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Gauge invariance and the choice of gauge for one-gluon-exchange corrections to quarkonium mass spectra

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The gauge invariance of the bound-state energy correction due to one-gluon exchange between a quark and antiquark is demonstrated under the assumption that, in zeroth order, the bound state is described by a Bethe-Salpeter equation with an instantaneous and local interaction kernel, which does not include one-gluon exchange. It is argued that the most convenient choice of gauge for the gluon propagator is the radiation gauge.

I. INTRODUCTION

The purpose of this note is to comment on some questions involving the choice of gauge for gluon propagators in the calculation of the mass spectrum of a bound quark-antiquark system ("quarkonium").

We recall that in the early models of the psions,[1] the gross structure was assumed to be describable by a nonrelativistic Schrödinger equation with a potential $V_{\text{conf}}(r)$ which is local, spin-independent, and confining. It was soon realized[1,2] that this simple picture had to be modified to account for the fine structure (splitting of the ^3P_J states) and the "hyperfine" structure (splitting of the ^3S and ^1S states). These splittings were assumed to arise from $(v/c)^2$ corrections to the nonrelativistic Schrödinger problem. The most natural way of including such corrections is to begin with a Bethe-Salpeter (BS) equation ($p=p_1-p_2$)

$$(\not{p}_1-M)(\not{p}_2-M)\psi(p)=\int I(p,p')\psi(p')d^4p' . \quad (1.1)$$

The kernel $I(p,p')$ is then assumed to contain a confining part I_{conf} (the long-range part), and a part I_{1g}, due to one-gluon exchange (the short-range part). The usual reduction to large components is carried out.[3,4] The zeroth-order potential is then given by $V_{\text{conf}}(r)$, coming from I_{conf}, and $V_{1g}^{\text{Coul}}\equiv-\frac{4}{3}\alpha_s/r$, the "Coulomb" term coming from I_{1g}. The potential to order $(v/c)^2$ will then have contributions from both I_{conf} and I_{1g} and these terms will contain spin-dependent and spin-independent parts. To accomplish the program, it is necessary to make an assumption about the Dirac character of the confining kernel I_{conf}. It is now common practice to assume that I_{conf} has a substantial scalar piece[5,6] I_{conf}^S (a part which is independent of the Dirac matrices associated with 1 and 2) as well as a vector piece I_{conf}^V bilinear in γ_1^μ and γ_2^μ. In this approach, the question of the choice of gauge for the gluon propagator does not arise in zeroth approximation. Of course, if one imagines that I_{conf}^S and I_{conf}^V arise from multigluon exchange, the functional form of these quantities would depend on the choice of gauge. However, at the present level of understanding of the forces which bind the q-$\bar{q}$ system, one usually takes a semiphenomenological point of view and regards both I_{conf}^S and I_{conf}^V as adjustable functions, so that the question of gauge for gluon propagators remains implicit.[7]

Nevertheless, even in this approach, the question of gauge arises explicitly when one considers the effects of one-gluon exchange. For example, if I_{conf} is imagined to arise from the exchange of two or more gluons, one can consider the effect of I_{1g} (without encountering the danger of double counting). The question of the choice of gauge for the gluon propagator then re-emerges explicitly since I_{1g} is of the form

$$I_{1g}(k)=\Gamma_1^\mu(k)\Gamma_2^\nu(-k)D_{\mu\nu}(k) \quad (k=p-p') , \quad (1.2)$$

where the $\Gamma_i(k)$ are the vertex functions for gluon emission by a quark and $D_{\mu\nu}$ is the dressed gluon propagator. This question is of interest, because of some recent progress in obtaining a fit to the quarkonium spectrum within such a framework. In that work,[6] the v^2/c^2 correction to the spin-independent part of the level shift $\Delta E^{(1)}$, arising from one-gluon exchange between the quark and antiquark, plays an important role and it is just this part which appears to depend on the choice of gauge for $D_{\mu\nu}$.

The major aim of this note is to point out that when I_{conf} is approximated by an instantaneous and local operator $[I_{\text{conf}}=\delta(p_0-p_0')\tilde{U}(\vec{p}-\vec{p}\,')]$, as has been the case in the literature, the quantity $\Delta E^{(1)}$ is independent of the gauge of the gluon propagator. Under these circumstances, there is no ambiguity in the calculation of $\Delta E^{(1)}$, and the question of the choice of gauge is just one of convenience. In the next two sections we exhibit the mechanism which assures the gauge invariance of $\Delta E^{(1)}$, first for a heliumlike system, then for quarkonium.

