear combination of the normal and rotation generators $\mathcal{H}$ and $\mathcal{H}^{(ab)}$:

$$H = N\mathcal{H} + \Omega_{(ab)}\mathcal{H}^{(ab)}$$

($N, \Omega_{(ab)}$ arbitrary).

Varying the action with respect to the Lagrange multipliers $N, \Omega_{(ab)}$ one gets four first-class constraints on the dynamical variables

$$\mathcal{H} \approx 0, \quad \mathcal{H}^{(ab)} \approx 0, \tag{2.22}$$

$$[\mathcal{H},\mathcal{H}]=0, \quad [\mathcal{H},\mathcal{H}^{(ab)}]=0,$$
$$[\mathcal{H}^{(ab)},\mathcal{H}^{(cd)}]=\tfrac{1}{2}(-\eta^{(ac)}\mathcal{H}^{(bd)}-\eta^{(bd)}\mathcal{H}^{(ac)} + \eta^{(ad)}\mathcal{H}^{(bc)}+\eta^{(bc)}\mathcal{H}^{(ad)}), \tag{2.23}$$

which are mere consequences of the various invariances of the type-I Einstein-Dirac system for the transformations depending on arbitrary functions of the time x^0 described above.

The canonical decomposition of the Einstein-Dirac action in the general case (no symmetry) has been studied by Dirac[10] and more recently by Nelson and Teitelboim[11] (see also Ref. 12). In fact, the work of these authors has been our starting point for getting (2.17) since we have simply inserted in their results the particular form (2.2) and (2.3) of the fields, a procedure that is always justified in type-I cosmologies.[13] The reader, unaware of these developments, will easily check that the equations derived from the action (2.17) are indeed equivalent to Einstein-Dirac equations. This is rather straightforward once the following key points are understood: (i) The Poisson brackets between the metric [viewed as a function of the triads—formula (2.2)] and π^{ab} [given by (2.18)] read

$$[g_{ab},g_{cd}]=0, \quad [g_{ab},\pi^{cd}]=\tfrac{1}{2}(\delta_a^c\delta_b^d+\delta_a^d\delta_b^c), \tag{2.24}$$

as in the usual canonical metric formulation of pure gravity, but

$$[\pi^{ab},\pi^{cd}]=\tfrac{1}{8}(g^{ac}\mathcal{H}_G^{bd}+g^{ad}\mathcal{H}_G^{bc}+g^{bc}\mathcal{H}_G^{ad}+g^{bd}\mathcal{H}_G^{ac})\neq 0. \tag{2.25}$$

The equations $g_{ab,0}=[g_{ab},H]$ thus reproduce the definition of the quantities π^{ab} in terms of the metric and its temporal derivatives, whereas the Hamiltonian equations for π^{ab} are just the G_{ab} Einstein equations (2.12) (the contribution to $[\pi^{ab},H]$ due to (2.25) is precisely the stress tensor of the spinor field, owing to the super angular momentum constraints (2.22)). (ii) If one expresses the Lagrange multipliers $\Omega_{(ab)}$ in terms of the triads $h_{(l)a}$ and their temporal derivatives with the help of the Hamiltonian equations $h_{(l)a,0}=[h_{(l)a},H]$, one finds the definition (2.7) again. This is because the super-Hamiltonian is a function of the momenta $\pi^{(l)a}$ through the combinations π^{ab} only. (iii) The Hamiltonian equations for $\pi^{(l)a}$ are basically equivalent to those for π^{ab} since, by virtue of the super angular momentum constraints (2.22), one has

$$\pi^{(l)a}=2h^{(l)}{}_b\pi^{ab}-\mathcal{H}_{1/2}^{(lm)}h_{(m)}{}^a, \tag{2.26}$$

a relation which contains the momenta $\pi^{(l)a}$ only through the symmetric combinations π^{ab} in its right member.

