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Calculation of the s-wave final-state interactions in the $N\pi\pi$ system

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The coupled integral equations for the $N\pi\pi$ system are considered for s-wave isobars in S states. For phenomenological objectives the equations are made tractable and the subenergy dependence of the isobar amplitudes is organized according to a collection of resolvents. These quantities satisfy coupled integral equations which are solved by means of a basis-function method. Calculations of the resolvents as functions of the subenergy variable are made for total energy $W = 1.3$, 1.4, and 1.5 GeV. All of the large number of results admit a useful physical parametrization in terms of a sort of complex scattering length. A remarkable sign regularity among the parameters is observed and interpreted.

INTRODUCTION

The physics of a three-hadron system is usefully described in terms of the pairwise interactions among the three particles. There arise three configurations, isobar channels, each consisting of a two-body isobar plus a third particle. The isobar subenergy dependence of the amplitude to each of these channels becomes the quantity of interest in this approach. The description in terms of subenergies is a natural one phenomenologically because distributions in these (Dalitz-plot) variables are generally focused upon in experimental situations.

Considerable recent progress has been made in the effort to develop a dynamical framework based on a subenergy perspective, the cornerstone of which is subenergy unitarity imposed upon the isobar expansion.[1,2] The isobar amplitudes are found to satisfy coupled linear integral equations in a single variable.[3] The form they take for the $N\pi\pi$ problem has been developed and discussed at length,[4] and has been applied in a set of medium-energy calculations.[5]

The purpose of this paper is to take the next logical step, to show how the coupled integral equations may be solved. It should be noted immediately that even at modest values of the total energy, the number of coupled amplitudes is rather large; therefore, the emphasis of the procedure is to avoid quadrature methods of obtaining the solution. Such a numerical approach would become cumbersome in a highly coupled application and might even swamp the capacity of the computing system. Instead, a set of suitable functions is identified and used as a basis set for the expansion of the desired solutions. By this means the coupled integral equations are solved in terms of simple algebra.

The scope of the problem has been limited somewhat as the title would indicate. Only s-wave $N\pi$ and $\pi\pi$ isobars in S states are considered so that only the waves having $J^P = \frac{1}{2}^+$ occur. This collection of isobar channels is sufficiently complex to illustrate the method, even though the p-wave Δ isobar in S and P states ought not to be left out of any comprehensive phenomenology. Despite this truncation of the problem, a multitude of quantities is to be determined according to the procedure. Calculations of these quantities are performed for total energy $W = 1.3$, 1.4, and 1.5 GeV. By way of interpreting the many numerical results, a parametrization of universal form is proposed which reveals a striking observation common to all the calculations.

EQUATIONS

The isobar amplitudes are identified as in Ref. 4, Eqs. (25) and (26), and the basic amplitudes of interest are organized in two-component form according to the isobar isospin:

$$f_S = \begin{pmatrix} f_{S_1} \\ f_{S_3} \end{pmatrix} \quad \text{and} \quad f_\epsilon = \begin{pmatrix} f_{\epsilon_0} \\ f_{\epsilon_2} \end{pmatrix}.$$

The notation for the s-wave isobars is the same as in Ref. 5: S_1 and S_3 for $N\pi$, ϵ_0 and ϵ_2 for $\pi\pi$.[6] The coupled integral equations, in the notation of Ref. 4, are

$$f_S(s_1) = c_S + \int_{-\infty}^{\hat{z}_2} dz_2 \Delta_{12}(s_1, z_2) D\zeta_S(z_2) f_S(z_2) + \int_{-\infty}^{\hat{z}_3} dz_3 \Delta_{13}(s_1, z_3) \bar{C}\zeta_\epsilon(z_3) f_\epsilon(z_3) \tag{1a}$$

and

$$f_\epsilon(s_3) = c_\epsilon + \int_{-\infty}^{\hat{z}_1} dz_1 2\Delta_{31}(s_3, z_1) C\zeta_S(z_1) f_S(z_1)\,; \tag{1b}$$

the variables s_i and z_j denote the three isobar sub-

energy variables. The isospin crossing matrices are

$$D=-\frac{1}{3}\begin{pmatrix}1 & 2\sqrt{2}\\ 2\sqrt{2} & -1\end{pmatrix} \text{ and } C=\frac{1}{\sqrt{3}}\begin{pmatrix}-1 & \sqrt{2}\\ 0 & 0\end{pmatrix}$$

for total isospin $T=\frac{1}{2}$, and

$$D=\frac{1}{3}\begin{pmatrix}2 & \sqrt{5}\\ \sqrt{5} & -2\end{pmatrix} \text{ and } C=\frac{1}{\sqrt{6}}\begin{pmatrix}0 & 0\\ \sqrt{5} & 1\end{pmatrix}$$

for $T=\frac{3}{2}$; the matrix $\overline{C}$ is the transpose of C. In Eqs. (1), ζ_S and ζ_ϵ are the diagonal matrices

$$\zeta_S=\begin{pmatrix}\zeta_{s_1} & 0\\ 0 & \zeta_{s_3}\end{pmatrix}$$

and

$$\zeta_\epsilon=\begin{pmatrix}\zeta_{\epsilon_0} & 0\\ 0 & \zeta_{\epsilon_2}\end{pmatrix},$$

the diagonal elements of which are the unitary two-body amplitudes describing elastic scattering in the isobar states.

