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One-loop calculation of the K_{l3} form factors in a renormalizable SU(3) σ model and tests of chiral-symmetry breaking

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The K_{l3} form factors are investigated in the one-loop approximation within the context of a linear renormalizable SU(3) σ model. The model incorporates SU(3) nonets of pseudoscalar (π,K,η,η') and scalar $(\epsilon,\kappa,\sigma,\sigma')$ mesons. The η', ϵ, κ, σ, and σ' mesons are associated with the $X^0(957)$, $\delta(980)$, $\kappa(1400)$, $S^*(980)$, and $\epsilon(1300)$, respectively. The Lagrangian contains the most general renormalizable chiral-SU(3) $\times$ SU(3)-invariant couplings as well as explicit linear symmetry-breaking terms belonging to the $(3,3^*)\oplus(3^*,3)$ representation of SU(3) $\times$ SU(3). All calculations are carried out in the one-loop approximation. With this model we first obtain a reasonable approximation to the scalar and pseudoscalar mass spectrum and the known leptonic decay constants. Most of the masses and decay constants are reproduced within 10%. In addition, the second-order corrections were usually in the neighborhood of 15–20% or less, supporting the conjecture that higher-order strong-interaction effects may, in most cases, be rather small. Employing these solutions, we calculate the K_{l3} form factors and compare these with recent experimental studies. The predictions for λ_+ are too small probably due to the fact that, at this level of approximation, the model contains no spin-one poles (vector mesons). However, the model predictions for λ_0 and $\xi(0)$ are fairly good. A number of theoretical predictions for the K_{l3} form factors based on current algebra and chiral perturbation theory are then investigated. In particular, we were interested in the magnitude of the corrections to various predictions derived using specific symmetry assumptions. As the model reproduces quite closely the model-independent calculation of $f_+(0)$ from chiral perturbation theory, we feel that our conclusions in this area may have a more general significance. For example, in the tree approximation the Callan-Treiman relation is an identity. In the one-loop approximation, the magnitude of the strong-interaction effects is much larger than the symmetry-breaking effects and the relation is still obeyed quite well (within $\sim 2\%$), reflecting the influence of the underlying SU(2) $\times$ SU(2) symmetry. Overall, our results support calculations based on chiral perturbation theory.

I. INTRODUCTION

The K_{l3} decays $K\to\pi+l+\nu_l$, where l stands for either an electron or a muon, have been enthusiastically studied[1] both theoretically and experimentally. This great interest stems in large part from the hope that they will reveal valuable information about the strong interactions. According to the Cabibbo theory the transition matrix element describing these decays can be factored into a known leptonic piece and a hadronic part given (up to unimportant normalization factors) by

$$\langle\pi(q')|V_\mu^{\Delta S=1}(0)|K(q)\rangle \propto f_+(t)(q+q')_\mu + f_-(t)(q-q')_\mu ,$$

where $V_\mu^{\Delta S=1}(x)$ is the weak, hadronic, strangeness-changing vector current and $t=(q'-q)^2$. The form factors $f_+(t)$ and $f_-(t)$ are manifestations of the underlying hadronic dynamics. Their experimental determination can thus serve to test some of our ideas about hadronic interactions. $f_+(t)$ and $f_-(t)$ may be especially useful as probes of approximate hadronic symmetries. This aspect will be of interest to us in the following.

The purpose of the present investigation is to study the K_{l3} form factors within the context of a renormalizable SU(3) σ model.[2] In particular $f_+(t)$ and $f_-(t)$ are calculated in the one-loop approximation. This gives an indication of the magnitude, not only of the higher-order strong-interaction contributions to the form factors, but also of the corrections to a number of predictions derived on the basis of various assumed symmetries.

It has been emphasized by a number of people[3-7] that chiral SU(3) $\times$ SU(3) may be a fairly good symmetry of the strong interactions, perhaps as good as SU(3), and that SU(2) $\times$ SU(2) is even better. Indeed, SU(2) $\times$ SU(2) is probably almost as good a symmetry as isospin,[7] corrections to it being of the order of 5–10%. Chiral symmetry is supposed to be spontaneously broken,[8] with the pion, or the entire pseudoscalar-meson octet, becoming massless Nambu-Goldstone bosons in the limit of exact SU(2) $\times$ SU(2) or SU(3) $\times$ SU(3), respectively. The smallness of chiral-symmetry-breaking terms in the strong-interaction Hamiltonian is implied by the relatively small kaon, η meson, and especially pion masses, as well as by other evidence[9] such as the success of many current-algebra predictions.[10]

According to the above philosophy, it makes sense to write the strong-interaction Hamiltonian density as

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\mathrm{SB}} \tag{1.1}$$

with

$$\mathcal{H}_{\mathrm{SB}}=\delta_1\mathcal{H}'+\delta_2\mathcal{H}'', \tag{1.2}$$

where $\mathcal{H}_0$ is SU(3)×SU(3) invariant, $\mathcal{H}'$ is SU(2)×SU(2) invariant but breaks SU(3), and $\mathcal{H}''$ breaks SU(2)×SU(2). Then we would expect that

$$\delta_1 \gg \delta_2\,. \tag{1.3}$$

SU(3) σ models can of course be constructed to have the structure[11] of (1.1) and (1.2). They also have the advantage of incorporating the SU(3)×SU(3) algebra of currents[12] as well as pseudoscalar- and scalar-meson pole dominance of amplitudes. In addition, such models provide a useful laboratory in which to explore various symmetry limits. In this connection solutions have been found[13,14] in which SU(2)×SU(2) and SU(3)×SU(3) are realized in the Nambu-Goldstone mode, thus lending support to the ideas discussed above.

The SU(3) σ model employed here has the structure of (1.1) and (1.2). It is based on a Lagrangian which is constructed out of SU(3) nonets of pseudoscalar (π,K,η,η') and scalar $(\epsilon,\kappa,\sigma,\sigma')$ meson fields which are assigned to the $(3,3^*)\oplus(3^*,3)$ representation of SU(3)×SU(3). The Lagrangian contains no term of degree greater than four in the fields, thus ensuring its renormalizability. In addition to SU(3)×SU(3)-invariant terms, the Lagrangian contains terms which are linear in the (σ and σ') fields giving rise to $(3,3^*)\oplus(3^*,3)$ symmetry breaking.[3,4,12]

This model has been studied in the tree approximation by many authors.[13,15] Of course, the virtue of working with a renormalizable model is that one can go beyond the tree (or effective Lagrangian) approximation to consider higher-order contributions. Crater[16] explicitly demonstrated the renormalizability of the SU(3)×SU(3)-invariant form of the model in the one-loop approximation. Lee[17] and Symanzik,[18] considering different chiral Lagrangian models in which the symmetry-breaking terms are linear in the fields, have shown that these models can be renormalized such that all divergent counterterms can be absorbed into the coupling constants of the symmetric part of the Lagrangian. Subsequently, Chan and Haymaker[19] carried out a renormalization of the present SU(3) σ model with linear symmetry-breaking terms, in which they calculated one-loop corrections to the one- and two-point functions. They have also[14] used the model to calculate meson-meson scattering amplitudes in the one-loop approximation.

The above applications of the σ model are examples of lower-order calculations in the loop expansion,[20] which is essentially a perturbation series in the number of closed loops in the Feynman diagram contributing to a given process. The loop expansion has the advantage that the symmetry properties of the Lagrangian are preserved order by order. For example, the Ward-Takahashi identities hold to each order in the number of loops, as do the relations obtained from them by the PCAC (partial conservation of the axial-vector current) substitution.

The present investigation is meant to shed light on two different aspects of the K_{l3} form factors. First, and most obviously, it is interesting to determine the higher-order contributions to $f_+(t)$ and $f_-(t)$ in a strong-interaction field theory. It is often supposed that such effects will be quite large and even that a perturbation series will not converge becasue of the large coupling constants. However, previous work indicates that, at least for the renormalizable SU(3) σ model, higher-order contributions are rather small.[14,19] This is born out by the present analysis as well.

At least as important as this general insight into strong-interaction effects is the estimation of the sizes of corrections to various symmetry predictions for the K_{l3} form factors. These corrections are generated by the presence of terms such as $\delta_1\mathcal{H}'$ and $\delta_2\mathcal{H}''$ [see Eq. (1.2)] in the strong-interaction Hamiltonian. Moreover, since several of the more basic predictions are trivially satisfied in the tree approximation, one must go to higher orders to investigate symmetry-breaking effects. For example, since $f_+(0)=1.0$ in the tree approximation, we do not obtain any information, at that level, about the size of the second-order SU(3)-breaking corrections called for by the Ademollo-Gatto theorem.[21]

Similarly, nothing is learned in the tree approximation about corrections to the current-algebra relation based on pion PCAC[22]

$$f_+(m_K{}^2)+f_-(m_K{}^2)\approx\frac{F_K}{F_\pi} \tag{1.4}$$

nor to its kaon PCAC counterpart[23]

$$f_+(m_\pi{}^2)-f_-(m_\pi{}^2)\approx\frac{F_\pi}{F_K}\,, \tag{1.5}$$

where m_π (m_K) and F_π (F_K) are the pion (kaon) mass and decay constant, respectively. Both of these sum rules are identically satisfied in lowest order for the on-mass-shell form factors.

One of our goals then will be to study the success of the above predictions in the one-loop approximation. We will also consider relations for the derivatives of the form factors which have been obtained[24-26] using current-algebra approaches. In addition, a number of other sum rules and relations will be examined. Not all of these involve the K_{l3} form factors, but rather serve as tests of, e.g., pole dominance and vertex-function smoothness in the one-loop approximation.

Because it is interesting in its own right and can also serve as a check on our calculations, we have also determined the pion electromagnetic form factor $F_\pi(t)$ in the one-loop approximation.

Additional tests of the calculations were afforded by the requirement that certain Ward-Takahashi identities should be satisfied in the tree and one-loop approximation.

The plan of the paper is the following: In Sec. II we briefly discuss the renormalizable SU(3) σ model and the method used to determine numerically the basic Lagrangian parameters to second order. The evaluation of the K_{l3} form factors and the pion electromagnetic form factor are discussed in Secs. III and IV, respectively. In Sec. V some details of the numerical analysis are provided. Finally, our results are presented and discussed in Sec. VI. Included are tests of the above-mentioned predictions[21-26] for the K_{l3} form factors based on SU(3) and current algebra with PCAC, as well as tests of other relations obtained from symmetry, pole-dominance, and smoothness assumptions.