As a final comment, we note the decoupled form of the super-Hamiltonian which is a sum of two terms, a purely gravitational one and a purely spinorial one. This rather remarkable property is due to the homogeneity requirements (which suppress the coupling terms containing spatial derivatives), of course, but also to our choice of the weighted spinors χ and $\phi^\dagger$ as spinorial variables (instead of ψ and $g^{1/2}\bar{\psi}\gamma^{(0)}$). In fact, the variables χ and $\phi^\dagger$ already play an interesting role in the general Einstein-Dirac system.[11]

III. CONSTANTS OF THE MOTION

The linear coordinate transformations

$$x'^0=x^0, \quad x'^a=(\Lambda^{-1})^a{}_b x^b,$$

where Λ is an arbitrary constant matrix possessing an inverse Λ^{-1}, induce the following changes in the canonical variables:

$$h'_{(l)a}=\Lambda^b{}_a h_{(l)b},$$
$$\pi'^{(l)a}=(\Lambda^{-1})^a{}_b \det\Lambda\, \pi^{(l)}{}_b, \tag{3.1}$$
$$\chi'=\chi\det{}^{1/2}\Lambda, \quad \phi^{\dagger\prime}=\phi^\dagger\det{}^{1/2}\Lambda$$

(remember that the momenta $\pi^{(l)a}$ are vector densities of unit weight). These transformations are not canonical, unless $\det\Lambda=1$, i.e., unless Λ belongs to SL(3). From the discussion of the previous paragraph, we expect the eight infinitesimal generators of SL(3),

$$P^a{}_b=\pi^{(l)a}h_{(l)b}-\tfrac{1}{3}\delta_b^a\pi^{(l)c}h_{(l)c}, \quad P^a{}_a=0, \tag{3.2}$$

to be conserved quantities and one indeed checks that they commute with the Hamiltonian (in the Poisson bracket sense):

$$[P^a{}_b,\mathcal{H}]=0, \quad [P^a{}_b,\mathcal{H}^{(cd)}]=0. \tag{3.3}$$

(3.3) follows from the fact that $\mathcal{H}$ and $\mathcal{H}^{(cd)}$ are densities of unit weight under changes of coordinates, and thus invariant for SL(3).

If the cosmological constant λ and the mass of the spinor field both vanish, the quantity $\pi^{(l)c}h_{(l)c}$, which generates the infinitesimal transformations

$$\delta h_{(l)a}=\epsilon[h_{(l)a},\pi^{(m)c}h_{(m)c}]=\epsilon h_{(l)a},$$
$$\delta\pi^{(l)a}=-\epsilon\pi^{(l)a} \tag{3.4}$$

(dilatations in which one forgets about the weight of the momenta $\pi^{(l)a}$) also gives rise to a conserved quantity. However, since the kinetic part, the potential $\lambda\sqrt{g}$, and the mass term have different weights for the linear transformations (3.4), $\pi^{(l)c}h_{(l)c}$ is not, in the general case ($\lambda\neq 0, m\neq 0$), a constant during the motion.

The conservation law obeyed by the eight generators of SL(3) (acting on the coordinate indices) is not peculiar to the spin-$\frac{1}{2}$ field. It is in fact a property valid for all type-I models filled (or not) with a matter field. What is peculiar to the spin-$\frac{1}{2}$ case is the completely decoupled form (2.11) of the super-Hamiltonian, and this leads to additional conserved quantities. These are the energy density (multiplied by $\sqrt{g}$) of the spinor field[14]

$$\tau_{tt}(\psi) = -m\phi^\dagger\gamma^{(0)}\chi \tag{3.5}$$

which generates "$\gamma^{(0)}$ transformations"

$$\delta\chi \sim \gamma^{(0)}\chi, \quad \delta\phi^\dagger \sim -\phi^\dagger\gamma^{(0)},$$

and the spinor part $\mathcal{H}_{1/2}^{(lm)}$ ($\approx -\mathcal{H}_G^{(lm)}$) of the super angular momentum. Let us remark, however, that $\mathcal{H}_{1/2}^{(lm)}$ commutes with the Hamiltonian H only when $\Omega_{(ab)}=0$ [see the commutation relations (2.23)]. We shall thus work in that gauge from now on.