Some concessions must be made to make progress with these equations. The attitude one takes about the inhomogeneous terms c_S and c_ϵ is the heart of the matter. In principle these terms may depend on s_i as long as they are free of the right-hand (elastic) unitarity cut which the integral terms contain; for example, production Born terms having left-hand cuts would show up here. These may be of interest as driving terms in a dynamical application of the equations. For the present purpose a phenomenological attitude is the one to adopt. Only physical values of s_i are of interest; for subenergies thus restricted c_S and c_ϵ may be assumed independent of s_i, although of course they must depend on $s=W^2$. The motives here would ultimately be to fit data, allowing for rescattering corrections to previous nonunitary fits using the isobar model.[7] In this context the c's are thought of as complex fitting parameters,

TABLE I. Parameters for fitting $I_{\alpha\beta}(s_i)$ according to Eq. (27).

		(a) Total isospin $T=\frac{1}{2}$						(b) Total isospin $T=\frac{3}{2}$			
$\alpha\beta$	W (GeV)	η		r (GeV^{-1})		$\alpha\beta$	W(GeV)	η		r (GeV^{-1})	
		Real	Imaginary	Real	Imaginary			Real	Imaginary	Real	Imaginary
S_1S_1	1.3	−0.064	0.001	−0.364	−2.040	S_1S_1	1.3	0.088	−0.009	−1.292	−1.203
	1.4	−0.077	−0.004	−0.262	−2.545		1.4	0.080	−0.020	−1.650	−1.228
	1.5	−0.090	−0.009	−0.456	−2.675		1.5	0.074	−0.031	−1.651	−1.135
S_1S_3	1.3	0.175	0.009	−2.029	−5.285	S_1S_3	1.3	−0.107	0.002	−1.738	−5.175
	1.4	0.227	0.011	−2.573	−4.325		1.4	−0.140	0.008	−2.542	−4.357
	1.5	0.284	0.007	−2.674	−3.411		1.5	−0.178	0.013	−2.730	−3.388
S_3S_1	1.3	−0.281	−0.031	−0.645	−3.498	S_3S_1	1.3	0.183	0.010	−0.334	−2.839
	1.4	−0.291	−0.049	−0.822	−3.070		1.4	0.196	0.018	−0.494	−2.737
	1.5	−0.299	−0.065	−0.844	−2.701		1.5	0.210	0.028	−0.553	−2.532
S_3S_3	1.3	−0.128	−0.042	−1.660	−5.474	S_3S_3	1.3	0.110	−0.001	−2.046	−5.299
	1.4	−0.163	−0.084	−1.860	−4.485		1.4	0.148	−0.001	−2.644	−4.354
	1.5	−0.195	−0.134	−1.861	−3.681		1.5	0.191	−0.001	−2.786	−3.409
$S_1\epsilon_0$	1.3	−0.064	−0.007	−2.743	−4.346	$S_1\epsilon_2$	1.3	−0.063	−0.001	−2.404	−3.655
	1.4	−0.075	−0.006	−2.649	−3.674		1.4	−0.072	0.003	−2.223	−3.015
	1.5	−0.091	−0.007	−2.581	−3.057		1.5	−0.081	0.006	−2.102	−2.407
$S_3\epsilon_0$	1.3	0.108	0.022	−2.278	−4.535	$S_3\epsilon_2$	1.3	−0.035	−0.004	−1.907	−3.940
	1.4	0.131	0.036	−2.160	−3.804		1.4	−0.042	−0.006	−1.634	−3.300
	1.5	0.156	0.058	−2.143	−3.197		1.5	−0.050	−0.010	−1.537	−2.729
ϵ_0S_1	1.3	−0.688	0.004	−0.160	−3.332	ϵ_2S_1	1.3	1.463	0.074	0.229	−4.209
	1.4	−0.636	0.039	−0.402	−2.896		1.4	1.477	0.122	0.339	−3.925
	1.5	−0.592	0.071	−0.456	−2.471		1.5	1.478	0.164	0.388	−3.606
ϵ_0S_3	1.3	−0.802	0.003	−1.775	−5.492	ϵ_2S_3	1.3	−0.491	−0.050	−1.528	−5.685
	1.4	−0.954	0.089	−2.188	−4.978		1.4	−0.603	−0.069	−1.708	−5.176
	1.5	−1.100	0.195	−2.312	−4.112		1.5	−0.714	−0.120	−1.737	−4.232
$\epsilon_0\epsilon_0$	1.3	−0.017	−0.012	−5.358	−7.152	$\epsilon_2\epsilon_2$	1.3	−0.042	−0.014	0.836	−8.001
	1.4	−0.038	−0.040	−3.983	−6.388		1.4	−0.056	−0.029	3.263	−7.372
	1.5	−0.054	−0.085	−3.263	−5.034		1.5	−0.064	−0.048	6.138	−6.480

constants for a given value of s. In any practical application of the results which follow, no more parameters than the original set of c's arise. This represents a distinct improvement over the state of affairs concluded in Ref. 5, wherein the effective parameter set was doubled in size.