II. THE "HELIUM" CASE

To show that the perturbed energy $\Delta E^{(1)}$ is independent of choice of gauge for the gluon propagator, we consider first the analogous but simpler problem of two charged spin $\frac{1}{2}$ fermions of equal mass bound to an external local potential $U(\vec{r})$. In addition, there is an interaction between the fermions due to the exchange of a photon ("gluon"). This interaction is treated in lowest-order perturbation theory. We may think of this system as helium or a heliumlike ion.

The zeroth-order solution to the bound-state problem is a product of single-particle solutions

$$\Psi_{n,m}=\psi_n^{(1)}(p_1)\psi_m^{(2)}(p_2)\,, \tag{2.1}$$

$$E_{n,m}^{(0)}=\epsilon_n+\epsilon_m\,, \tag{2.2}$$

where

$$\psi_n^{(j)}(p_j)=\delta(p_{0j}-\epsilon_n)\phi_n^{(j)}(\vec{p}_j),\quad j=1,2\,, \tag{2.3}$$

with $\phi_n^{(j)}(\vec{p}_j)$ satisfying the Dirac equation

$$(\vec{\alpha}_j\cdot\vec{p}_j+\beta_jM-\epsilon_n)\phi_n^{(j)}(\vec{p}_j)=-\int\tilde{U}(\vec{q})\phi_n^{(j)}(\vec{p}_j+\vec{q})d^3q\quad(j=1,2)\,. \tag{2.4}$$

Here $\vec{\alpha}$ and β are the usual Dirac matrices and $\tilde{U}(\vec{q})$ is the Fourier transform of $U(\vec{r})$ which, although Hermitian, can have nontrivial spinor properties.

The first-order energy correction due to one-gluon exchange in perturbation theory is illustrated in Fig. 1 and is given by

$$\Delta E^{(1)}=N\int\overline{\Psi}_{n,m}(p_1,p_2)\gamma_\mu^{(1)}D^{\mu\nu}(p_1'-p_1)\gamma_\nu^{(2)}\Psi_{n,m}(p_1'p_2')\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2')\prod_{j=1}^{2}(d^4p_j)(d^4p_j')\,, \tag{2.5}$$

where, for a general gauge (including the possibility of both covariant and noncovariant gauges),

$$D_{\mu\nu}(k)=g_{\mu\nu}/k^2+f(k)k_\mu k_\nu+g(k)(n_\mu k_\nu+k_\mu n_\nu)\,, \tag{2.6}$$

with n_μ being a timelike vector. The quantity N in (2.5) is a constant which depends on the normalization of Ψ; it need not concern us here.

The proof of gauge invariance in this order will consist in showing that the contributions of terms containing the gauge functions f and g vanish. It will be adequate to demonstrate this by illustration with the term proportional to $k_\mu n_\nu$. The contribution of this term to $\Delta E^{(1)}$ is

$$\mathcal{T}_{n,m}\equiv N\int\overline{\phi}_n(\vec{p}_1)g(0;\vec{p}_1'-\vec{p}_1)(\vec{p}\!\!/_1'-\vec{p}\!\!/_1)\phi_n(\vec{p}_1')R_m(\vec{p}_2,\vec{p}_2')\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2')\prod_{j=1}^{2}(d^3p_j)(d^3p'_j)\,, \tag{2.7}$$

where we need not be concerned with the form of $R_m(\vec{p}_2,\vec{p}_2')$ and

$$\vec{p}\!\!/\equiv\gamma_ip^i\,. \tag{2.8}$$

Use of the Dirac equation (2.4) in (2.7) yields

$$\mathcal{T}_{n,m}=N\int[-\phi_n^\dagger(\vec{p}_1)\tilde{U}(\vec{q})\phi_n(\vec{p}_1'+\vec{q})+\phi_n^\dagger(\vec{p}_1-\vec{q})\tilde{U}(\vec{q})\phi_n(\vec{p}_1')]d^3q$$
$$\times g(0;\vec{p}_1'-\vec{p}_1)R_m(\vec{p}_2,\vec{p}_2')\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2')\prod_{j=1}^{2}(d^3p_j)(d^3p_j')\,. \tag{2.9}$$

If one makes the transformations $\vec{p}_1'\to\vec{p}_1'+\vec{q}$, $\vec{p}_1\to\vec{p}_1+\vec{q}$ in the second term in Eq. (2.9), $\mathcal{T}_{n,m}$ is seen to vanish. The same holds for all of the other gauge terms. The gauge invariance of $\Delta E^{(1)}$ has thus been established.