Before proceeding further, let us rewrite the super-Hamiltonian $\mathcal{H}$ in terms of the conserved generators. Since

$$\pi^a{}_b - \tfrac{1}{3}\delta^a{}_b\pi = \tfrac{1}{2}P^a{}_b + \tfrac{1}{2}\mathcal{H}_G^{(lm)}h_{(l)}{}^a h_{(m)b}, \tag{3.6}$$

we get

$$\mathcal{H} = g^{-1/2}(\tfrac{1}{4}P^a{}_b P^b{}_a + \tfrac{1}{4}\mathcal{H}_G^{(lm)}\mathcal{H}_{G(lm)} - \tfrac{1}{6}\pi^2) + 2\lambda g^{1/2} - m\phi^\dagger\gamma^{(0)}\chi \approx 0, \tag{3.7}$$

and we conclude that the super-Hamiltonian constraint (the G_{00} equation) can be used to express π as a function of $g^{1/2}$ and the constants of the motion $P^a{}_b$, $\mathcal{H}_{1/2}^{(lm)}$, $\tau_{tt}(\psi)$.

IV. THE TETRADS AND THE METRIC

Let $2\mu^a{}_b$, $2C^{(lm)}$, and e be the actual values of the conserved quantities $P^a{}_b$, $\mathcal{H}_{1/2}^{(lm)}$, and τ_{tt} for a given motion.

Hamilton's equations for the tetrads and the metric read, using the relation (3.6),

$$h_{(l)a,0} = Ng^{-1/2}(h_{(l)b}\mu^b{}_a - C_{(l}{}^{p)}h_{(p)a} - \tfrac{1}{6}h_{(l)a}\pi), \tag{4.1}$$

$$g_{ab,0} = Ng^{-1/2}(g_{ac}\mu^c{}_b + g_{cb}\mu^c{}_a - \tfrac{1}{3}g_{ab}\pi), \tag{4.2}$$

from which one deduces

$$(g^{1/2})_{,0} = -\tfrac{1}{2}N\pi. \tag{4.3}$$

The problem of solving the Einstein equations is thus reduced, owing to the constants of the motion, to the problem of solving the (redundant) first-order differential equations (4.1)–(4.3) and (3.7).

The structure of these equations suggests making the following conformal decomposition, familiar from the study of the vacuum anisotropic models,[6]

$$\begin{aligned} &g_{ab} = g^{1/3}f_{ab}, \quad \det f_{ab} = 1, \\ &h_{(l)a} = g^{1/6}k_{(l)a}, \quad k_{(l)a}k^{(l)}{}_a = f_{ab}. \end{aligned} \tag{4.4}$$

Indeed, in the gauge

$$N = g^{1/2}, \quad g^{1/2}dx^0 = dt \tag{4.5}$$

(which is a good gauge since $g^{1/2} > 0$), the eight differential equations for the eight unimodular triads $k_{(l)a}$,

$$k_{(l)a,0} = k_{(l)b}\mu^b{}_a - C_{(l}{}^{p)}k_{(p)a} \tag{4.6}$$

(which are linear homogeneous equations) are immediately integrated by matrix exponentiation. This gives

$$k_{(l)a} = k^0_{(p)b}(e^{\mu x^0})^b{}_a(e^{-Cx^0})_{(l)}{}^{(p)}, \tag{4.7}$$

where $k^0_{(p)b}$ are a set of eight integration constants (they are restricted by the preserved-in-time condition $\det k_{(p)b} = 1$).

Equation (4.7) leads to the following expression for the unimodular metric f_{ab}:

$$f_{ab} = f^0_{cd}(e^{\mu x^0})^c{}_a(e^{\mu x^0})^d{}_b, \tag{4.8}$$

with

$$f^0_{cd} = k^0_{(l)c}k^0_{(m)d}\eta^{(lm)}. \tag{4.9}$$

Since the signature of a metric is preserved by the linear transformations, f_{ab} is always positive definite if it is positive definite at the time $x^0 = 0$.

The integration constants $k^0_{(l)c}$, $C^{(lm)}$, and $\mu^a{}_b$ are not all independent. This follows from the fact that (3.6) with the index a lowered is symmetric in a, b. Hence we have the relations (true at any instant if true initially)

$$f^0_{ac}\mu^c{}_b - f^0_{bc}\mu^c{}_a = 2C^{(lm)}k^0_{(l)a}k^0_{(m)b}, \tag{4.10}$$

which can be used, for example, to express the constants $C^{(lm)}$ in terms of the 16 $h^0_{(l)a}$, $\mu^a{}_b$. The relations (4.10) have an important consequence: they imply that the matrix $f^0_{ac}\mu^c{}_b$ is in general asymmetric. Consequently, one can show the following.