II. BASIC NUMERICAL CALCULATIONS IN THE ONE-LOOP APPROXIMATION

As indicated above, we chose the linear SU(3) σ model developed by Chan and Haymaker[14,19] for our numerical calculations. The basic Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M \partial^\mu M^\dagger) - \tfrac{1}{2}\mu^2\,\mathrm{Tr}(MM^\dagger) + g(\det M + \det M^\dagger) + f_1[\mathrm{Tr}(MM^\dagger)]^2 + f_2\,\mathrm{Tr}(MM^\dagger MM^\dagger) - \epsilon_0\sigma_0 - \epsilon_8\sigma_8\,, \tag{2.1}$$

where M and $M^\dagger$ are 3×3 matrices of fields that transform as the $(3,3^*)$ and $(3^*,3)$ representations of chiral SU(3)×SU(3), respectively. M can be rewritten as

$$M = \frac{1}{\sqrt{2}}\,\lambda^i(\sigma_i + i\phi_i)\,, \tag{2.2}$$

where σ_i and ϕ_i represent nonets of scalar $(\epsilon,\kappa,\sigma,\sigma')$ and pseudoscalar (π,K,η,η') mesons, respectively. This Lagrangian contains the most general renormalizable chiral-invariant couplings plus a linear $(3,3^*)\oplus(3^*,3)$ symmetry-breaking term.

For details of the model concerning the translation of the scalar fields, the treatment of particle mixing, the transformation of the Lagrangian into a more convenient form for numerical calculation and the type of perturbation theory used see Chan and Haymaker.[14,19] We follow their notation whenever possible. In this section we outline our method of determining the basic Lagrangian parameters to second order. The Feynman rules for this model are given in Fig. 1.

Eight parameters $(\mu^2, g, f_1, f_2, \xi_0, \xi_8, \epsilon_0, \epsilon_8)$ must be fixed in the true approximation. We input various masses and leptonic decay constants to determine the parameters and adjust the input somewhat to obtain acceptable overall values for the mass spectrum and the decay constants. Our criteria for accepting solutions and the details of the numbers obtained are discussed in Sec. V. In this section we only consider the method used for fixing the Lagrangian parameters for a given numerical input.

Two constraints are immediately available from the condition that the vacuum expectation value of

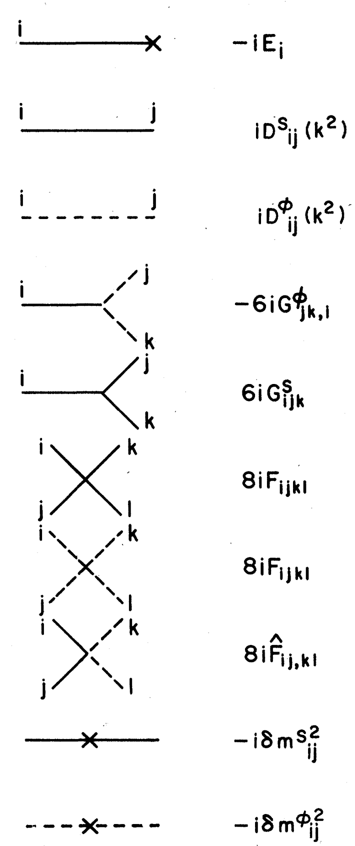

FIG. 1. Feynman-diagram rules for the Lagrangian of Eq. (2.1). Solid and dashed lines represent scalars and pseudoscalars, respectively.

the scalar fields vanish by construction. Two nontrivial constraints are afforded by

$$E_i = 0, \quad i = 0, 8\,. \tag{2.3}$$

The remaining six parameters are evaluated using F_π, F_K/F_π, m_π, m_K, m_η, and m_σ.[27] The latter mass is input as only the $I=0$ scalar masses provide sufficient information to allow the evaluation of both μ^2 and f_1. The quantities

$$a = \frac{1}{\sqrt{2}}\,\frac{\epsilon_8}{\epsilon_0} \tag{2.4}$$

and

$$b = \frac{1}{\sqrt{2}}\,\frac{\xi_8}{\xi_0} \tag{2.5}$$

provide a measure of the relative strengths of SU(3) and SU(2)×SU(2) breaking, and of the SU(3) invariance of the vacuum, respectively.

The second-order Lagrangian parameters are fixed by again using the one- and two-point functions and inputting masses and leptonic decay constants. Several points regarding the second-order calculations are worth reiterating.

(1) Although only the parameters of the symmetric Lagrangian acquire divergent second-order part, all parameters can have finite second-order corrections.

(2) The analysis is simplified if a basis for the fields is chosen for the internal meson loops such that the mass matrix is diagonal in the tree approximation. The orthogonal matrix

$$U^{\phi}_{\alpha i} = \begin{array}{c c} & \begin{array}{cccc} \pi & K & 8 & 0 \end{array} \\ \begin{array}{c} \pi \\ K \\ \eta \\ \eta' \end{array} & \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \cos\theta_P & -\sin\theta_P \\ & & \sin\theta_P & \cos\theta_P \end{bmatrix} \end{array} \tag{2.6}$$

is defined such that

$$U^{\phi}_{\alpha i} m^{\phi^2}_{ij} \tilde{U}^{\phi}_{j\beta} = m^{\phi^2}_{\alpha}\delta_{\alpha\beta}\,. \tag{2.7}$$

For any tensor T,

$$T_\alpha = U^{\phi}_{\alpha i} T_i\,, \tag{2.8}$$

where Latin indicies indicate the old basis and Greek indices denote the new one. A similar matrix is defined for the scalar case ($\theta_P \to \theta_S$).

(3) In the evaluation of any quantity to second order, the second-order corrections (the Δ terms) can only appear linearly. This often requires the $\Delta\xi_i$ terms to be treated separately. The resulting numerical problem is simpler than the tree approximation, however, as only linear equations need be solved.

In this case two constraints on the parameters are again provided by required vanishing of the vacuum expectation values of the scalar fields. The remaining parameters are fixed by inputting second-order values for F_π, m_π, m_K, m_η, $m_{\eta'}$, and m_σ. There values are not necessarily fixed at their tree-approximation values, but are adjusted to give an acceptable mass spectrum overall.

III. THE K_{l3} DECAY FORM FACTORS IN THE ONE-LOOP APPROXIMATION

The K_{l3} form factors are defined in the usual manner:

$$[(2\pi)^6 4\omega_i\omega_j]^{1/2}\langle\phi_j(q')|V^{\mu}_k(0)|\phi_i(q)\rangle = i f_{ijk}[(q+q')^{\mu} f_+(t) + p^{\mu} f_-(t)]\,, \tag{3.1}$$

where

$$p = q - q'\,, \tag{3.2}$$

$$t = p^2\,, \tag{3.3}$$

and the appropriate indices are chosen.

The free-field relations

$$[(2\pi)^3 2\omega_i]^{1/2}\langle 0|\partial^{\mu}s_i|s_i(p)\rangle = -ip^{\mu} \tag{3.4}$$

and

$$[(2\pi)^6 4\omega_i\omega_j]^{1/2}\langle s_j(q')|s_i\overleftrightarrow{\partial}^{\mu}s_j|s_i(q)\rangle = i(q+q')^{\mu} \quad (i\neq j) \tag{3.5}$$

lead to the vector-current Feynman-diagram contributions given in Fig. 2. From the form of the vector current the unmixed vertex coefficients are

$$\rho_{k,i} = i\xi_8 f_{8ki} \tag{3.6}$$

and

$$\rho^{\phi}_{k,ij} = \rho^{s}_{k,ij} = i f_{kij}\,, \tag{3.7}$$

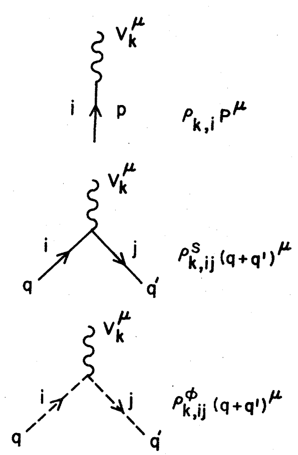

FIG. 2. Feynman-diagram rules for the vector-current–field vertices. The current is denoted by the wiggly line. The $\rho_{k,i}$ and $\rho_{k,ij}$ factors are given in Eqs. (3.6) and (3.7).

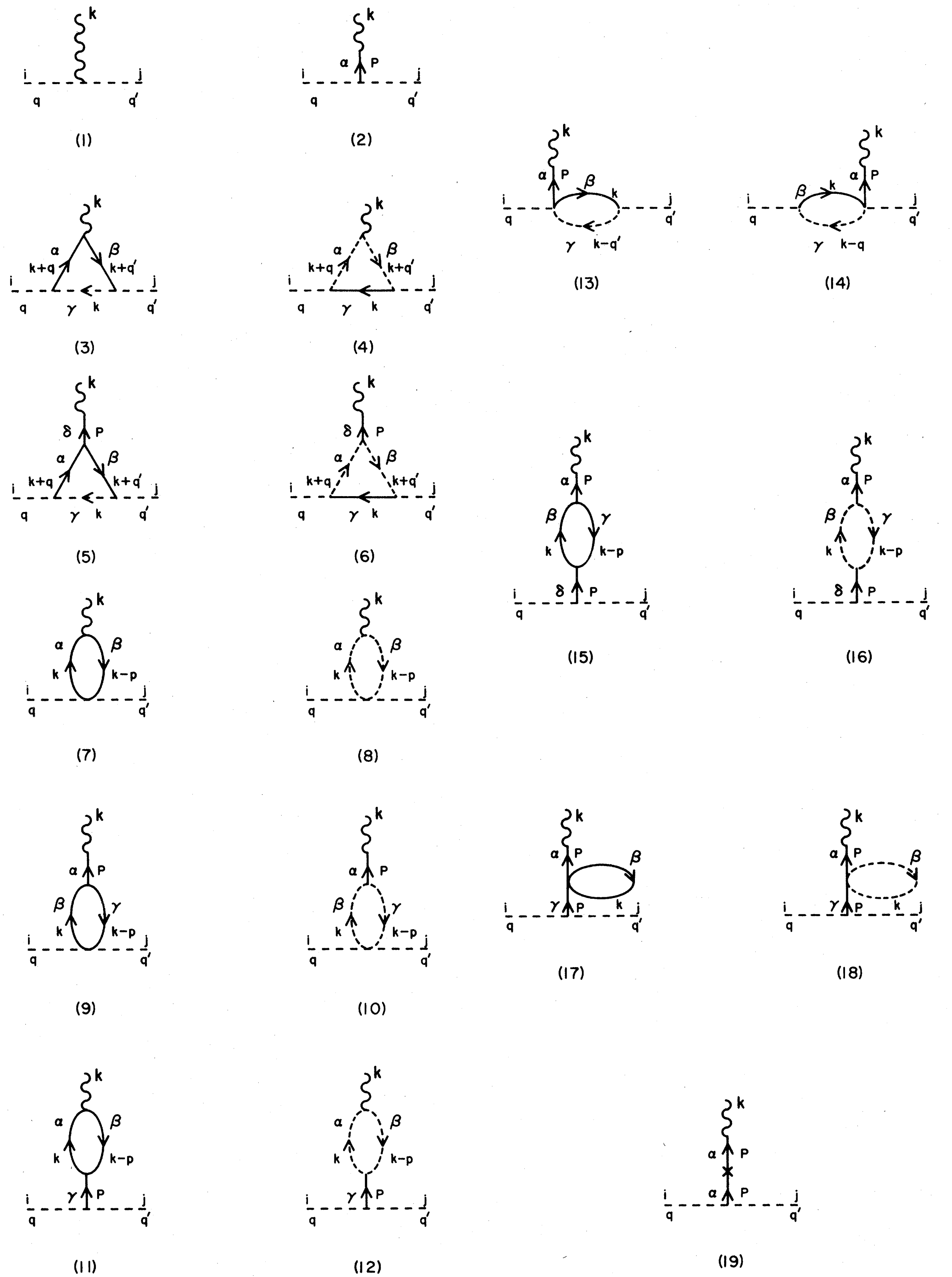

FIG. 3. Feynman diagrams for the K_{l3} decay amplitude to second order. The current V_k^μ is denoted by the wiggly line.

where k is the current index and i and j are the field indices. The mixing of the $I=0$ fields is incorporated via

$$\rho^{\phi,s}_{k,\alpha j} = U^{\phi,s}_{\alpha i}\rho^{\phi,s}_{k,ij} \,. \tag{3.8}$$

Clearly,

$$\rho^{\phi,s}_{k,\alpha\beta} = U^{\phi,s}_{\alpha i}U^{\phi,s}_{\beta j}\rho^{\phi,s}_{k,ij} = 0 \,. \tag{3.9}$$

Finally, note that ρ^{ϕ} and ρ^{s} are antisymmetric in the field indices.