III. QUARKONIUM

We are now ready to turn to the problem of interest, that of quarkonium. We shall assume that the zeroth-order BS kernel is instantaneous in relative time and local in space. This implies that, in momentum space, it is a function of the relative three-momentum transfer, and not of the total four-momentum P_μ. [We take $P_\mu\equiv 2b_\mu=(2b,\vec{0})$.] The BS equation for the zeroth-order Green's function becomes

$$S_1^{-1}(b+p)S_2^{-1}(b-p)G_b(p,p')$$
$$=\delta^{(4)}(p-p')+\int\tilde{U}(\vec{p}-\vec{p}'')G_b(p'',p')d^4p''\,. \tag{3.1}$$

where

$$S^{-1}(q)=q\!\!/-M\,. \tag{3.2}$$

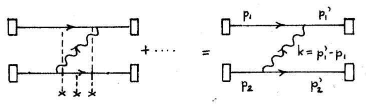

FIG. 1. Level shift $\Delta E^{(1)}$ arising from one-gluon exchange in the helium case. The effect of the interaction of the constituents of the bound state with the external field (dashed lines) is completely contained in the bound-state wave function $\Psi_{n,m}$ as shown in Eq. (2.5) of the text. The wavy line denotes the gluon propagator in a general gauge defined by Eq. (7.6). The boxes denote bound-state wave functions.

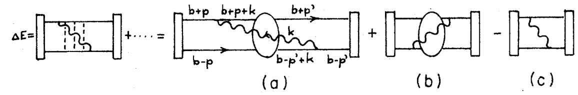

FIG. 2. Level shift $\Delta E^{(1)}$ arising from one-gluon exchange in the quarkonium case. The dashed lines denote a local, instantaneous binding interaction and the open circles symbolize the Green's function defined by Eq. (3.1) of the text. The third term on the right-hand side of the equality sign is necessary to avoid double counting of one-gluon exchange *without* binding corrections.

There is a corresponding homogeneous equation [see (1.1)] for the wave function with $b \to b_n$ (the bound-state energy).

We can now use the "helium" case as a guide. The diagram analogous to Fig. 1 is that of Fig. 2. The exchange of a gluon in the presence of the zeroth-order interaction involves any number of "rungs" of a U ladder crossing the gluon, just as in Fig. 1. However, while the ladders collapse into the wave function for an external field (due to the product nature of the zeroth-order wave function for "helium"), Green's functions as well as wave functions survive in the quarkonium case. This phenomenon is known as correction for binding[8] and will turn out to play an essential role in the proof of gauge invariance.

Our argument follows the lines of the one for "helium". We concern ourselves with the term $k_\mu n_\nu$ in the propagator $D_{\mu\nu}$ and consider its successive contributions to $\Delta E^{(1)}$ which arise from Figs. 2(a), 2(b), and 2(c). The contribution due to Fig. 2(a) is

$$\mathcal{T}^{(a)}_{b_n} = \int \bar{\Psi}_{b_n}(p) \not{k}^{(1)} S_2^{-1}(b_n - p) G_{b_n+k/2}(p+\tfrac{1}{2}k, p'+\tfrac{1}{2}k) \not{n}^{(2)} S_1^{-1}(b_n+p') \Psi_{b_n}(p') g(k) d^4p\, d^4p'\, d^4k\,. \tag{3.3}$$

Now insert

$$S_1^{-1}(b_n+p)\, S_1(b_n+p)$$

to the left and

$$S_1(b_n+p+k)\, S_1^{-1}(b_n+p+k)$$

to the right of $\not{k}^{(1)}$ in (3.3) and use the identity (i.e., the Ward identity)

$$S(q)\not{k}S(q+k) = S(q) - S(q+k)\,. \tag{3.4}$$

We obtain

$$\mathcal{T}^{(a)}_{b_n} = \int \bar{\Psi}_{b_n}(p) [S_1^{-1}(b_n+p+k) S_2^{-1}(b_n - p) - S_1^{-1}(b_n+p) S_2^{-1}(b_n - p)] G_{b_n+k/2}(p+\tfrac{1}{2}k, p'+\tfrac{1}{2}k) \not{n}_2 \cdots d^4k\,. \tag{3.5}$$