Theorem: the metric $f_{ab}(x^0)$—and thus also the conformally related metric $g_{ab}(x^0)$—can be diagonalized by a constant coordinate transformation $x'^a = \Lambda^a{}_b x^b$ if and only if one of the following conditions holds:

(i) The three constants $C^{(lm)}$ all vanish (this implies severe restrictions on the spinor ψ), or (ii) the metric admits a revolution axis, z say, and can thus be written

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2) + b^2(t)dz^2.$$

In that case, only $C_{xy} = -C_{yx}$ can be different from zero. When a and b are equal, the model is isotropic ($k=0$ Robertson-Walker cosmology) and there are no restrictions on the spinor angular momentum $C^{(lm)}$.

The demonstration of this theorem is straightforward and left to the reader. It uses some algebra and the relations (4.8), (4.10).

From this theorem, we conclude that the general case is not diagonal, unlike the vacuum or perfect-fluid solutions. Nevertheless, the coordinate transformations $x'^a = \Lambda^a{}_b x^b$ and the rotations of the tetrads can be used to simplify the "initial" data (for example, f_{ab} initially diagonal, or $\epsilon_{(klm)}C^{(lm)}$ along $h_{(3)}, \ldots$).

To get the metric completely, we still need to find the square root $g^{1/2} \equiv y$ of its determinant. This is accomplished with the help of the super-Hamiltonian constraint (3.7) and the relation (4.3) which, in the gauge (4.5), lead to

$$\frac{dy}{y(\alpha + ey + 2\lambda y^2)^{1/2}} = \pm\left(\tfrac{3}{2}\right)^{1/2} dx^0, \tag{4.11}$$

where

$$\begin{aligned}\alpha &\equiv \mu^a{}_b\mu^b{}_a + C^{(lm)}C_{(lm)} \\ &= \tfrac{1}{4}(\mu^{ab}+\mu^{ba})(\mu_{ab}+\mu_{ba})\end{aligned} \tag{4.12}$$

is strictly positive, except in the isotropic case (characterized by $\alpha = 0$). This equation, which is the same as in the pressureless perfect fluid type-I models, gives $g^{1/2}$ as a function of x^0 and is easily integrated. For the sake of clarity, we shall discuss here the simpler equation

$$\frac{dy}{(\alpha + ey + 2\lambda y^2)^{1/2}} = \pm\, dt', \quad t' = \left(\tfrac{3}{2}\right)^{1/2} t \tag{4.13}$$

which relates the square root of $\det g_{ab}$ to the more physical proper time t, and we shall assume that the physical condition $e \geq 0$ holds. The cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ need to be considered separately. One gets, with a suitable choice for the sign and origin of the t coordinate,

(a) $\lambda > 0$ (i) $\sigma \equiv e^2 - 8\alpha\lambda > 0$: $y = \dfrac{\sqrt{\sigma}}{4\lambda}\left[\cosh(\sqrt{2\lambda}\,t' + k) - \dfrac{e}{\sqrt{\sigma}}\right]$,

$\cosh k = \dfrac{e}{\sqrt{\sigma}} \geq 1, \; k \geq 0$

(ii) $\sigma = 0$: $y = \dfrac{e}{4\lambda}[\exp(\sqrt{2\lambda}\,t') - 1], \; e > 0$

$y = \exp(\sqrt{2\lambda}\,t'), \; e = 0$

(iii) $\sigma < 0$: $y = \dfrac{\sqrt{-\sigma}}{4\lambda}\left[\sinh(\sqrt{2\lambda}\,t' + k) - \dfrac{e}{\sqrt{-\sigma}}\right]$,

$\sinh k = \dfrac{e}{\sqrt{-\sigma}}$;

(b) $\lambda = 0$ (i) $e > 0$: $y = \dfrac{1}{e}\left[\left(\dfrac{et'}{2} + \sqrt{\alpha}\right)^2 - \alpha\right]$,

(ii) $e = 0$: $y = \sqrt{\alpha}\,t', \; \alpha \neq 0$

$y = \text{const}, \; \alpha = 0$

(c) $\lambda < 0$ $(\sigma > 0)$: $y = -\dfrac{\sqrt{\sigma}}{4\lambda}\left[\sin(\sqrt{-2\lambda}\,t' + k) + \dfrac{e}{\sqrt{\sigma}}\right]$,

$\sin k = -\dfrac{e}{\sqrt{\sigma}} \; (\geq -1)$,

$-\dfrac{\pi}{2} \leq k \leq 0$.