In the same spirit, the integral terms in Eqs. (1) may be put into a more practical form. Let each integral be written as

$$\int_{-\infty}^{\hat{z}_j} = \int_{-\infty}^{0} + \int_{0}^{\hat{z}_j} ,$$

where $\hat{z}_j$ lies at the upper edge of the Dalitz-plot boundary. It may then be recognized that, for s_i physical, the second piece spans the Dalitz plot where it is expected to make the dominant contribution to variations in s_i. If the first piece has comparatively negligible dependence on s_i, then it may be treated as constant and absorbed into the unknown c term. By this argument the integral equations to solve for physical s_i are just Eqs. (1) with constant c's and with the lower limits $-\infty$ replaced by 0. Once this has been done, then simple effective range forms are admissible for ζ_S and ζ_ϵ. In Sec. IV of Ref. 5, expressions for these have been given in Eqs. (14) and (19); for the purposes of the calculations to follow, the parameters used, which provide an excellent fit to the available elastic $N\pi$ and $\pi\pi$ data, are as given numerically in Table I of Ref. 5.

Every one of the c's contributes according to its weight in the determination of how each f depends on its s_i. The way to sort out all the possible ingredients to the s_i dependence is to introduce *resolvents* and deduce the integral equations they satisfy. The s_i dependence of each of these quantities is obtained independently of the c's.

A simple one-component example serves to illustrate. Let f satisfy

$$f(x) = c + \int K(x,y) f(y) dy . \tag{2}$$

The solution in terms of the resolvent kernel is

$$f(x) = c + \int G(x,y) c dy , \tag{3}$$

where G satisfies

$$G(x,z) = K(x,z) + \int K(x,y) G(y,z) dy , \tag{4}$$

independently of c. The special feature of this well-known construction is that a further step can be taken if c is a constant. Then (3) becomes

$$f(x) = c + I(x) c , \tag{5}$$

where I satisfies

$$I(x) = I^0(x) + \int K(x,y) I(y) dy \tag{6}$$

in which

$$I^0(x) = \int K(x,z) dz . \tag{7}$$

Thus the dependence of the resolvent I on its single variable is obtained from a knowledge of the kernel K alone. The simple step (5) then provides f in terms of c.

The problem at hand involves a set of several coupled f's and a corresponding collection of c's. Because c_S and c_ϵ are constants, the same procedure goes through, with allowance made for the greater dimensionality. Thus, because there are two kinds of isobar channels, $(N\pi)\pi$ and $N(\pi\pi)$, there must be four resolvents: I_{SS}, $I_{S\epsilon}$, $I_{\epsilon S}$, and $I_{\epsilon\epsilon}$. Further, each of these must be a 2×2 matrix in the isobar isospin. Step (5) becomes therefore

$$f_S(s_1) = c_S + I_{SS}(s_1) c_S + I_{S\epsilon}(s_1) c_\epsilon \tag{8}$$

and

$$f_\epsilon(s_3) = c_\epsilon + I_{\epsilon S}(s_3) c_S + I_{\epsilon\epsilon}(s_3) c_\epsilon . \tag{9}$$

The problem of determining the subenergy dependence of the f's comes down to obtaining the s_i dependence of the I's by solving the integral equations they satisfy, independently of the c's. It is easy to show from (1), (8), and (9) that these equations are

$$I_{SS}(s_1) = DI^0_{SS}(s_1) + D\int_0^{\hat{z}_2} dz_2 \Delta_{12}(s_1,z_2)\zeta_S(z_2) I_{SS}(z_2) + \overline{C}\int_0^{\hat{z}_3} dz_3 \Delta_{13}(s_1,z_3)\zeta_\epsilon(z_3) I_{\epsilon S}(z_3) , \tag{10a}$$

$$I_{S\epsilon}(s_1) = \overline{C} I^0_{S\epsilon}(s_1) + D\int_0^{\hat{z}_2} dz_2 \Delta_{12}(s_1,z_2)\zeta_S(z_2) I_{S\epsilon}(z_2) + \overline{C}\int_0^{\hat{z}_3} dz_3 \Delta_{13}(s_1,z_3)\zeta_\epsilon(z_3) I_{\epsilon\epsilon}(z_3) , \tag{10b}$$

$$I_{\epsilon S}(s_3) = CI^0_{\epsilon S}(s_3) + C\int_0^{\hat{z}_1} dz_1 2\Delta_{31}(s_3,z_1)\zeta_S(z_1) I_{SS}(z_1) , \tag{10c}$$

$$I_{\epsilon\epsilon}(s_3) = C\int_0^{\hat{z}_1} dz_1 2\Delta_{31}(s_3,z_1)\zeta_S(z_1) I_{S\epsilon}(z_1) . \tag{10d}$$

The inhomogeneous terms are, apart from the isospin crossing matrices,

$$I^0_{SS}(s_1)=\int_0^{\hat{z}_2} dz_2\Delta_{12}(s_1,z_2)\zeta_S(z_2)\,, \tag{11a}$$

$$I^0_{S\epsilon}(s_1)=\int_0^{\hat{z}_3} dz_3\Delta_{13}(s_1,z_3)\zeta_\epsilon(z_3)\,, \tag{11b}$$

$$I^0_{\epsilon S}(s_3)=\int_0^{\hat{z}_1} dz_1 2\Delta_{31}(s_3,z_1)\zeta_S(z_1)\,. \tag{11c}$$

Note that the I^0's are *diagonal* because the ζ's are, and that there occurs no $I^0_{\epsilon\epsilon}(s_3)$.