The Feynman diagrams which contribute, in the one-loop approximation, to the K_{l3} hadronic matrix element are shown in Fig. 3. We write this matrix element as

$$[(2\pi)^6 4\omega_i\omega_i]^{1/2}\langle\phi_j(q')|V^{\mu}_k(0)|\phi_i(q)\rangle = \sum_{a=1}^{19} T^{a\mu}_{k,ij} \,, \tag{3.10}$$

where $T^{a\mu}_{k,ij}$ denotes the contribution from diagram a. These are[28]

$$T^{1\mu}_{k,ij} = (q+q')^{\mu}\rho^{\phi}_{k,ij}[1+\tfrac{1}{2}\Sigma_i'(M_i^2)+\tfrac{1}{2}\Sigma_j'(M_j^2)] \,, \tag{3.11}$$

$$\begin{aligned} T^{2\mu}_{k,ij} = \sum_{\alpha}\frac{6p^{\mu}}{p^2-m^{s2}_{\alpha}}\big(\{\rho_{k,\alpha}[1+\tfrac{1}{2}\Sigma_i'(M_i^2)+\tfrac{1}{2}\Sigma_j'(M_j^2)] \\ +\Delta\rho_{k,\alpha}\}G^{\phi}_{ij,\alpha} \\ +\rho_{k,\alpha}\Delta G^{\phi}_{ij,\alpha}\big) \,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} T^{3\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma} 36 G^{\phi}_{i\gamma,\alpha}G^{\phi}_{\gamma j,\beta}\rho^{s}_{k,\alpha\beta} \\ \times C^{\mu}(q,q',p^2,m^{\phi2}_{\gamma},m^{s2}_{\alpha},m^{s2}_{\beta}) \,, \end{aligned} \tag{3.13}$$

$$\begin{aligned} T^{4\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma} 36 G^{\phi}_{i\alpha,\gamma}G^{\phi}_{\beta j,\gamma}\rho^{\phi}_{k,\alpha\beta} \\ \times C^{\mu}(q,q',p^2,m^{s2}_{\gamma},m^{\phi2}_{\alpha},m^{\phi2}_{\beta}) \,, \end{aligned} \tag{3.14}$$

$$\begin{aligned} T^{5\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma,\delta}(-216)p^{\mu}\rho_{k,\delta}\frac{G^{\phi}_{i\gamma,\alpha}G^{\phi}_{\gamma j,\beta}G^{s}_{\alpha\beta\delta}}{p^2-m^{s2}_{\delta}} \\ \times C(q^2,q'^2,p^2,m^{\phi2}_{\gamma},m^{s2}_{\alpha},m^{s2}_{\beta}) \,, \end{aligned} \tag{3.15}$$

$$\begin{aligned} T^{6\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma,\delta} 216p^{\mu}\rho_{k,\delta}\frac{G^{\phi}_{i\alpha,\gamma}G^{\phi}_{\beta j,\gamma}G^{\phi}_{\alpha\beta,\delta}}{p^2-m^{s2}_{\delta}} \\ \times C(q^2,q'^2,p^2,m^{s2}_{\gamma},m^{\phi2}_{\alpha},m^{\phi2}_{\beta}) \,, \end{aligned} \tag{3.16}$$

$$T^{7\mu}_{k,ij} = \sum_{\alpha,\beta}(-4)\hat{F}_{ij,\alpha\beta}\rho^{s}_{k,\alpha\beta}R^{\mu}(p,m^{s2}_{\alpha},m^{s2}_{\beta}) \,, \tag{3.17}$$

$$T^{8\mu}_{k,ij} = \sum_{\alpha,\beta}(-4)F_{ij\alpha\beta}\rho^{\phi}_{k,\alpha\beta}R^{\mu}(p,m^{\phi2}_{\alpha},m^{\phi2}_{\beta}) \,, \tag{3.18}$$

$$\begin{aligned} T^{9\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma} 24p^{\mu}\rho_{k,\alpha}\frac{G^{s}_{\alpha\beta\gamma}\hat{F}_{ij,\beta\gamma}}{p^2-m^{s2}_{\alpha}} \\ \times\overline{B}(p^2,m^{s2}_{\beta},m^{s2}_{\gamma}) \,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} T^{10\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-24)p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{\beta\gamma,\alpha}F_{ij,\beta\gamma}}{p^2-m^{s2}_{\alpha}} \\ \times\overline{B}(p^2,m^{\phi2}_{\beta},m^{\phi2}_{\gamma}) \,, \end{aligned} \tag{3.20}$$

$$\begin{aligned} T^{11\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-18)\rho^{s}_{k,\alpha\beta}\frac{G^{\phi}_{ij,\gamma}G^{s}_{\alpha\beta\gamma}}{p^2-m^{s2}_{\gamma}} \\ \times R^{\mu}(p,m^{s2}_{\alpha},m^{s2}_{\beta}) \,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} T^{12\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma} 18\rho^{\phi}_{k,\alpha\beta}\frac{G^{\phi}_{ij,\gamma}G^{\phi}_{\alpha\beta,\gamma}}{p^2-m^{s2}_{\gamma}} \\ \times R^{\mu}(p,m^{\phi2}_{\alpha},m^{\phi2}_{\beta}) \,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} T^{13\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-48)p^{\mu}\rho_{k,\alpha}\frac{\hat{F}_{i\gamma,\alpha\beta}G^{\phi}_{\gamma j,\beta}}{p^2-m^{s2}_{\alpha}} \\ \times\overline{B}(q'^2,m^{s2}_{\beta},m^{\phi2}_{\gamma}) \,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} T^{14\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-48)p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{i\gamma,\beta}\hat{F}_{\alpha\beta,\gamma j}}{p^2-m^{s2}_{\alpha}} \\ \times\overline{B}(q^2,m^{s2}_{\beta},m^{\phi2}_{\gamma}) \,, \end{aligned} \tag{3.24}$$

$$\begin{aligned} T^{15\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma,\delta} 108p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{ij,\delta}G^{s}_{\alpha\beta\gamma}G^{s}_{\beta\gamma\delta}}{(p^2-m^{s2}_{\alpha})(p^2-m^{s2}_{\delta})} \\ \times\overline{B}(p^2,m^{s2}_{\beta},m^{s2}_{\gamma}) \,, \end{aligned} \tag{3.25}$$

$$\begin{aligned} T^{16\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma,\delta} 108p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{ij,\delta}G^{\phi}_{\beta\gamma,\alpha}G^{\phi}_{\beta\gamma,\delta}}{(p^2-m^{s2}_{\alpha})(p^2-m^{s2}_{\delta})} \\ \times\overline{B}(p^2,m^{\phi2}_{\beta},m^{\phi2}_{\gamma}) \,, \end{aligned} \tag{3.26}$$

$$\begin{aligned} T^{17\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-24)p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{ij,\gamma}F_{\alpha\gamma\beta\beta}}{(p^2-m^{s2}_{\alpha})(p^2-m^{s2}_{\gamma})} \\ \times(m^{s2}_{\beta}-\nu^2)\overline{B}(0,m^{s2}_{\beta},\nu^2) \,, \end{aligned} \tag{3.27}$$

$$\begin{aligned} T^{18\mu}_{k,ij} = \sum_{\alpha,\beta,\gamma}(-24)p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{ij,\gamma}\hat{F}_{\alpha\gamma,\beta\beta}}{(p^2-m^{s2}_{\alpha})(p^2-m^{\phi2}_{\gamma})} \\ \times(m^{\phi2}_{\beta}-\nu^2)\overline{B}(0,m^{\phi2}_{\beta},\nu^2) \,, \end{aligned} \tag{3.28}$$

and

$$T^{19\mu}_{k,ij} = \sum_{\alpha} 6p^{\mu}\rho_{k,\alpha}\frac{G^{\phi}_{ij,\alpha}}{(p^2-m^{s2}_{\alpha})^2}\Delta m^{s2}_{\alpha} \,. \tag{3.29}$$

The $\overline{B}$ integral is evaluated in Ref. 19.

The integrals

$$\begin{aligned} C(q^2,q'^2,p^2,x^2,y^2,z^2) \\ = i\int\frac{d^4l}{(2\pi)^4}\frac{1}{[l^2-x^2]}\frac{1}{[(l+q)^2-y^2]}\frac{1}{[(l+q')^2-z^2]} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} C^{\mu}(q,q',p^2,x^2,y^2,z^2) \\ = i\int\frac{d^4l}{(2\pi)^4}\frac{(2l+q+q')^{\mu}}{[l^2-x^2][(l+q)^2-y^2][(l+q')^2-z^2]} \,, \end{aligned}$$

where

$$p^2 = (q-q')^2 \,, \tag{3.31}$$

are discussed in Ref. 14. The linearly divergent integral

$$R^\mu(p,x^2,z^2)=i\int\frac{d^4l}{(2\pi)^4}\frac{(2l-p)^\mu}{[l^2-x^2][(l-p)^2-z^2]} \tag{3.32}$$

is evaluated using a regularization procedure in Ref. 17.

IV. THE PION ELECTROMAGNETIC FORM FACTOR

Our general method of attack in this section follows that of the K_{l3} form-factor analysis. The pion electromagnetic form factor $F_\pi(t)$ is defined in the usual manner:

$$2(2\pi)^3\sqrt{\omega_q\omega_{q'}}\langle\pi^+(q')|V^\mu_{\rm em}(0)|\pi^+(q')\rangle=(q+q')^\mu F_\pi(t)\,, \tag{4.1}$$

where $t=(q-q')^2=p^2$. The electromagnetic current $V^\mu_{\rm em}$ is related to the octet of vector currents via

$$V^\mu_{\rm em}=V^\mu_3+\frac{1}{\sqrt{3}}V^\mu_8\,. \tag{4.2}$$

In the present model

$$V^\mu_{\rm em}=\frac{1}{2}\left(f_{3ij}+\frac{1}{\sqrt{3}}f_{8ij}\right)(s_i\overleftrightarrow{\partial}^\mu s_j+\phi_i\overleftrightarrow{\partial}^\mu\phi_j)\,. \tag{4.3}$$

The current-field vertices in this case are given in Fig. 4. These vertices are characterized by the factors $\rho^{\rm em}_{ij}$ which are defined as

$$\rho^{\rm em}_{ij}=f_{3ij}+\frac{1}{\sqrt{3}}f_{8ij}\,. \tag{4.4}$$

Of course, $V^\mu_{\rm em}$ only couples to identical, charged mesons.