All these models possess a singularity ($\sqrt{g} = 0$) at $t = 0$ [except when $\alpha = 0$, $e = 0$, but $\lambda > 0$ corresponds to the de Sitter model, whereas $\lambda = 0$ is Minkowski space; in both cases $T_{\alpha\beta}(\psi) = 0$]. When the cosmological constant is negative, the expansion of the universe is followed by a reconstruction and another singularity. Near the initial singularity, i.e., to first order in t, the volume $g^{1/2}$ of the universe grows as t in the anisotropic case $\alpha \neq 0$, and as t^2 in the isotropic case $\alpha = 0$.

If one sets $C^{(lm)} = 0$ and $e = 0$, the solutions reduce to the vacuum solutions[15] as is easily checked.

V. THE SPINOR FIELD

It is straightforward to integrate the equations (2.9) for the weighted spinor χ (with $N=1$, $\Omega_{(ab)} = 0$):

$$\chi(t) = \exp(-m\gamma^{(0)}t)\chi^0 \tag{5.1}$$

(χ^0 = constant spinor). From (5.1) and (2.14) one then gets

$$\phi^\dagger(t) = i\chi^{0\dagger}\exp(m\gamma^{(0)}t). \tag{5.2}$$

The integration constants χ^0, e, and $C^{(lm)}$ are of course related by

$$e = -m\chi^{0\dagger}\gamma^{(0)}\chi^0$$

and

$$C^{(lm)} = i\chi^{0\dagger}S^{(lm)}\chi^0.$$

Near the singularity, the trace of the energy-momentum tensor $T_{\alpha\beta}$ $(= -m\bar{\psi}\psi)$ behaves as $eg^{-1/2}$ i.e., as t^{-1} in the anisotropic case $\alpha \neq 0$ or as t^{-2} in the isotropic case $\alpha = 0$. This demonstrates that the singularity is a true physical one.

VI. A SOLUTION TO THE EQUATIONS PROPOSED BY WEYL[16]

In the zero-mass case, which we shall consider from now on, the above method can still be applied when a torsion term is included. The physical and theoretical motivations for such a modification of the Einstein-Dirac theory have been discussed elsewhere,[16,17] and we shall not repeat them here. Although we take $m=0$, we do not assume that the spinor field is an eigenstate of $\gamma_{(5)}$ (with $\gamma_{(5)}=\gamma_{(0)}\gamma_{(1)}\gamma_{(2)}\gamma_{(3)}$).

As noted in Ref. 16, the modified equations for the metric and the spinor field can be deduced from the usual Einstein-Dirac Lagrangian to which the following spinor quartic interaction $\mathcal{L}_{\mathrm{I}}=-\mathcal{H}_{\mathrm{I}}$,

$$\mathcal{H}_{\mathrm{I}}=\tfrac{3}{32}\sqrt{-{}^{(4)}g}\,(\bar{\psi}\gamma_{(\alpha)}\gamma_{(5)}\psi)(\bar{\psi}\gamma^{(\alpha)}\gamma_{(5)}\psi)\,, \tag{6.1}$$

has been added. In the canonical formalism, this contact term simply adds to the super-Hamiltonian which, for type-I models, reads

$$\mathcal{H}=g^{-1/2}(\pi^{ab}\pi_{ab}-\tfrac{1}{2}\pi^2)+2\lambda g^{1/2} +\tfrac{3}{32}g^{-1/2}(\phi^{\dagger}\gamma_{(0)}\gamma_{(\alpha)}\gamma_{(5)}\chi)(\phi^{\dagger}\gamma_{(0)}\gamma^{(\alpha)}\gamma_{(5)}\chi)\,. \tag{6.2}$$