SOLUTION

The I^0's in Eqs. (10) and (11) are among the many rescattering integrals which have been calculated in Ref. 5. In fact, it is the further study of their detailed behavior which reveals the method by which the integral equations are to be solved.

In Fig. 1 the integrals $I^0_{s_1s_1}$ and $I^0_{s_3s_3}$ are shown as functions of s_1 for $W=1.5$ GeV. Their behavior is representative of all of the I^0's in (11). The real and imaginary parts cross in the region above the subenergy threshold, a fact which has been noted in Ref. 5 by way of suggesting a useful parametrization and interpretation of the results.[8] For the present purpose, it is more to the point to extend the calculation below threshold and observe the behavior of the I^0's over the whole range of s_i from 0 to $\hat{z}_i$. The dominant feature is the pronounced kink at threshold. A standard function[9] which has such a kink is

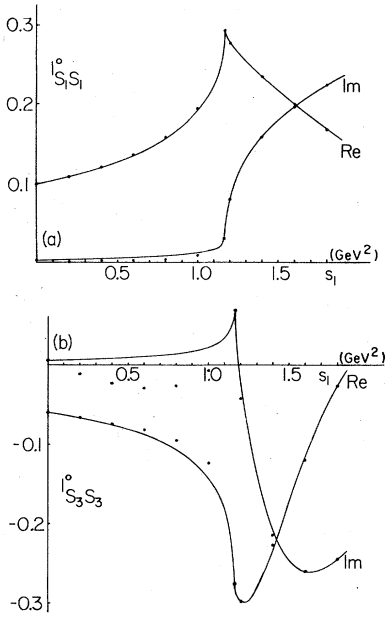

FIG. 1. Rescattering integrals, real and imaginary parts, for $W=1.5$ GeV: (a) $I^0_{s_1s_1}(s_1)$ and (b) $I^0_{s_3s_3}(s_1)$. The dots indicate the quality of the fit obtained using the basis set (13) in Eq. (15a).

$$g(z)=\frac{[z-(M+\mu)^2]^{1/2}[z-(M-\mu)^2]^{1/2}}{z}\left\{\ln\frac{[z-(M-\mu)^2]^{1/2}+[z-(M+\mu)^2]^{1/2}}{[z-(M-\mu)^2]^{1/2}-[z-(M+\mu)^2]^{1/2}}-i\pi\right\}$$
$$+\left(\frac{M^2-\mu^2}{z}-\frac{M-\mu}{M+\mu}\right)\ln\frac{M}{\mu}\,, \tag{12a}$$

where z corresponds to an $N\pi$ subenergy, and

$$g_3(z)=\left(\frac{z-4\mu^2}{z}\right)^{1/2}\left[\ln\frac{z^{1/2}+(z-4\mu^2)^{1/2}}{z^{1/2}-(z-4\mu^2)^{1/2}}-i\pi\right], \tag{12b}$$

where z corresponds to a $\pi\pi$ subenergy. Other functions having a sharp break at threshold are the roots $[z-(M+\mu)^2]^{1/2}$ and $(z-4\mu^2)^{1/2}$. It would appear that any of the I^0's could be represented to very good approximation by taking a combination of Eqs. (12a) or (12b) with the proper root function to take the kink out, and then fitting the smooth remainder of I^0 with a low-order polynomial in z. To implement this idea a set of four independent functions is introduced to use as a basis set for expressing each I^0. With $A=0$ to 3 as an index, let

$$\{u_A(z)\}=\{1,[z-(M+\mu)^2],\ g(z),[z-(M+\mu)^2]^{1/2}\} \tag{13}$$

for z an $N\pi$ variable, and

$$\{v_A(z)\}=\{1,(z-4\mu^2),\ g_3(z),\ (z-4\mu^2)^{1/2}\} \tag{14}$$

for z a $\pi\pi$ variable. Thus the I^0's are written as

$$I^0_{SS}(s_1)=\sum_{A=0}^{3}(I^0_{SS})_A u_A(s_1)\,, \tag{15a}$$

$$I^0_{S\epsilon}(s_1)=\sum_{A=0}^{3}(I^0_{S\epsilon})_A u_A(s_1)\,, \tag{15b}$$

$$I^0_{\epsilon S}(s_3)=\sum_{A=0}^{3}(I^0_{\epsilon S})_A v_A(s_3)\,. \tag{15c}$$

The four complex 2×2 diagonal matrix coefficients in each of Eqs. (15) are obtained by matching each I^0 at four values of s_i.