The Feynman diagrams that can contribute to $F_\pi(t)$ are given in Fig. 5. The diagram contributions are denoted by $\overline{T}^{a\mu}_{ij}$, where a is the diagram number. Then

$$\overline{T}^{1\mu}_{ij}=i(q+q')^\mu\rho^{\rm em}_{ij}Z_i^{1/2}Z_j^{1/2}\,, \tag{4.5}$$

$$\overline{T}^{2\mu}_{ij}=\sum_{\alpha,\beta,\gamma}36iG^\phi_{i\alpha,\gamma}G^\phi_{\beta j,\gamma}\rho^{\rm em}_{\alpha\beta}\times C^\mu(q,q',p^2,m^{s^2}_\gamma,m^{\phi^2}_\alpha,m^{\phi^2}_\beta)\,, \tag{4.6}$$

$$\overline{T}^{3\mu}_{ij}=\sum_{\alpha,\beta,\gamma}36iG^\phi_{i\gamma,\alpha}G^\phi_{\gamma j,\beta}\rho^{\rm em}_{\alpha\beta}\times C^\mu(q,q',p^2,m^{\phi^2}_\gamma,m^{s^2}_\alpha,m^{s^2}_\beta)\,, \tag{4.7}$$

$$\overline{T}^{4\mu}_{ij}=\sum_{\alpha,\beta}-4iF_{ij\alpha\beta}\rho^{\rm em}_{\alpha\beta}R^\mu(p,m^{\phi^2}_\alpha,m^{\phi^2}_\beta)\,, \tag{4.8}$$

and

$$\overline{T}^{5\mu}_{ij}=\sum_{\alpha,\beta}-4i\hat{F}_{ij,\alpha\beta}\rho^{\rm em}_{\alpha\beta}R^\mu(p,m^{s^2}_\alpha,m^{s^2}_\beta)\,. \tag{4.9}$$

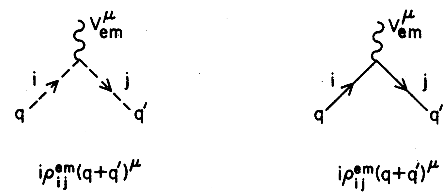

FIG. 4. Feynman-diagram rules for the electromagnetic-current–field vertices. The current is denoted by the wiggly line. The factor $\rho^{\rm em}_{ij}$ is given in Eq. (4.4).

The contributions from the last two diagrams will vanish identically upon regularization, as only pairs of particles of equal masses can contribute.

We will be interested primarily in the t dependence of $F_\pi(t)$. The consequence of electromagnetic current conservation, $F_\pi(0)=1$, is used as a check on the calculations.

V. THE NUMERICAL ANALYSIS

We will now present a brief discussion of our numerical solutions for the model. The reader should consult Chan and Haymaker[14,19] for additional comments on the properties of other possible numerical solutions. Our approach consisted of attempting to find values of the model param-

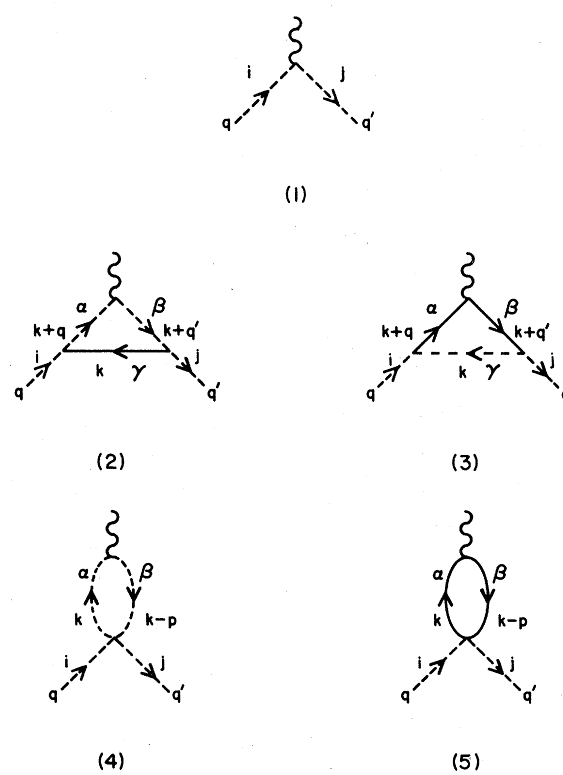

FIG. 5. Feynman diagrams for the pion electromagnetic form factor to second order. The electromagnetic current is denoted by the wiggly line.

eters which would give a reasonable account of the meson mass spectrum, and the decay constants F_π and F_K, and then to investigate the K_{l3} form factors for these values.

The pseudoscalar-meson spectrum is well known[29] with the possible exception of the η'; we shall assume here that the X^0 (957) is the η'. The status of the scalar meson nonet is much less clear. The best-established member is the $I=0$ $S^*(980)$.[29] The location of the other $I=0$ state is uncertain. It was once thought to be the $\epsilon(660)$.[30] More recently it has been identified with the broad $\epsilon(1300)$ (Ref. 29) (whose mass may be as high as 1700 MeV) with the broad $\epsilon(660)$ disappearing.[31] Nevertheless, for the sake of comparison, we have considered solutions in which the S^* is paired both with the $\epsilon(660)$ and with the $\epsilon(1300)$ as the σ and σ' of the model. The $I=1$ ϵ and the $I=\frac{1}{2}$ κ of the model are associated respectively with the $\delta(980)$ (Ref. 29) and the $\kappa(1400)$.[29]

We considered it desirable to attempt to obtain a value of ~1.25 for the ratio F_K/F_π. (The justification for this value will be given in the following section where we discuss the experimental data for the K_{l3} decays.)

A large number of solutions were considered. We have chosen to present six of these as being representative. The tree and one-loop approximation solutions are given in Tables I and II, respectively. It is not intended that the tree approximation solutions be taken seriously in their own right as they were used only as a starting point to obtain the one-loop solutions.

Our method of calculation was tested and our solutions verified using several Ward-Takahashi identities. We set $\nu^2=|\mu^2|$ throughout our numerical work.

The discussion of our results should be prefaced by the following remarks. The present model was not able to accommodate all of the masses and decay constants at their "physical" values in the one-loop approximation. Of course, it is neither necessary nor desirable to achieve exactly these values to any finite order in perturbation theory. For a perturbative approach to make sense, the percentage difference between three and one-loop values of the masses and decay constants should not be too large, say 10–15%; the differences between the target values of these quantities and their corresponding one-loop values should probably be no greater than this.

We were successful in finding solutions which largely satisfied these requirements. Owing to the rather time-consuming nature of the numeri-

TABLE I. The tree-approximation calculations for the six cases considered.

Quantity	Solution 1	2	3	4	5	6
m_π (MeV)	137.5	137.5	137.5	137.5	148.2	228.1
m_K (MeV)	497.0	495.0	495.0	495.0	490.0	507.8
m_η (MeV)	548.8	548.8	548.8	548.8	543.2	556.7
$m_{\eta'}$ (MeV)	924.1	1084.4	1115.2	1115.2	1095.4	1115.3
m_ϵ (MeV)	874.4	1071.1	1124.0	1124.0	1109.5	1131.6
m_κ (MeV)	841.3	1108.2	1172.9	1172.9	1160.8	1179.1
m_σ (MeV)	525.0	800.0	900.0	800.0	898.8	941.8
$m_{\sigma'}$ (MeV)	1087.7	1248.2	1324.9	1280.5	1314.5	1325.9
F_π (MeV)	95.0	110.0	100.0	110.0	99.3	103.5
F_K (MeV)	142.0	135.3	120.0	132.0	119.0	122.4
F_K/F_π	1.495	1.230	1.200	1.200	1.198	1.182
$F_{8\eta}$ (MeV)	160.5	143.5	126.2	138.7	125.1	128.2
$F_{8\eta'}$ (MeV)	−32.70	−25.30	−21.50	−23.65	−21.57	−20.53
F_κ (MeV)	−47.03	−25.30	−20.00	−22.00	−19.67	−18.82
θ_P (deg)	4.19	−0.569	−1.199	−1.20	−1.39	−1.24
θ_S (deg)	−52.02	−57.93	−55.71	−64.0	−55.1	−54.5
a	−0.9251	−0.9088	−0.9066	−0.9066	−0.8901	−0.7641
b	−0.2481	−0.1329	−0.1176	−0.1176	−0.1166	−0.1081
ξ_8 (MeV)	−54.30	−29.21	−23.09	−25.40	−22.71	−21.73
ξ_0 (MeV)	154.7	155.4	138.8	152.7	137.7	142.2
μ^2 (GeV^2)	0.0349	−0.0524	−0.1459	−0.0030	−0.1540	−0.1521
f_1	−2.779	−3.644	−5.485	−3.042	−5.478	−5.455
f_2	−1.604	−5.604	−8.814	−7.284	−8.957	−8.564
g (GeV)	1.286	1.887	2.252	2.047	2.181	2.157
ϵ_8 (GeV^3)	0.0384	0.0358	0.0318	0.0349	0.0306	0.0302
ϵ_0 (GeV^3)	−0.0294	−0.0279	−0.0248	−0.0273	−0.0243	−0.0280

TABLE II. Six one-loop-approximation solutions for the tree-approximation solutions presented in Table I.