Let us take the first operator in the square brackets to be acting to the right on $G_{b_n+k/2}$ and the second as acting to the left on $\bar{\Psi}_{b_n}(p)$ and use the equations of motion, just as in the "helium" case. The two terms proportional to the kernel U will cancel (just as for "helium") after a suitable change of momentum variable is made in one of them. It is crucial for this cancellation that the kernel U be both independent of the total energy variable b and also a function only of the relative momentum transfer. A term remains, due to the inhomogeneous part of the Green's function equation and $\mathcal{T}^{(a)}_{b_n}$ becomes

$$\mathcal{T}^{(a)}_{b_n} = \int \bar{\Psi}_{b_n}(p) S_1^{-1}(b_n+p') \not{n}^{(2)} \Psi_{b_n}(p') \times g(p-p') d^4p\, d^4p'\,. \tag{3.6}$$

We repeat the same calculation for the contribution of Fig. 2(b), obtaining

$$\mathcal{T}^{(b)}_{b_n} = -\int \bar{\Psi}_{b_n}(p) S_1^{-1}(b_n+p) \not{n}^{(2)} \Psi_{b_n}(p') \times g(p-p') d^4p\, d^4p'\,. \tag{3.7}$$

Figure 2(c) gives a contribution directly:

$$\mathcal{T}^{(c)}_{b_n} = -\int \bar{\Psi}_{b_n}(p) (\not{p}^{(1)\prime} - \not{p}^{(1)}) \not{n}^{(2)} \Psi_{b_n}(p') \times g(p-p') d^4p\, d^4p'\,. \tag{3.8}$$

Equation (3.8) cancels the sum of Eqs. (3.6) and (3.7) and once again, after carrying out the same procedure for the other gauge terms in $D_{\mu\nu}$, we have succeeded in demonstrating the gauge invariance of $\Delta E^{(1)}$. [Note that the proof holds also if $\tilde{U}$ in (3.1) depends on $p_0 - p_0''$, i.e., if the kernel I is not instantaneous.]

IV. ADDITIONAL COMMENTS

In order to emphasize the importance of all terms (large and small components, retardation, binding corrections, etc.) in the proof of the gauge invariance of $\Delta E^{(1)}$, we shall look at the problem of the calculation of $\Delta E^{(1)}$ in a slightly different way. Let us examine (in two separate gauges) the contribution to $\Delta E^{(1)}$ arising from the exchange of a single gluon in the non-retardation limit (i.e., we neglect binding corrections). In this limit, the interaction can be written in the form of potentials in position space. Let us consider the Coulomb-Breit potential

$$V_{\mathrm{CB}}(r) = \frac{\lambda}{r}[1 - \tfrac{1}{2}(\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \vec{\alpha}_1 \cdot \hat{r}\vec{\alpha}_2 \cdot \hat{r}] \tag{4.1}$$

and the Møller-Feynman (MF) potential

$$V_{\mathrm{MF}}(\vec{r}) = \frac{\lambda}{r}(1 - \vec{\alpha}_1 \cdot \vec{\alpha}_2). \tag{4.2}$$

These can be obtained by Fourier transformations of the c.m. system one-gluon exchange matrix amplitudes, T_{CB} and T_{MF}, calculated in the radiation and Feynman gauges, respectively.

Since

$$\langle u_1', u_2' | T_{\mathrm{CB}} | u_1, u_2 \rangle = \langle u_1', u_2' | T_{\mathrm{MF}} | u_1, u_2 \rangle, \tag{4.3}$$

where the u's are Dirac spinors, these two potentials make the same contribution to the on-shell scattering amplitude of fermions 1 and 2 to lowest order in λ. If 1 and 2 are each bound to a local external potential $U(\vec{r}_i)$ (helium problem) then the lowest-order level shifts

$$\Delta E_{\mathrm{CB}} = \langle \phi | V_{\mathrm{CB}} | \phi \rangle \tag{4.4}$$

and

$$\Delta E_{\mathrm{MF}} = \langle \phi | V_{\mathrm{MF}} | \phi \rangle \tag{4.5}$$

are still equal. This follows from the fact that[4(b)]

$$C \equiv V_{\mathrm{MF}} - V_{\mathrm{CB}} = -\tfrac{1}{2}\lambda[\vec{\alpha}_1 \cdot \vec{p}_1, [\vec{\alpha}_2 \cdot \vec{p}_2, r]]. \tag{4.6}$$