If the I^0's are adequately represented in terms of the basis sets (13) and (14), then the same might be assumed for the solutions of the integral equations (10) in which the I^0's appear as inhomogeneous terms. Obviously, if all the functions of the subenergy variables are represented by means of the basis sets then the problem reduces to one of solving simple algebraic equations to determine all the representation coefficients. This is the strategy to be adopted in solving Eqs. (10).

Some comment is necessary about the matching procedure used in representation (15). The four matching points chosen are 0, threshold, $\hat{z}_i$, and a point intermediate between threshold and $\hat{z}_i$. The quality of the fit to $I^0_{s_1s_1}$ and $I^0_{s_3s_3}$ is shown in Fig. 1. Clearly, the fit is excellent above threshold but less so below; in fact $I^0_{s_3s_3}$ is deliberately shown as an example of one of the poorest-quality fits. If a fifth basis function were used in (13) and (14), a matching point between 0 and threshold could be included for overall improvement. Short of doing this, an alternative test can be made to evaluate the sensitivity of the solution of the problem to a choice of a matching point intermediate between 0 and threshold rather than at 0 itself. The ultimate results prove to be remarkably unaffected by this change of procedure, so that a four-function basis set can be regarded as adequate.

The resolvents are expanded as in (15); summation symbols are suppressed from here on:

$$\begin{aligned} I_{SS}(s_1) &= (I_{SS})_A u_A(s_1)\,, \\ I_{S\epsilon}(s_1) &= (I_{S\epsilon})_A u_A(s_1)\,, \\ I_{\epsilon S}(s_3) &= (I_{\epsilon S})_A v_A(s_3)\,, \\ I_{\epsilon\epsilon}(s_3) &= (I_{\epsilon\epsilon})_A v_A(s_3)\,. \end{aligned} \tag{16}$$

When Eqs. (16) are inserted into Eqs. (10) there arise certain integrals, functions of s_i, which according to the method are also expanded:

$$\int_0^{\hat{z}_2} dz_2 \Delta_{12}(s_1,z_2)\zeta_S(z_2)u_B(z_2) = u_A(s_1)\Gamma_{AB}\,, \tag{17a}$$

$$\int_0^{\hat{z}_3} dz_3 \Delta_{13}(s_1,z_3)\zeta_\epsilon(z_3)v_B(z_3) = u_A(s_1)\tilde{\Gamma}_{AB}\,, \tag{17b}$$

$$\int_0^{\hat{z}_1} dz_1 2\Delta_{31}(s_3,z_1)\zeta_S(z_1)u_B(z_1) = v_A(s_3)\hat{\Gamma}_{AB}\,. \tag{17c}$$

Note that for each A and B, ranging from 0 to 3, the coefficients Γ_{AB}, $\tilde{\Gamma}_{AB}$, and $\hat{\Gamma}_{AB}$ are 2×2 *diagonal* matrices in the isobar isospin (again, diagonal because the ζ's are). For $B=0$ a comparison of Eqs. (11), (15), and (17) leads to the observation that

$$\Gamma_{A0} = (I^0_{SS})_A\,, \tag{18a}$$

$$\tilde{\Gamma}_{A0} = (I^0_{S\epsilon})_A\,, \tag{18b}$$

$$\hat{\Gamma}_{A0} = (I^0_{\epsilon S})_A\,. \tag{18c}$$

The numerical values of all the Γ's are obtained by the same four-point matching procedure as described above for the I^0's. No tabulation of these complex numbers is given here; suffice it to say that the fit (17) is very good especially above threshold and that a different choice of four matching points makes no significant difference in the ultimate results.

The solution of the integral equations (10) for the resolvents $I_{SS}(s_1)$, etc., has become the problem of determining the coefficients $(I_{SS})_A$, etc. It is easy to show from (10) that these satisfy the *algebraic* equations

$$(I_{SS})_A = D\Gamma_{A0} + D\Gamma_{AB}(I_{SS})_B + \overline{C}\tilde{\Gamma}_{AB}(I_{\epsilon S})_B\,, \tag{19a}$$

$$(I_{S\epsilon})_A = \overline{C}\,\tilde{\Gamma}_{A0} + D\Gamma_{AB}(I_{S\epsilon})_B + \overline{C}\,\tilde{\Gamma}_{AB}(I_{\epsilon\epsilon})_B\,, \tag{19b}$$

$$(I_{\epsilon S})_A = C\hat{\Gamma}_{A0} + C\hat{\Gamma}_{AB}(I_{SS})_B\,, \tag{19c}$$

$$(I_{\epsilon\epsilon})_A = C\hat{\Gamma}_{AB}(I_{S\epsilon})_B\,. \tag{19d}$$

Clearly, the primitive quantities are $(I_{SS})_A$ and $(I_{S\epsilon})_A$ since $(I_{\epsilon S})_A$ and $(I_{\epsilon\epsilon})_A$ are obtained from them. By insertion of (19c) into (19a) and (19d) into (19b), the two equations to solve are

$$R_{AC}(I_{SS})_C = D\Gamma_{A0} + \overline{C}\,\tilde{\Gamma}_{AB}C\hat{\Gamma}_{B0} \tag{20}$$

and

$$R_{AC}(I_{S\epsilon})_C = \overline{C}\tilde{\Gamma}_{A0}\,, \tag{21}$$

in which

$$R_{AC} = \delta_{AC} - D\Gamma_{AC} - \overline{C}\tilde{\Gamma}_{AB}C\hat{\Gamma}_{BC}\,. \tag{22}$$