Quantity	Solution 1	2	3	4	5	6
M_π (MeV)	137.5	137.5	137.5	137.5	142.2	193.5
M_K (MeV)	497.0	505.0	510.0	510.0	504.2	546.9
M_η (MeV)	548.8	548.8	548.8	548.8	545.9	595.2
$M_{\eta'}$ (MeV)	958.0	958.0	958.0	958.0	993.3	995.9
M_ϵ (MeV)	885.2	863.0	910.4	956.7	905.8	979.5
M_κ (MeV)	921.2	1056.7	1287.3	1305.3	1229.5	1333.4
M_σ (MeV)	450.0	990.0	990.0	990.0	874.1	953.1
$M_{\sigma'}$ (MeV)	1038.0	1653.6	1608.6	1578.7	1455.4	1546.4
$(F+\Delta F)_\pi$ (MeV)	95.0	95.0	95.0	95.0	97.5	100.8
$(F+\Delta F)_K$ (MeV)	118.5	117.0	108.5	108.1	112.5	114.9
$F_K/F_\pi+\Delta(F_K/F_\pi)$	1.248	1.231	1.145	1.146	1.155	1.141
$(F+\Delta F)_{8\eta}$ (MeV)	124.9	122.1	109.3	108.5	115.0	126.1
$(F+\Delta F)_{8\eta'}$ (MeV)	−24.45	−44.20	−54.1	−54.06	−41.58	−50.27
$(F+\Delta F)_\kappa$ (MeV)	−32.04	−33.37	−19.51	−17.54	−22.49	−20.81
	$-i12.99$	$-i20.62$	$-i20.09$	$-i17.87$	$-i20.55$	$-i16.13$
a	−0.9135	−0.9134	−0.9098	−0.9098	−0.9060	−0.8899
b	−0.1578	−0.1433	−0.1049	−0.0995	−0.1097	−0.0985
θ_η (deg)	−1.29	−6.25	−11.6	−11.0	−7.96	18.9
$\theta_{\eta'}$ (deg)	1.77	−6.96	−15.2	−14.7	−10.3	−14.5
θ_σ (deg)	−52.4	−53.9	−5.05	−2.03	−11.8	−13.7
$\theta_{\sigma'}$ (deg)	−67.2	−44.2	−81.2	−69.8	−91.6	−91.6
$Z_\pi^{1/2}$	0.9314	0.8502	0.7719	0.8295	0.7741	0.7961
$Z_K^{1/2}$	0.9256	0.8516	0.7779	0.8298	0.7804	0.7977
$Z_\eta^{1/2}$	0.9267	0.8525	0.7791	0.8303	0.7811	0.7936
$Z_{\eta'}^{1/2}$	0.9086	1.020	1.097	1.0889	0.9694	0.9772
$Z_\epsilon^{1/2}$	0.9423	0.8710	0.6914	0.5523	0.6870	0.5128
$Z_\kappa^{1/2}$	1.022	1.124	1.034	1.0239	1.0590	1.018
$Z_\sigma^{1/2}$	1.093	0.3540	0.6815	0.3136	0.7264	0.8031
$Z_{\sigma'}^{1/2}$	0.9074	0.8706	0.5274	0.3676	0.7776	0.6768
$\Delta\xi_8$ (MeV)	29.95	5.682	9.215	12.21	7.036	7.098
$\Delta\xi_0$ (MeV)	−28.99	−42.36	−40.32	−49.75	−34.48	−33.85
$\Delta\mu^2$ (GeV^2)	0.1815	−0.0736	2.448	−0.5008	3.184	2.999
Δf_1	22.88	58.54	107.76	157.6	111.5	106.9
Δf_2	1.181	28.10	37.48	102.6	41.47	36.30
Δg (GeV)	−0.3573	0.5115	−0.1500	6.259	0.1201	−0.0479
$\Delta\epsilon_8$ (GeV^3)	−0.0038	0.0025	0.0069	0.0025	0.0078	0.0146
$\Delta\epsilon_0$ (GeV^3)	0.0026	−0.0018	−0.0053	−0.0019	−0.0058	−0.0089

cal analysis, we usually input the known masses at or very near to their physical values and, as indicated in Sec. II, adjusted these somewhat to obtain an acceptable solution overall. This resulted in spreads of ~20% between tree and one-loop results for some quantities. However, solution 6 represents an example in which there is perhaps a more realistic averaging of the percentage differences between tree and one-loop values.

Let us now look at some of the more important features of the solutions. Since the $\epsilon(1300)$ is now preferred to the $\epsilon(660)$ as the partner of the $S^*(980)$, we have included only one example in which the σ is assumed to be a (broad) low-lying state and the S^* is the σ'. This is solution 1 in Tables I and II. In the remaining solutions, 2–6, the σ is identified with the S^* and the σ' with the $\epsilon(1300)$.

Solution 1 is the only one (displayed) in which the tree and one-loop values of the pion mass and decay constant and the kaon and η masses are each chosen to have the same physical values. In solutions 2–6, $\Delta F_\pi \neq 0$ and for a least one of π, K, or η there is a second-order mass adjustment.

It can be seen from Table II that solution 1 comes reasonably close to achieving the goals of a low-lying σ coupled with a σ' at ~1 GeV. However, the one-loop κ mass comes out ~50% below that of the $\kappa(1400)$.[32] On the other hand, the target value of 1.25 for F_K/F_π,[27] could be approached arbitrarily closely.

Turning now to solutions 2–6 one sees from Table II that all but solution 2 give quite respectable results for all of the masses, as well as for F_π and F_K/F_π. Solution 2 has the lowest κ and the highest σ' of these solutions. While the one-loop values of the σ' mass are quite a bit higher than 1300 MeV in most of these solutions, this is probably not a major problem, because the $\epsilon(1300)$ is fairly broad. Although solution 6 is perhaps the most satisfactory overall (see the above discussion), it still has a flaw in that its one-loop pion mass is 25–30% higher than the physical value. An investment of additional time and effort would undoubtedly lead to an improvement of this situation through further averaging over all the masses, without seriously affecting the qualitative features of solution 6.

Most of the particle masses as well as the known decay constants are in fact reproduced to within ~10%, especially in solutions 4–6. It may not be overly optimistic then to expect that one-loop calculations of other quantities, such as the K_{l3} form factors, would typically be that accurate as well, although some may be no closer than ~20% to the exact model value. The assumption here is that an exact solution to the model would yield all masses and decay constants at or very near their physical values.

As a final point, we note that in our solutions the coupling constants undergo large shifts from their tree to one-loop values. However, it should be realized that large shifts in the coupling constants[3] are not manifested directly as large shifts in the decay widths of the unstable particles ($\sigma, \kappa, \ldots$); these widths may be determined by means of phase-shift calculations such as those carried out in Ref. 14.

The parameter a is shifted by an insignificant amount in all our solutions, save solution 6. In general the model solutions appear to support the conjecture that the higher-order corrections to the tree-approximation results are relatively small and that the perturbation series will in fact converge.

With our solutions chosen, we then calculated the K_{l3} and pion electromagnetic form factors. Table III lists the contributions to $f_+(t)$ and $f_-(t)$ at $t=0$ and $t=m_K^2$ from the individual Feynman diagrams of Fig. 3 for solution 4 of Tables I and II. A comparison between the tree and one-loop approximation values of $f_+(t)$ and $f_-(t)$ is given in Table IV (again using solution 4) for a number of t values. Our predictions for the K_{l3} form-factor parameters, defined in the next section, are listed in Table V. The calculated K_{l3} for factors were checked numerically using Ward-Takahashi identities.

TABLE III. The individual diagram contributions to $f_+(t)$ and $f_-(t)$ for $t=0$ and $t=12m_{\pi^0}^2$ (where $m_{\pi^0}=140$ MeV) for solution 4.

	$t=0$		$t=12m_{\pi^0}^2$	
Diagram	f_+	f_-	f_+	f_-
1	0.6592	0	0.6592	0
2	0	−0.7236	0	−0.8729
3	0.1826	0.0104	0.1847	0.0108
4	0.1426	−0.0348	0.1500	−0.0470
5	0	0.1033	0	0.1273
6	0	−0.0406	0	−0.0571
7	0	−0.0131	0	−0.0133
8	0	−0.1208	0	−0.1631
9	0	0.4721	0	0.5667
10	0	0.2321	0	0.2463
11	0	0.0411	0	0.0507
12	0	0.1315	0	0.2138
13	0	0.0101	0	0.0122
14	0	0.1229	0	0.1482
15 + 16	0	−1.4431	0	−2.0451
17 + 18	0	−1.7853	0	−2.5976
19	0	3.1374	0	4.5650
Total	0.9844	0.0997	0.9939	0.1450

The pion electromagnetic form factor $F_\pi(t)$ is given in Table VI for selected values of t. In Table VII we give the diagram contributions to $F_\pi(t)$ for solution 4. In all cases, $F_\pi(0)=1.0$, as required by electromagnetic-current conservation.

VI. PREDICTIONS OF THE SU(3) σ MODEL

Now that its parameters have been fixed by the requirement of a reasonable mass spectrum (and

TABLE IV. The values of $f_+(t)$ and $f_-(t)$ in the tree and one-loop approximation for solution 4. The tree-approximation form factors are given in Eq. (6.10). The last row corresponds to $t=m_K^2$ ($m_{\pi^0}=140$ MeV).

	Tree		One-loop	
t ($m_{\pi^0}^2$)	$f_+(t)$	$f_-(t)$	$f_+(t)$	$f_-(t)$
−2	1	0.160	0.9830	0.0953
−1	1	0.162	0.9837	0.0974
0	1	0.164	0.9844	0.0997
1	1	0.167	0.9851	0.1021
2	1	0.169	0.9858	0.1047
3	1	0.172	0.9865	0.1074
4	1	0.174	0.9873	0.1104
5	1	0.177	0.9880	0.1135
6	1	0.180	0.9888	0.1170
7	1	0.183	0.9896	0.1207
8	1	0.186	0.9904	0.1247
9	1	0.189	0.9912	0.1291
10	1	0.192	0.9921	0.1339
11	1	0.195	0.9930	0.1392
12	1	0.198	0.9939	0.1450
13.6	1	0.203	0.9960	0.1607

TABLE V. The K_{l3} form-factor parameters to second order for the solutions presented in Tables I and II.

	1	2	3	4	5	6
$f_+(0)$	0.9848	0.9847	0.9803	0.9844	0.9813	0.9876
$f_-(0)$	0.1918	0.1584	0.0923	0.0997	0.1001	0.0977
λ_0	0.0173	0.0140	0.0087	0.0090	0.0097	0.0094
λ_+	0.0004	0.0007	0.0010	0.0007	0.0010	0.0008
λ_0'	0.0003	0.0003	0.0002	0.0002	0.0003	0.0005
λ_+'	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
λ_-	0.0214	0.0232	0.0213	0.0194	0.0241	0.0452
$\xi(0)$	0.2035	0.1673	0.0990	0.1060	0.1010	0.0597
Λ	0.0036	0.0039	0.0024	0.0023	0.0027	0.0015

decay constants), the SU(3) σ model can be put to a number of interesting uses. In subsection A the calculated K_{l3} form factors will be compared with experiment. This will be followed in subsection B by a study, within the context of the present model, of a number of theoretical predictions for the K_{l3} form factors and for other quantities of interest such as masses and decay constants. Let us first introduce several additional form factors and the conventional parametrization which will be needed below.