We remark readily that $g^{1/2}\mathcal{H}$ has a completely decoupled form; we thus conclude that, besides $P^a{}_b=2\mu^a{}_b$, the quantities $\mathcal{H}_{1/2}^{(lm)}=2C^{(lm)}$ and $g^{1/2}\mathcal{H}_{\mathrm{I}}=\xi\geqslant 0$ commute with the Hamiltonian. Accordingly, the integration for the tetrads and metric proceeds as before: the unimodular $k_{(a)b}$ and f_{ab} are given by the same expressions as in the torsionless case, whereas $g^{1/2}$ is determined by the super-Hamiltonian constraint $\mathcal{H}\approx 0$ which becomes here

$$\frac{dy}{(\alpha+\xi+2\lambda y^2)^{1/2}}=\pm(\tfrac{3}{2})^{1/2}dt\,. \tag{6.3}$$

This equation has been solved in the previous paragraphs. Note that the "torsion energy" is on the same footing as the anisotropy parameter α. Thus, even when $\alpha=0$ (isotropy) the behavior of the volume $g^{1/2}$ near the singularity is in t, not in t^2 [the isotropic universe with torsion grows more rapidly—or collapses more rapidly if one reverses the sign of t—than the torsionless one (see Ref. 18 in this context)].

Let us now compute the Poisson brackets of the real vector density components $v_{(\alpha)}$ defined by

$$v_{(\alpha)}=\phi^{\dagger}\gamma_{(0)}\gamma_{(\alpha)}\gamma_{(5)}\chi\,. \tag{6.4}$$

One finds

$$[v_{(0)},v_{(\alpha)}]=0,\quad [v_{(a)},v_{(b)}]=-8\mathcal{H}_{1/2\,(ab)}\,. \tag{6.5}$$

The Hamiltonian equations for $v_{(\alpha)}$ thus read, in the gauge $\Omega_{(ab)}=0$,

$$v_{(0),0}=0,\quad v_{(a),0}=-3Ng^{-1/2}C_{(a}{}^{b)}v_{(b)}\,,$$

and are immediately integrated if one chooses $N=g^{1/2}$ [condition (4.5)]. They give

$$v_{(0)}=v^0_{(0)},\quad v_{(a)}=(e^{-3Cx^0})_{(a)}{}^{(b)}v^0_{(b)}\,, \tag{6.6}$$

where $v^0_{(\alpha)}$ are a set of integration constants. We can at present solve for the spinor χ, which satisfies the now linear equations

$$\chi_{,0}=\tfrac{3}{16}v^{(\alpha)}\gamma_{(0)}\gamma_{(\alpha)}\gamma_{(5)}\chi\,. \tag{6.7}$$

These imply

$$\chi(x^0)=e^{-3C_{(ab)}S^{(ab)}x^0}\times\exp[(3C_{(ab)}S^{(ab)}+\tfrac{3}{16}v^{0(\alpha)}\gamma_{(0)}\gamma_{(\alpha)}\gamma_{(5)})x^0]\,\chi^0\,, \tag{6.8}$$

where χ^0 is a constant spinor. To get (6.8), go first to "rotating" tetrads in which $v_{(\alpha)}$ is constant (perform the rotation e^{3Cx^0} which does not affect the components $C_{(a}{}^{b)}$), solve the equations for the spinor χ in these frames, and finally come back to the original tetrads satisfying $\Omega_{(ab)}=0$.

Since the spacetime scalar $g^{-1}v^{(\alpha)}v_{(\alpha)}$ ($\sim{}^{(4)}R$ near $t=0$) blows up as t^{-2} as one goes towards the initial singularity, one concludes that this singularity is a true physical one.

As a final comment, we would like to emphasize once more the power and elegance of the Hamiltonian techniques, which plainly relate the conserved generators to geometrical and physical properties of the type-I Einstein-Dirac models.

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ing different purposes in mind, they only consider the case when the matrix $\mu^a{}_b$ (introduced in Sec. IV) can be diagonalized and possesses real eigenvalues.]

[8]The invariance properties of spinor fields have been studied recently in a paper by M. Henneaux, Gen. Relativ. Gravit. (to be published). We use the same conventions for the γ matrices as in that paper, namely,

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$$\gamma_{(0)}=-\gamma^{\dagger}_{(0)},\quad \gamma_{(a)}=\gamma^{\dagger}_{(a)}.$$

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