At this point it should be recalled that all the quantities above are 2×2 matrices in the isobar isospin. If these are indexed with small letters, Eqs. (20), (21), and (22) are written in detail as

$$(R_{AC})_{ik}[(I_{SS})_C]_{kj} = D_{ij}\Gamma^j_{A0} + \overline{C}_{ik}\,\tilde{\Gamma}^k_{AB}C_{kj}\,\hat{\Gamma}^j_{B0}\,, \tag{20'}$$

$$(R_{AC})_{ik}[(I_{S\epsilon})_C]_{kj} = \overline{C}_{ij}\,\tilde{\Gamma}^j_{A0}\,, \tag{21'}$$

$$(R_{AC})_{ik} = \delta_{AC}\delta_{ik} - D_{ik}\,\Gamma^k_{AC} - \overline{C}_{im}\tilde{\Gamma}^{\,m}_{AB}C_{mk}\,\hat{\Gamma}^k_{BC}\,, \tag{22'}$$

in which the Γ's are diagonal and so Γ^j denotes the jth diagonal element.

The equations are readily solved by finding 2×2 matrices T_{BA} such that

$$(T_{BA})_{ni}(R_{AC})_{ik} = \delta_{nk}\delta_{BC}\,. \tag{23}$$

Then the results are

$$[(I_{SS})_B]_{nj} = (T_{BA})_{ni}(D_{ij}\Gamma^j_{A0} + \bar{C}_{ik}\tilde{\Gamma}^k_{AB}C_{kj}\hat{\Gamma}^j_{B0}) \tag{24}$$

and

$$[(I_{Se})_B]_{nj} = (T_{BA})_{ni}\bar{C}_{ij}\tilde{\Gamma}^j_{A0}\,. \tag{25}$$

The most convenient way to proceed to these results is to express R as an 8×8 complex matrix and to invert it, by computer, to obtain an 8×8 matrix T. In this view the determinant of R, a function only of W, plays the role of the Fredholm denominator. As noted already, insertion of (24) into (19c) and (25) into (19d) provides the remaining quantities $(I_{eS})_A$ and $(I_{ee})_A$.

CALCULATION

The algebraic solution for all the coefficients appearing in Eqs. (16) may be carried out, with the aid of a computer, for any given value of W. The s_i dependence of all the resolvents is then assembled by means of Eqs. (16).

Some highlights of the solution should be noted. The I^0's do not depend on the total isospin T while the I's do. The I^0's are diagonal but the I's are not, indicating isobar isospin coupling. The origin of both the T dependence and the off-diagnal coupling is of course attributed to the isospin crossing matrices D, C, and $\bar{C}$. Neither of these features is recognized in the calculations of Ref. 5, where the full solution of the integral equations is not addressed. Some elements of the I's vanish identically because of isospin conservation. The crossing matrix C ($\bar{C}$) has one vanishing row (column) reflecting this. This feature may be traced through the solution to secure the vanishing of ϵ_2 effects for $T=\frac{1}{2}$, and of ϵ_0 effects for $T=\frac{3}{2}$.

Graphs of the s_i dependence of the real and imaginary parts of the nonvanishing resolvents are shown in Figs. 2–5 for total energy $W=1.3$, 1.4, and 1.5 GeV. The results are plotted for subenergies between threshold and the physical upper limit[10] $\hat{z}_i$.