The divergence form factor $f_0(t)$ is defined by[34]

$$i[(2\pi)^6 4\omega_\pi\omega_K]^{1/2}\langle\pi^0(q')|\partial_\gamma V^\gamma_{4+i5}(0)|K^+(q)\rangle = \frac{1}{\sqrt{2}}(m_K^2-m_\pi^2)f_0(t). \quad (6.1)$$

From Eq. (3.1) it follows that

$$f_0(t)=f_+(t)+\frac{t}{m_K^2-m_\pi^2}f_-(t). \quad (6.2)$$

Next, we define in the usual way

$$\xi(t)=f_-(t)/f_+(t). \quad (6.3)$$

In experimental analyses of the K_{l3} decays it is customary to employ the parametrization [in the physical region $m_l^2 \le t \le (m_K-m_\pi)^2$]

$$f_+(t)=f_+(0)\left(1+\lambda_+\frac{t}{m_\pi^2}+\lambda_+'\frac{t^2}{m_\pi^4}\right), \quad (6.4)$$

$$f_-(t)=f_-(0)\left(1+\lambda_-\frac{t}{m_\pi^2}\right), \quad (6.5)$$

$$f_0(t)=f_+(0)\left(1+\lambda_0\frac{t}{m_\pi^2}+\lambda_0'\frac{t^2}{m_\pi^4}\right), \quad (6.6)$$

and

$$\xi(t)=\xi(0)+\Lambda\frac{t}{m_\pi^2}. \quad (6.7)$$

Note that, from Eqs. (6.3)–(6.6),

$$\xi(0)=\frac{m_K^2-m_\pi^2}{m_\pi^2}(\lambda_0-\lambda_+), \quad (6.8)$$

and (for small λ_+)

$$\Lambda=\xi(0)(\lambda_- -\lambda_+). \quad (6.9)$$

It is usually assumed that $f_+(t)$ and $f_0(t)$ are linear in the decay region; this requires $\lambda_-=0$ for consistency.

A. Comparison of K_{l3} predictions with experiment

Tables III–V contain our predictions for the K_{l3} form factors based on the one-loop approximation calculations of Section III. A measure of the higher-order effects of the strong interactions is provided by Table IV in which the tree-approximation form factors

$$f_+(t)=1.0$$

and

$$f_-(t)=\frac{12F_K G^\phi_{43,4}}{t-m_K^2} \quad (6.10)$$

are compared with their one-loop approximation

TABLE VI. The pion electromagnetic form factor $F_\pi(t)$ for various values of t for the solutions given in Tables I and II. $F_\pi(0)=1.0000$ in all cases ($m_{\pi^0}=140$ MeV).

t ($m_{\pi^0}^2$)	1	2	3	4	5	6
4	1.0105	1.0128	1.0172	1.0130	1.0150	1.0077
−20	0.9814	0.9720	0.9605	0.9709	0.9617	0.9731
−40	0.9720	0.9560	0.9373	0.9540	0.9388	0.9548

TABLE VII. The individual diagram contributions to $F_\pi(t)$ for $t=4m_{\pi^0}^2$ and $t=40m_{\pi^0}^2$ for solution 4 ($m_{\pi^0}=140$ MeV).

Diagram	$t=4m_{\pi^0}^2$	$t=-40m_{\pi^0}^2$
1	0.6589	0.6589
2	0.2358	0.1810
3	0.1183	0.1141
4	0.0000	0.0000
5	0.0000	0.0000
Total	1.0130	0.9540

counterparts at a number of different t values. The one-loop correction to $f_+(t)$ is seen to be extremely small over the range of t considered; on the other hand the one-loop corrections to $f_-(t)$ are quite substantial.

In Table V we have listed our predictions for the parameters appearing in Eqs. (6.4)–(6.7). First, with regard to $f_+(0)$, the best experimental information on this quantity is its correlation with F_K and F_π given by[35]

$$\frac{F_K}{F_\pi f_+(0)}=1.25\pm 0.03\,. \tag{6.11}$$

We used (6.11) and our expectation[21] that $f_+(0) \approx 1.0$ to establish the target value of F_K/F_π in our solutions. The predicted one-loop values of $F_K/F_\pi f_+(0)$ lie within ~10% of the experimental range. Also, the one-loop values of $f_+(0)$ are consistent with the rather inaccurate result $f_+(0)=0.96\pm 0.07$ of Buchanan *et al.*[36]

The one-loop values of λ_+ are seen to range from 0.0004 to 0.0010. These are much smaller than the recent, accurate value of $\lambda_+=0.030\pm 0.003$ obtained by Donaldson *et al.*[35] in a Dalitz-plot analysis of 1.6×10^6 $K_L^0\to\pi\mu\nu$ events. Perhaps more significantly, they are small compared with the world averages of K_{l3}^+ and K_{l3}^0 Dalitz-plot experiments as reported by Donaldson *et al.* and with which their own value is consistent. In fact, the experimental values of λ_+ are remarkably consistent with simple K^* dominance of $f_+(t)$ and, since in the present model $f_+(t)$ does not contain any spin-one poles in the one-loop approximation,[37] it is not surprising that the predicted t dependence is small.[38]

With respect to the parameter λ_0 the situation is quite different. Our predictions run from $\lambda_0=0.0087$ to $\lambda_0=0.0173$. Most are lower than, but not inconsistent with, the result $\lambda_0=0.019\pm 0.004$ of Donaldson *et al.*[35] and are also consistent with the value of $\lambda_0=0.024\pm 0.013$ found by Buchanan *et al.*[36] in a constrained $[f_+(0)=1.0]$ fit. They are not consistent with the result $\lambda_0=0.032\pm 0.010$ obtained by Buchanan *et al.* using an unconstrained fit. Donaldson *et al.* also obtained acceptable results with a separate, two-pole (K^*,κ) fit to $f_+(t)$ and $f_0(t)$ with $m_\kappa=1109\pm 42$ MeV. It can be checked that in our case the κ pole accounts for the largest portion of the slope; e.g., in solution 4 $\lambda_0=0.0090$, while $(m_\pi/m_\kappa)^2=0.0111$.

The positive values we obtain for $\xi(0)$ are correlated with our small λ_+ [see Eq. (6.8)]. A more realistic λ_+ would bring our $\xi(0)$ closer to the results $\xi(0)=-0.11\pm 0.03$ of Donaldson *et al.*[35] and $\xi(0)=-0.20\pm 0.15$ of Buchanan *et al.*[36] These latter values are among the least negative of the experimental determinations of $\xi(0)$. Until a short time ago, of all the world averages, only that of the $(K_{\mu 3}^0/K_{e3}^0)$ branching-ratio analyses yields a (small) positive value for $\xi(0)$.[35] However, a recent analysis[39] of the muon polarization in the decay $K_L^0\to\pi^-\mu^+\nu_\mu$, based on more than 200 000 events, obtained $\xi(0)=0.178\pm 0.105$, which is close to our model predictions.

B. Model tests of various theoretical predictions

Using the present σ model as a laboratory, we shall now examine a number of theoretical predictions both for the K_{l3} form factors and for pseudoscalar- and scalar-meson masses, decay, and renormalization constants, which have been derived using various symmetry and smoothness assumptions. We will be especially interested in tests of current-algebra relations and the insight they provide into the effects of chiral symmetry breaking.

We commented in the last section that one might expect the one-loop calculation of any given quantity to be accurate to within ~20% and perhaps even ~10%, since the mass spectrum and know decay constants were reproduced to that accuracy. In this section we will assume further that, when a symmetry relation holds between several quantities, the corrections to this relation, which arise from symmetry breaking, are predicted to within ~20% by the present model. This appears reasonable, because quantities related by a symmetry property should receive correspondingly related contributions at each order of the loop expansion.[20] It will be seen below that this behavior usually does manifest itself in lowest order, as the differences between one-loop and tree-approximation values of both the right- and left-hand sides of most relations considered have the same sign and rough magnitude. Exceptions to this will be pointed out when we come to them. More conservatively, the following tests can probably at least be trusted as a guide to the relative validity of the various symmetry predictions.

As was mentioned in the Introduction, the present model has the structure of (1.1) and (1.2). For example, the symmetry-breaking interaction (2.1) can be re-expressed as (with $\mathcal{H}_{SB} = -\mathcal{L}_{SB}$)

$$\mathcal{H}_{SB} = -\frac{\epsilon_8}{\sqrt{2}}(\sigma_0 - \sqrt{2}\,\sigma_8) + \left(\epsilon_0 + \frac{\epsilon_8}{\sqrt{2}}\right)\sigma_0\,, \qquad (6.12)$$

where $\sigma_0 - \sqrt{2}\sigma_8$ is SU(2)×SU(2) invariant, so that

$$\delta_1 = -\frac{\epsilon_8}{\sqrt{2}}$$

and (6.13)

$$\delta_2 = \epsilon_0 + \frac{\epsilon_8}{\sqrt{2}}\,.$$

From our analysis it turns out (see Tables I and II) that $|\epsilon_0 + \epsilon_8/\sqrt{2}\,| \ll |\epsilon_8/\sqrt{2}\,|$ so that (1.3) is satisfied, as expected.[3,13]

When considering a particular prediction below we will indicate, where appropriate, in which limit, $\delta_1 \to 0$ or $\delta_2 \to 0$, it becomes exact.[40] The predictions will be examined first in the tree and then in the one-loop approximation of the σ model. The tests will be carried out using solutions 1 and 4 of Tables I and II which are representative of solutions with low and high σ mass, respectively. The results of these tests are presented in Table VIII.

Let us first look at one of the earliest most general of the predictions for the K_{l3} form factors, namely the Ademollo-Gatto theorem[21]

$$f_+(0) = 1.0 + O(\delta_1^2)\,. \qquad (6.14)$$

Actually Langacker and Pagels have shown[41] that

$$f_+(0) = 1.0 + O(\delta_1^2/\delta_1) = 1.0 + O(\delta_1) \qquad (6.15)$$

when SU(3)×SU(3) symmetry is realized in the Nambu-Goldstone mode (as in the present model[14]). The correction linear in δ_1 has been calculated[41,42] leading to

$$f_+(0) = 1.0 - \frac{m_K^2}{64\pi^2 F_\pi^2}\left(\tfrac{5}{2} - 6\ln\tfrac{4}{3}\right). \qquad (6.16)$$

Relation (6.16) could serve as a useful test of the capability of the σ model (in the present, one-loop approximation) to furnish accurate estimates of corrections to chiral symmetry predictions. However, there are difficulties of interpretation in any comparison of this relation with the model prediction for $f_+(0)$ as calculated according to Sec. IV.

Although calculated quantities in the present model contain terms both analytic and nonanalytic in the pseudoscalar-meson masses (equivalently δ_1 and δ_2), there is not a one-to-one correspondence between the loop expansion and the different sorts of terms present in a perturbation series in δ_1 and/or δ_2. Thus for example the one-loop approximation to a particular quantity may contain terms which behave like $\delta\ln\delta$, $\delta^{1/2}$, δ, $\delta^{3/2}$, ... The contribution of the additional, nonanalytic terms to the σ-model value of $f_+(0)$ will cause it to differ from the prediction in (6.16) to any order in the loop expansion.

As can be seen from Table VIII the one-loop σ model values for $f_+(0)$ are all quite close to those obtained[43] from (6.16). The corrections arising in the model to this order are of the same sign and approximate magnitude as that given in Eq. (6.16). The discrepancies undoubtedly stem from the sources noted above. At any rate we are encouraged to hope that the estimate of chiral-symmetry-breaking effects which follow are at least qualitatively and perhaps quantitatively meaningful. In fact, if our estimates are indeed good to within ~20%, our results may give a better indication of chiral-symmetry-breaking effects than can be obtained by simply isolating a correction term with a particular δ dependence.