We then have

$$\langle \phi | V_{\mathrm{MF}} - V_{\mathrm{CB}} | \phi \rangle = \tfrac{1}{2}\lambda \langle \phi | [H_1, [\vec{\alpha}_2 \cdot \vec{p}_2, r]] | \phi \rangle = 0, \tag{4.7}$$

where

$$H_j = \vec{\alpha}_j \cdot \vec{p}_j + \beta_j M + U(\vec{r}_j), \quad j = 1, 2 \tag{4.8}$$

and we have used the fact that ϕ is a product wave function $\phi^{(1)}(\vec{r}_1)\phi^{(2)}(\vec{r}_2)$ and $U(\vec{r}_j)$ is a local potential. This gauge independence is precisely what was found in Sec. II.

This equivalence of the right-hand sides of Eqs. (4.4) and (4.5) no longer holds if particles 1 and 2 are bound to *each other* by a potential, say $U_{12}(\vec{r})$. In this case, the zeroth-order equal-time wave function $\chi(\vec{r})$ is no longer a product function. To show that the level shifts due to V_{CB} and V_{MF} are *not* the same in this case, let us assume U_{12} is small compared to M and let χ_{nr} denote the wave function in the nonrelativistic limit. Then

$$\chi \approx \left(1 + \frac{\vec{\alpha}_1 \cdot \vec{p}}{2M}\right)\left(1 - \frac{\vec{\alpha}_2 \cdot \vec{p}}{2M}\right)\chi_{\mathrm{nr}}, \tag{4.9}$$

and to leading order in $(v/c)^2$ we have

$$\langle \chi | C | \chi \rangle \approx \langle \chi_{\mathrm{nr}} | C_{\mathrm{red}} | \chi_{\mathrm{nr}} \rangle, \tag{4.10}$$

where

$$C_{\mathrm{red}} = \frac{-\lambda}{8M^2}[\vec{p}^2, [\vec{p}^2, r]] \tag{4.11}$$

is independent of spin. Thus, V_{CB} and V_{MF} are equivalent in this approximation insofar as spin-dependent terms are considered, but not with regard to spin-independent terms.[4(b)]

V. CONCLUSIONS

Now that we have shown (in Sec. III) that the level shift $\Delta E^{(1)}$ due to one-gluon exchange in quarkonium is gauge invariant, we are free to do our calculation in any gauge. The question is: Which gauge is most convenient to use? The answer is the radiation gauge. This is so because binding corrections cause the least complication for it. In this gauge, for the transverse propagator part, to order $(v/c)^2$ we can take

$$D_{ij}(k) = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \approx \frac{1}{\vec{k}^2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right), \tag{5.1}$$

and of course $D_{00}(k)$ is already a function of $\vec{k}^2$ only. The binding corrections now do not appear since the relevant gluon propagators are instantaneous in relative time and if we tried to repeat the construction of Fig. 2 we would find that no U rungs would cross the gluon propagator. This means that, in this gauge, the entire contribution to $\Delta E^{(1)}$ would be given by an expression of the form (4.4), with $\phi \to \chi$.

Using a different gauge, say the Feynman gauge, there would still be no difficulty with D_{ij}, but D_{00} would be a function of the four-vector k squared, k^2. This means that binding corrections involving Green's functions would have to be used to calculate a gauge invariant $\Delta E^{(1)}$. We would be faced with a much more tedious calculation than in the radiation gauge and the analog of Eq. (4.5) would not be correct. For the case of positronium, such a calculation in the Feynman gauge has been attempted.[9] In that case, $(v/c)^2 \sim \alpha^2$, and the spin-dependent parts of the potential are independent of the gauge through order $m\alpha^4$. However, the spin-independent part of the potential gives (in the Feynman gauge) not only gauge-dependent, but even spurious $m\alpha^3$, $m\alpha^3 \ln\alpha$, and $m\alpha^4 \ln\alpha$ contributions. Only if one takes binding correction effects painstakingly into account[9] are these terms seen to vanish. On the other hand, in the radiation gauge the lowest-order spin-independent $(v/c)^2$ energy contribution is of order $m\alpha^4$ and no binding corrections are needed to obtain it.

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