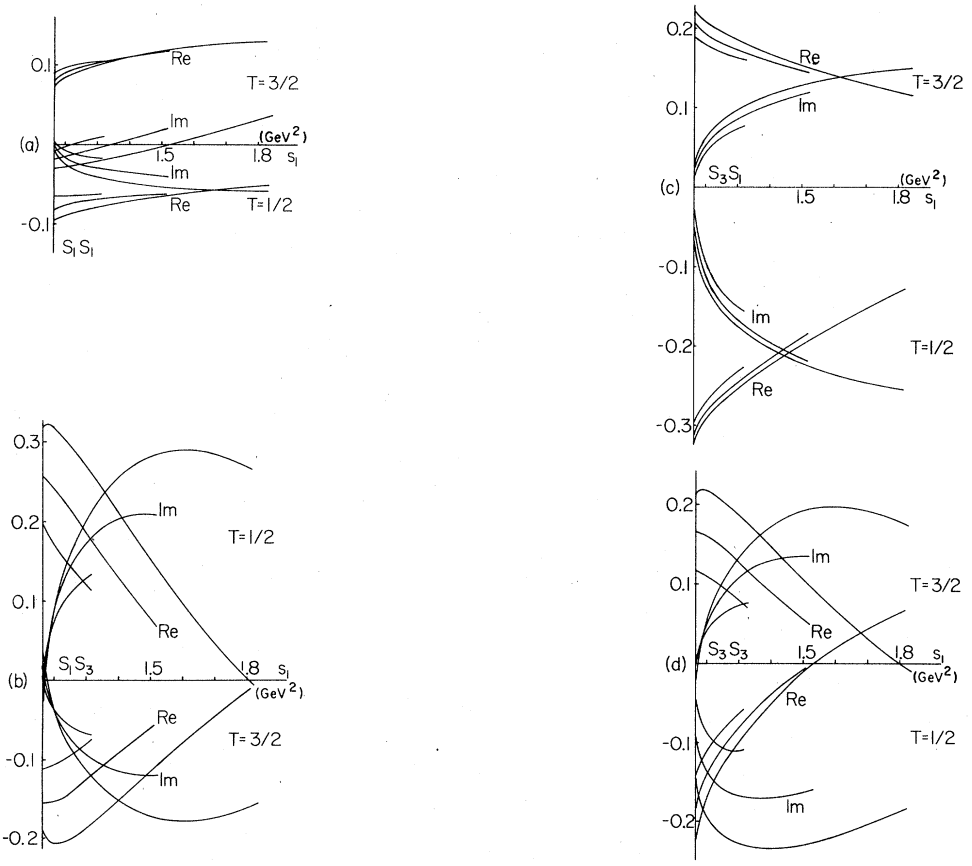

FIG. 2. Real and imaginary parts of the resolvents $I_{ss}(s_1)$ for $W=1.3$, 1.4, and 1.5 GeV: (a) $I_{S_1S_1}$, (b) $I_{S_1S_3}$, (c) $I_{S_3S_1}$, (d) $I_{S_3S_3}$. Results for total isospin $T=\frac{1}{2}$ and $\frac{3}{2}$ are displayed. The behavior is shown over the physical subenergy region $(M+\mu)^2 \leq s_1 \leq (W-\mu)^2$; therefore, the curves are distinguished by their increasing length as W increases.

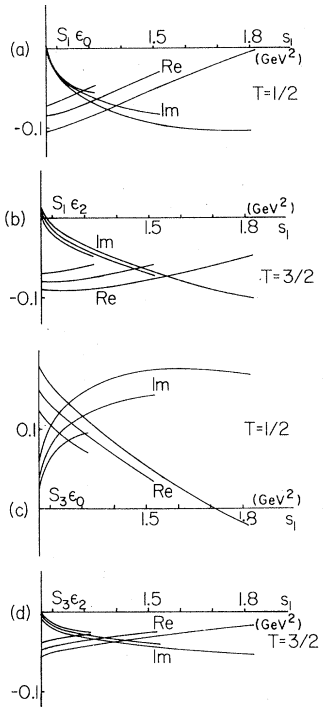

FIG. 3. Resolvents $I_{S\epsilon}(s_1)$ for $W=1.3$, 1.4, and 1.5 GeV: (a) $I_{S_1\epsilon_0}$, (b) $I_{S_1\epsilon_2}$, (c) $I_{S_3\epsilon_0}$, (d) $I_{S_3\epsilon_2}$.

CONCLUSION

The figures drawn in the previous section show the subenergy dependence of final-state interactions in the $N\pi\pi$ system. These have been restricted for purposes of illustrating the method to the s-wave isobar channels $S\pi$ and $N\epsilon$ in S states. The ingredients of the s_i dependence have been organized according to the contributing resolvents. Even for this somewhat truncated system they represent a rather bewildering array of effects. Fortunately there is a single unifying interpretation by which all of them may be comprehended.

The immediate qualitative observation to make is that in almost all the graphs the real and imaginary parts exhibit a crossover. This feature was shown in Ref. 5 to suggest a scattering length parametrization for s-wave effects. The same idea works rather well here, but with a slight modification in the way the scattering length is introduced. In Ref. 5 the form $(1-iqa)^{-1}$ was used, where

$$q^2=\begin{cases}[s_1-(M+\mu)^2][s_1-(M-\mu)^2]/4s_1 \\ \qquad\qquad \text{for } S\pi \text{ channels} \\ (s_3-4\mu^2)/4 \text{ for } N\epsilon \text{ channels.}\end{cases} \tag{26}$$