Before leaving $f_+(0)$, however, we will consider two other predictions for it. The first, derived by Lee[44] on the basis of a chiral Lagrangian model incorporating field-current identities, is

$$f_+(0) = \frac{1}{2}\left(\frac{F_K}{F_\pi} + \frac{F_\pi}{F_K}\right). \qquad (6.17)$$

One sees from Table VIII that for solution 1 (6.17) is much better satisfied in the one-loop than in the tree approximation. In solution 4, however, the convergence of the prediction is slower. In both solutions (6.17) holds to ~2%.

The second prediction

$$f_+(0) = \frac{1}{2F_\pi F_K}\left(F_\pi^2 + F_K^2 - F_\kappa^2\right) \qquad (6.18)$$

was first derived by Glashow and Weinberg[4] from Ward-Takahashi identities and the assumptions of $(3,3^*)\oplus(3^*,3)$ symmetry breaking and smoothness. Relation (6.18) is identically satisfied in the tree approximation, of course, but Table VIII shows that the right- and left-hand sides differ by $\lesssim 1.5\%$ in the one-loop approximation.

Our one-loop results are consistent with the right-hand sides of both (6.17) and (6.18) converging to values $\lesssim 1.0$, which would make them good approximations as judged by the present model or by the more general prediction (6.16).

We will now examine a number of current-algebra theorems for the K_{l3} form factors. The first states that[22]

$$f_+(m_K^2) + f_-(m_K^2) = \frac{F_K}{F_\pi} + O(\delta_2)\,. \qquad (6.19)$$

Since this relation is satisfied identically in the tree approximation, one of the interesting features of the present investigation is that it allows a

TABLE VIII. Tests in the SU(3) σ model of various theoretical predictions discussed in the text. The first column contains the equation numbers where the predictions are given. Opposite each of these are two rows, the first containing the values of the left- and right-hand sides of the predictions as calculated in the tree approximation of the present model, and the second containing the corresponding values as calculated in the one-loop approximation. The tests were carried out using solutions 1 and 4 of Tables I and II. Note that only the terms appearing explicitly in each equation have been calculated ($m_{\pi 0}=140$ MeV).

	Solution 1		Solution 4	
Prediction	Left-hand side	Right-hand side	Left-hand side	Right-hand side
(6.16)	1.0000	1.0000	1.0000	1.0000
	0.9848	0.9664	0.9844	0.9752
(6.17)	1.0000	1.0819	1.0000	1.0167
	0.9848	1.0136	0.9844	1.0084
(6.18)	1.0000	1.0000	1.0000	1.0000
	0.9848	0.9704	0.9844	0.9932
(6.19)	1.4950	1.4950	1.2000	1.2000
	1.2482	1.2477	1.1484	1.1460
(6.20)	0.6689	0.6689	0.8333	0.8333
	0.7899	0.7795	0.8830	0.8708
(6.21)	0.8268	0.4131	0.2517	0.1833
	0.4046	0.2341	0.2863	0.1376
(6.22)	1.4755	1.4950	1.1967	1.2000
	1.2399	1.2477	1.1411	1.1460
(6.23) ($m_{\pi 0}^2$)	18.641	18.196	14.276	14.198
	15.148	14.972	14.543	14.368
(6.24) ($m_{\pi 0}^2$)	240.35	236.82	187.33	186.76
	199.04	197.44	202.91	201.01
(6.26) ($m_{\pi 0}^2$)	18.786	19.486	15.680	15.805
	16.209	16.476	15.833	16.048
(6.28)	1.4484	1.3855	1.1919	1.1554
	1.2288	1.1929	1.1310	1.1153
(6.29)	0.5516	1.9353	0.8081	1.3778
	0.7497	1.4680	0.8546	1.2767
(6.32)	1.4484	1.4157	1.1919	1.1616
	1.2288	1.2023	1.1310	1.1192
(6.33)	0.5516	1.9413	0.8081	1.3790
	0.7497	1.4976	0.8546	1.2838
(6.35)	1.4950	1.2227	1.2000	1.0954
	1.2477	1.3850	1.1460	1.1832
(6.36)	1.5360	1.3934	1.2167	1.0781
	1.2695	1.1302	1.1605	1.0317
(6.37) ($m_{\pi 0}^3$)	0.6546	0.6546	0.7579	0.7579
	0.6995	2.0294	0.7838	1.8061
(6.38) ($m_{\pi 0}$)	0.6786	0.6786	0.7857	0.7857
	0.6320	0.5351	0.5446	0.4823
(6.39) ($m_{\pi 0}^4$)	68.177	68.177	51.386	51.386
	41.797	37.943	35.183	40.254
(8.40)	0.6896	0.6600	0.2624	0.2667
	0.3147	0.3303	0.1583	0.1947
(6.41)	0.0000	0.0000	0.0000	0.0000
	−0.0047	−0.0077	0.0008	0.0004
(6.42)	0.0000	0.0550	0.0000	0.0222
	−0.0058	0.0275	0.0003	0.0162

comparison, in the one-loop approximation, of chiral-symmetry-breaking corrections on the one hand with general higher-order strong-interaction effects on the other. It is clear from Table VIII that the latter, while not sizeable, are much larger than the former. On the basis of Eq. (6.13) and Table III we would expect the $O(\delta_2)$ corrections to (6.19) to be about 10 times smaller[45] than the $O(\delta_1)$ corrections to (6.15). This is true for solution 4, but in solution 1 they are 50 times smaller.

The experimental data, especially the high-statistics results of Donaldson *et al.*[35] support our finding of a small correction to (6.19). In particular, using Eqs. (6.2) and (6.6) (with $\lambda_0'=0$) the relation (6.19) leads to $\lambda_0 \cong 0.02$, while Donaldson *et al.* find $\lambda_0 = 0.019 \pm 0.004$.

The soft-kaon counterpart of (6.19) is[23]

$$f_+(m_\pi^2) - f_-(m_\pi^2) = \frac{F_\pi}{F_K} + O(\delta_1). \tag{6.20}$$

Again this relation is satisfied identically in the tree approximation and one sees from Table VIII that, as for (6.19), the symmetry-breaking corrections to (6.20) in the one-loop approximation are much smaller than the one-loop contribution to each side, which are 7–15%. The $O(\delta_1)$ corrections to (6.20) are close to what one might expect from our analysis of the Ademollo-Gatto theorem, namely ~2%.

If one accepts the predicted size of the corrections to (6.20) as realistic, as those for (6.15), and presumably (6.19), seem to be, then an interesting situation arises. For a plausible value of $F_K/F_\pi \cong 1.25$ [see Eq. (6.11)] (6.20), with $f_+(m_\pi^2) \cong f_+(0) \cong 1.0$, leads to $\xi(0) \cong 0.2$. This is higher than most of the experimental determinations,[35] although their trend has been toward less negative values (see, e.g., Ref. 39). In addition, a value of $\xi(0) \approx 0.2$ implies from Eq. (6.8) that $\lambda_0 - \lambda_+ \approx 0.016$. Now it would be surprising if λ_+ turned out to have a value much different from its "world average"[35] of ~0.03. This is turn forces λ_0 to be 0.046, which clearly is inconsistent with what is required by the soft-pion theorem (6.19), unless a simple linear expansion is not valid for $f_0(t)$ [or perhaps, for $f_+(t)$, but this seems less likely] in the decay region. One would then need, at least, to use the more general parametrization of Eq. (6.4), together with a nonvanishing λ_- in (6.5). Most experimental analyses including those of Donaldson *et al.*[35] and Buchanan *et al.*[36] use a linear parametrization for $f_0(t)$. It would be extremely interesting to look for nonlinearity in this form factor, but the effect would be quite difficult to detect.

It is certainly not inconceivable that the slope of $f_0(t)$ is higher at $t=0$ than at $t=m_K^2$. On the other hand, as Langacker and Pagels[6] have pointed out, perturbation about the SU(3)×SU(3)-symmetric limit may not always be reliable. An example in which its reliability is questionable follows.

Let us consider a theorem[25] for the slope of $f_0(t)$ which states that

$$(m_K^2 - m_\pi^2)\frac{d}{dt} f_0(t)\bigg|_{m_K^2+m_\pi^2} = \tfrac{1}{2}(F_K/F_\pi - F_\pi/F_K) + O(\delta_1). \tag{6.21}$$

Although the $O(\delta_1)$ corrections to (6.21) are model dependent, Dashen *et al.*[25] expected them to be small. As Table VIII indicates, the percentage correction is large in both the tree and one-loop approximations. In solution 4 the shifts in going from the tree to the one-loop approximation are of opposite sign for the two sides of (6.21). Our results agree with the conclusions of Auvil and Pritchett,[26] who generalized the theorem (6.21) to all values of t and estimated the correction at $t = m_K^2 + m_\pi^2$ to be ~50%.

Based on an incorrect version of (6.21) Dashen and Weinstein[24] argued that a variation of (6.19) which should be better satisfied is

$$f_+(m_K^2 - m_\pi^2) + f_-(m_K^2 - m_\pi^2) = \frac{F_K}{F_\pi} + O(\delta_1\delta_2) + O(\delta_2^2). \tag{6.22}$$

One can see from Table VIII that (6.22), though well satisfied, is no better than half as well so as (6.19).

Generalizations of (6.19) and (6.20) have been derived by Mathur, Okubo, and Yang[46-48] on the basis of assumptions about the smoothness of various physical quantities as functions of the chiral-symmetry-breaking parameters and their ratios. The first were[46]

$$(m_K^2 - m_\pi^2)f_+(\Delta) + (m_K^2 + m_\pi^2)f_-(\Delta) = m_K^2\frac{F_K}{F_\pi} - m_\pi^2\frac{F_\pi}{F_K} + O(\delta_1) \tag{6.23}$$

or an alternative suggested by Mathur and Okubo

$$(m_K^4 - m_\pi^4)f_+(\Delta) + (m_K^4 + m_\pi^4)f_-(\Delta) = m_K^4\frac{F_K}{F_\pi} - m_\pi^4\frac{F_\pi}{F_K} + O(\delta_1), \tag{6.24}$$

where

$$\Delta = m_K^2 + m_\pi^2. \tag{6.25}$$

Table VIII shows that for solution 1 these relations become better satisfied in going from the tree to the one-loop approximation, with (6.24) more nearly so than (6.23). In this solution (6.24) is better satisfied than (6.20), but not so well as (6.19). In solution 4, in which there may be accidental cancellations of $O(\delta_1)$ terms in the tree approximation, (6.23) and (6.24) are more poorly satisfied in the next order. Table VIII implies that these relations may still be quite well obeyed in solution 4, but perhaps no better than (6.20).