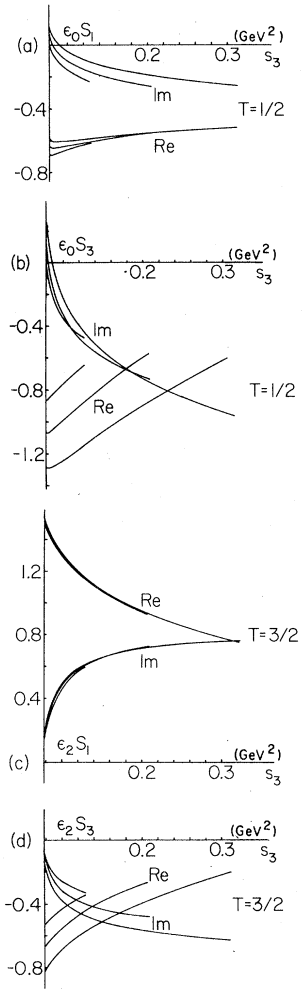

FIG. 4. Resolvents $I_{\epsilon S}(s_3)$ for $W=1.3$, 1.4, and 1.5 GeV: (a) $I_{\epsilon_0 S_1}$, (b) $I_{\epsilon_0 S_3}$, (c) $I_{\epsilon_2 S_1}$, (d) $I_{\epsilon_2 S_3}$. The upper limit of the physical subenergy region increases with W as $4\mu^2 \leq s_3 \leq (W-M)^2$. Only ϵ_0 occurs for $T=\frac{1}{2}$, and only ϵ_2 occurs for $T=\frac{3}{2}$.

The scattering-length parameter a was taken to be real and fit to the calculated results. Of course the parameter must depend on W. Because of this it seems more appropriate physically to let the parameter be complex, the idea being that any quantity having W dependence ought to have both real and imaginary parts reflecting the existence of the unitarity cut in W. Accordingly, each of the resolvents plotted in Figs. 2–5 is assumed to have the form

$$I(s_i)=\eta(1+q\gamma)^{-1} \tag{27}$$

for physical values of s_i. The parameters η and γ are W dependent and complex.

Fitting Eq. (27) to the results in Figs. 2–5 is a simple enough matter as long as one admits the

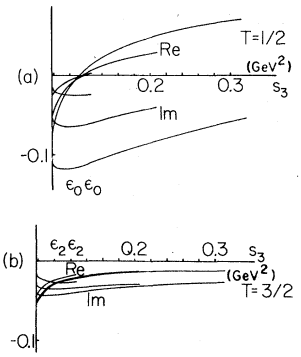

FIG. 5. Resolvents $I_{\epsilon\epsilon}(s_3)$ for $W=1.3$, 1.4, and 1.5 GeV: (a) $I_{\epsilon_0\epsilon_0}$, (b) $I_{\epsilon_2\epsilon_2}$.

limitations of a two-parameter form. In some cases the parametrization affords a very accurate fit, while in others only a rough fit is possible. In all cases the elementary two-parameter expression is representative of the size and shape of the results, and accomplishes the purpose of comprehending a large number of calculations in penetrable physical terms. In Table I the values obtained for η and r are listed for all the cases plotted. Clearly, the multiplicative factors, the η's, have little meaning since in practice they are effectively absorbed by the unknown fitting parameters, the c's, in Eqs. (8) and (9). The r's, on the other hand, play the role of the aforementioned *complex* scattering lengths; therefore, their physical significance is noteworthy.

In fact, a remarkable regularity is to be noticed in Table I concerning the r's: In every case Imr is *negative*. The function $(1+qr)^{-1}$ has a phase δ, given by

$$\tan\delta = -q\,\mathrm{Im}r(1+q\,\mathrm{Re}r)^{-1} \tag{28}$$

which grows from threshold positively for Im$r<0$. This effect in the dependence on subenergy is comparable to that of a wave function in an *attractive* potential; here it arises from the coupling to the other isobar channels as required by subenergy unitarity. Thus the interplay among the coupled isobar channels is seen to "pull in the wave function" in all these cases. It would appear that a general principle is in evidence here.

Although the system of channels considered here is adequate to demonstrate a method of solving the coupled equations, the practical significance of the calculations is somewhat more doubtful. If there were a regime of total energy W where the only waves needed phenomenologically were these, then Eqs. (8) and (9) would serve to unitarize the standard sort of fit without the need for any additional fitting parameters. Unfortunately, even near threshold the $\Delta\pi$ channels can not be neglected. A recent study around 1.3 GeV by Arndt *et al.*[11] allows for $\Delta\pi$ although it concludes that $\frac{1}{2}^+$ $N\epsilon$ production is more important. The $\Delta\pi$ channels could be readily included in this analysis and the basis function method of solution could be applied at the expense of making the number of computed quantities a great deal larger. It should finally be noted that the $S\pi$ isobar channels have never appeared in the standard fits[7]; it would seem that especially near threshold these ought to be included in the future.

ACKNOWLEDGMENT

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[6]Both f_S and f_ϵ are dimensionless. A factor $4M$ has been extracted from f_3 of Ref. 4 to achieve this: $f_\epsilon = 4Mf_3$, where M is the nucleon mass.
[7]E.g., D. J. Herndon *et al.*, Phys. Rev. D 11, 3183 (1975).
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[9]The functions in Eqs. (12) have been used in Ref. 5 to express the ξ's analytically; there, the notation was J and J_3, related to g and g_3 by $g=J-J_{\text{threshold}}$ and $g_3=J_3$. The forms given in (12) are for z above threshold; they must be continued analytically for other z. Note that g and g_3 have only the threshold branch point and vanish at threshold; μ is the pion mass.
[10]Recall that $\hat{z}_1=(W-\mu)^2$ and $\hat{z}_3=(W-M)^2$.
[11]R. A. Arndt *et al.*, Phys. Rev. D 20, 651 (1979).