An "improved" version of (6.23) was suggested by Mathur and Okubo,[47] namely

$$(m_K^2 + m_\pi^2)f_+(\Delta_0) + (m_K^2 - m_\pi^2)f_-(\Delta_0) = m_K^2\frac{F_K}{F_\pi} + m_\pi^2\frac{F_\pi}{F_K} \tag{6.26}$$

with

$$\Delta_0 = \frac{(m_K{}^2 - m_\pi{}^2)^2(4m_K{}^2 + m_\pi{}^2)}{(2m_K{}^2 + m_\pi{}^2)^2} \cong 0.8 m_K{}^2 . \qquad (6.27)$$

However, one sees from Table VIII that (6.26) is more poorly satisfied than (6.23) in second order. Finally, Mathur and Yang[48] derived the relations

$$f_+(\Delta_0) + f_-(\Delta_0) = 1 - \epsilon - 2a\epsilon + \text{const} \times \delta_1{}^4 \qquad (6.28)$$

and

$$f_+(\Delta_0) - f_-(\Delta_0) = 1 + \epsilon - a\epsilon + \text{const} \times \delta_1{}^4 , \qquad (6.29)$$

where Δ_0 is given in Eq. (6.27),

$$\epsilon = \frac{F_K}{F_\pi} - 1 \qquad (6.30)$$

and

$$a = \frac{m_\pi{}^2 - m_K{}^2}{m_K{}^2 + \frac{1}{2} m_\pi{}^2} \cong -0.89 . \qquad (6.31)$$

Although, according to Mathur and Yang, the *complete* corrections through $O(\delta_1{}^3)$ cancel in these relations, Table VIII shows that the remaining corrections are surprisingly sizeable. The corrections to (6.29) are especially so, being much larger than those to (6.20). Relation (6.29) is another example in which the differences between one-loop and tree values have opposite signs on the right- and left-hand sides. Thus, either this prediction, like (6.21), is not very good, or for some reason our model calculations do not fairly reflect its eventual convergence. Relation (6.28) is about as well satisfied as (6.20) in the one-loop approximation.

Improved versions of (6.28) and (6.29), which include $O(\epsilon^2)$ terms, were also presented by Mathur and Yang.[48] These are

$$f_+(\Delta_0) + f_-(\Delta_0) = -ax + (1+a)/x + \tfrac{2}{3}(1+a)y \qquad (6.32)$$

and

$$f_+(\Delta_0) - f_-(\Delta_0) = \left(1 - \frac{a}{2}\right) x + \frac{a}{2x} + \frac{1}{3}\left(1 - \frac{a}{2}\right) y , \qquad (6.33)$$

where again Δ_0 is given in Eq. (6.27), $x = 1 + \epsilon$, and

$$y = \frac{F_\kappa{}^2}{F_\pi F_K} . \qquad (6.34)$$

The same general comments which were made above for (6.28) and (6.29) apply as well to (6.32) and (6.33), respectively. Inspection of Table VIII shows that (6.32) is indeed somewhat better satisfied than (6.28) in the one-loop approximation. However, (6.33) is slightly more poorly satisfied than (6.29).

As an aside, there is another example of large $O(\delta_1)$ corrections among the predictions of Mathur *et al.* The relation

$$\frac{F_K}{F_\pi} = \left(\frac{1 - \frac{1}{2}b}{1 + b}\right)^{1/2} + O(\delta_1) \qquad (6.35)$$

is seen from Table VIII to be badly violated in the tree approximation, but considerably better satisfield in the next order. Although this suggests that (6.35) may converge to a more approximate validity, this may be misleading, since the one-loop contributions to the tree values of each side have opposite signs. All that can be stated with certainty is that (6.35) is not as well satisfied as (6.20) [or for that matter (6.28) or (6.32) in the one-loop approximation].

The last K_{l3} prediction we shall examine was derived by Gaillard.[49] Using $(3,3^*)\oplus(3^*,3)$ symmetry breaking, the Bjorken limit, and smoothness assumptions, she found

$$f_0(m_K{}^2) = \frac{m_K{}^2}{m_K{}^2 - m_\pi{}^2} \frac{F_K}{F_\pi} \left(1 - \frac{c + \sqrt{2}}{\sqrt{2} + \frac{1}{2}c}\right) + O(\delta_2) , \qquad (6.36)$$

where c is equal to $\sqrt{2}$ times our a of Eq. (2.4). In the limit of spontaneously broken $SU(2)\times SU(2)$ ($c \to -\sqrt{2}$, $m_\pi \to 0$) (6.36) reduces to (6.19). Note, though, that whenever the latter is well satisfied, as in the present case, the relation (6.36) cannot be, unless c is renormalized substantially away from its usual value.[3] Such a large shift in c appears to be unlikely.[6,50] Looking at Table VIII we see indeed that (6.36) is poorly satisfied in the tree approximation and there is only a tiny improvement in the next order. It seems doubtful to us that this prediction would improve if higher orders in the loop expansion were included.

We will conclude this section by considering a number of miscellaneous sum rules and relations which involve masses and decay and renormalization constants. The first group,

$$m_\pi{}^2 F_\pi Z_\pi{}^{-1/2} = m_K{}^2 F_K Z_K{}^{-1/2} + m_\kappa{}^2 F_\kappa Z_\kappa{}^{-1/2}, \qquad (6.37)$$

$$F_\pi Z_\pi{}^{1/2} = F_K Z_K{}^{1/2} + F_\kappa Z_\kappa{}^{1/2} , \qquad (6.38)$$

and

$$4[m_K{}^2 F_K{}^2 + m_\kappa{}^2 F_\kappa{}^2] = 3[m_\eta{}^2 F_\eta{}^2 + m_{\eta'}{}^2 F_{\eta'}{}^2] + m_\pi{}^2 F_\pi{}^2 , \qquad (6.39)$$

was derived by Glashow and Weinberg,[4] as was (6.18), using Ward-Takahashi identities, smoothness assumptions, and $(3,3^*)\oplus(3^*,3)$ symmetry breaking. (6.37)–(6.39) are identities in the tree approximation, but not in higher order. The failure of relation (6.37) to be identically satisfied in the one-loop approximation is an artifact of our renormalization procedure.[51] As Table VIII shows, neither (6.38) nor (6.39) is particularly well satisfied (compared with most of the above K_{l3} predictions) in the one-loop approximation. The $O(\delta_1)$ corrections to (6.38) and (6.39) are apparently larger than those in (6.18).

The last set of predictions we shall consider here were derived using chiral perturbation theory.[7] The first relation[6]

$$\frac{F_\eta}{F_\pi} - 1 = \frac{4}{3}\left(\frac{F_K}{F_\pi} - 1\right) + \text{const} \times \delta_1 + \cdots \qquad (6.40)$$

was obtained by explicitly evaluating and then eliminating the $\delta_1 \ln\delta_1$ corrections to the decay constants.

That is, it follows from the fact that a perturbation expansion in δ_1 yields

$$F_\eta/F_\pi - 1 = a\delta_1 \ln\delta_1 + b\delta_1 + \cdots$$

and

$$F_K/F_\pi - 1 = \tfrac{3}{4}a\delta_1 \ln\delta_1 + c\delta_1 + \cdots ,$$

with a, b, and c constants. Note that the correction terms $F_\eta/F_\pi - 1$ and $F_K/F_\pi - 1$, when evaluated in the one-loop approximation of the model receive contributions from δ_1-dependent terms in addition to those written explicitly above. If it is assumed that our one-loop determination of the right- and left-hand sides of (6.40) is reasonably accurate, then Table VIII shows our results to be quite consistent with the conclusion of Langacker and Pagels[6] that the $\delta_1 \ln\delta_1$ corrections dominate those linear in δ_1. The former are roughly four times the latter in magnitude.

The final predictions we will examine involve the renormalization constants associated with the pseudoscalar fields. First, one sees from Table II that Z_π and Z_K differ by ~1% in the one-loop approximation. This is consistent with the results of Langacker and Pagels[6] who found that the $O(\delta_1)$ corrections to the Z's are ≲5%. The relations

$$Z_\eta^{1/2}/Z_\pi^{1/2} - 1 = \tfrac{4}{3}\,(Z_K^{1/2}/Z_\pi^{1/2} - 1) + \text{const} \times \delta_1 \qquad (6.41)$$

and

$$Z_K^{1/2}/Z_\pi^{1/2} - 1 = \tfrac{1}{9}(F_K/F_\pi - 1) + \text{const} \times \delta_1 \qquad (6.42)$$

were derived[6,52] by evaluating the $\delta_1 \ln\delta_1$ corrections to the renormalization constants. According to Table VIII (6.41) is quite well satisfied in the one-loop approximation. (The correction terms should be compared with 1 here.) Again the $\delta_1 \ln\delta_1$ terms, which are responsible for the differences $Z - 1$, are seen to be substantially larger than the terms linear in δ_1.

(6.42) improves in going from the tree to the one-loop approximation. Although not as well satisfied as (6.41), it seems good to within ~1–3%. Note that in solution 4 the one-loop contributions to the right- and left-hand sides of (6.42) have opposite signs.

IX. CONCLUSIONS

In summary, we have employed a renormalizable σ model in the one-loop approximation to analyse pseudoscalar and scalar-meson masses and decay constants and the K_{l3} and pion electromagnetic form factors. A reasonable mass spectrum and values for F_π and F_K were obtained. Second-order strong-interaction effects were typically found to be small, thus justifying the use of the one-loop approximation. Possibly due to its inherent limitations, namely the lack of spin-1 poles in the one-loop approximation, the model was not able to reproduce the experimentally determined t dependence of $f_+(t)$ and the pion electromagnetic form factor. Nevertheless, the model has provided a useful laboratory in which to study a number of chiral-symmetry predictions. Our results for these are in general agreement with the conclusions of Pagels and co-workers,[7] who have employed chiral perturbation theory to calculate the leading symmetry-breaking corrections to a number of quantities.

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[32]It is possible, but probably not likely, that there is a broad κ centered ~ 1 GeV. The κ(1400) might then be its first radial excitation.

[33]The magnitudes of the second-order coupling constants (Δf_1, Δf_2, Δg) depend on the value chosen for ν^2. We have set $\nu^2 = |\mu^2|$. Consequently, there is no direct relation between the size of these couplings and the resulting decay widths.

[34]Small m's, which up until now have denoted the tree-approximation masses, will be used to represent physical masses in this section. Similarly, F_π, F_K, etc. will refer to the physical decay constants, rather than only to their tree-approximation values.

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[37]There is reason to believe that the effects of $J=1$ resonances would appear in higher order. Chan and Haymaker (Ref. 14) studied meson-meson scattering in the one-loop approximation of the present model. Using Padé approximants, they calculated P- and D-wave phase shifts which agree, at least qualitatively, with the physical particle spectrum.

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[40]Pagels and his co-workers (see Ref. 7) have found that, if chiral symmetry is spontaneously broken, then a perturbation series about the chiral-symmetry limit will contain nonanalytic terms in δ_2 (or δ_1). In general, the leading term will be $O(\delta \ln \delta)$ or $O(\delta^{1/2})$ as well as $O(\delta)$, except when the expansion is about a current-algebra result involving exceptional values of the momenta, in which cases the leading corrections are analytic in δ. In most of the following no distinction will be made between analytic and nonanalytic terms in δ_1 or δ_2. The notation $O(\delta)$ will simply denote terms which vanish in the limit $\delta \to 0$.

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[43]Another problem concerns the question of just what values of m_K and F_π to use in Eq. (8.16). To be at least partially consistent we have used the tree values of these quantities, since in the σ model the Feynman diagrams contributing to the correction term in (8.16) would have one or more loops.

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