Two-Reggeon cut in the ordered phase of Reggeon field theory

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The ordered phase of Reggeon field theory is analyzed in the framework of the Reggeon quantum spin model with three- and four-point couplings. We work at a particular value of the bare-intercept gap, $\Delta_0 = \Delta_{0M}$, for which the spin model is exactly solvable in one transverse dimension. We include the higher single-site states by developing a perturbative expansion for this spin model. The expansion is shown to be characterized by a small dimensionless parameter $\epsilon = 16\alpha_0' \Delta_{0M}{}^3/r_0{}^4$. At each stage of calculation we are able to relate the spin model to the continuum Reggeon field theory by letting the lattice spacing go to zero. The two-point function is then calculated to order ϵ. We find that the necessary condition for satisfying Reggeon unitarity, of Regge-Mandelstam singularities appearing in the complex angular momentum plane, is satisfied in the form of the two-Reggeon cut.

I. INTRODUCTION

Reggeon field theory[1] (RFT) has been shown to exhibit the behavior associated with a second-order phase transition.[2] The theory has been extensively studied at the critical point[3,4] Δ_{0c} and in the disordered phase $\Delta_0 > \Delta_{0c}$ by use of the renormalization group. All known t-channel and s-channel constraints are satisfied for these cases.[5] A great deal of progress in understanding the ordered phase $\Delta_0 < \Delta_{0c}$ has followed from the introduction of analog spin models, believed to be in the same universality class of the continuum RFT.[6-8] These models maintain a continuous rapidity variable while introducing a lattice in the transverse impact-parameter space. The intersite interaction is proportional to the slope of the Pomeron trajectory α_0' and is treated as a perturbation on the single-site solution since α_0' is believed to be a small parameter.[9] For large values of rapidity, the single-site dynamics is dominated by the two lowest-lying states. This motivates the spin-model approximation of truncating the Hilbert space to include only these two lowest states. While much knowledge of the nature of the vacuum and obeyance of s-channel constraints has been demonstrated with these models, the notable absence of Regge-Mandelstam singularities has created doubts as to the validity of the ordered phase of RFT. This follows because RFT is explicitly constructed to satisfy Reggeon unitarity which requires the appearance of Regge-Mandelstam cut singularities in the complex angular momentum plane.

In this paper we will develop a perturbative formalism for including the higher single-site states. We will utilize the spin model presented by Bronzan and Sugar,[8] in which a four-Pomeron interaction is included in addition to only the three-Pomeron interaction of the earlier models. This is not supposed to effect the high-energy behavior of the scattering amplitude but does provide a number of important advantages. This Reggeon quantum spin model does not break down for fixed Δ_0 and r_0 as the lattice spacing goes to zero. This allows one to take the continuum limit at any point in the calculation and is necessary for checking whether Reggeon unitarity is satisfied. We are also given the option of choosing a value of Δ_0, termed the "magic value" Δ_{0M}, which corresponds to the ordered phase of RFT and allows a similarity transformation to be applied that converts the non-Hermitian Hamiltonian into a Hermitian Hamiltonian.

We will determine the leading correction to the two-level spin model induced by the higher single-site states. At each stage of the calculation, the lattice spacing can be taken to zero, with a finite result. The perturbative expansion will be shown to be characterized by a small dimensionless parameter $\epsilon = 16\alpha_0'\Delta_{0M}{}^3/r_0{}^4$. This parameter indicates that our treatment corresponds to a strong-coupling expansion, and that the Reggeon quantum spin model is the strong-coupling limit of RFT. We then evaluate the two-point function (the propagator) in RFT to the first two orders in ϵ. In lowest order we have the two-level Reggeon quantum spin model result, which has both poles and a cut in the angular momentum plane. In the next order in ϵ we expect these moving singularities to produce further cuts in the angular momentum plane. What we find is a minor shift in the trajectories found in the two-level Reggeon quantum spin model and the production of a new singularity that can be identified with the two-Reggeon cut. This is extremely important because it shows that it is possible for RFT to satisfy Reggeon unitarity in the ordered phase. To check Reggeon unitarity quantitatively, we would have to calculate the appropriate Reggeon amplitudes. This is a formidable task and is beyond the scope of this paper.

Our paper is organized as follows. In Sec. II we introduce the Hamiltonian formulation of RFT and show how one may transform to a Hermitian version when Δ_0 is at its magic value. We then introduce a lattice in impact-parameter space and the spin-model approximation. The spin-model Hamiltonian is then diagonalized on the subspace of states which contribute to the propagator. These states are termed the "box states." We then expand the Hilbert space to include the next higher single-site state. In Sec. III we develop the perturbation theory for the Reggeon quantum spin model. The box states appear as the correct linear combinations of degenerate perturbation theory. This formalism differs from the standard formalism, in that the box states are characterized by a pair of continuous variables and their energies are degenerate. In Sec. IV we determine the propagator to order ϵ and discuss its implications. A brief summary and our conclusions are presented in Sec. V.

II. THE REGGEON QUANTUM SPIN MODEL

The Reggeon quantum spin model[9] is derived from the Hamiltonian formulation of RFT. We work in the Schrödinger picture so that the field operators are independent of rapidity. The Hamiltonian is given by

$$H=\int d^Dx\{\alpha_0'\vec{\nabla}\psi^\dagger(x)\cdot\vec{\nabla}\psi(x)+\Delta_0\psi^\dagger(x)\psi(x)+\tfrac{1}{2}ir_0[\psi^\dagger(x)^2\psi(x)+\psi^\dagger(x)\psi(x)^2]+\tfrac{1}{4}\lambda_0\psi^\dagger(x)^2\psi(x)^2\}\,, \tag{2.1}$$

where $\psi^\dagger(x)$ and $\psi(x)$ are the creation and annihilation operators of the bare Pomeron and satisfy the commutation relation

$$[\psi(x),\psi^\dagger(x')]=\delta^D(\vec{x}-\vec{x}')\,. \tag{2.2}$$

As mentioned in the Introduction, owing to the non-Hermiticity of H, $\psi^\dagger(x)$ may be taken to be the adjoint of $\psi(x)$ at only one particular value of rapidity. We choose this to be the value of y at which the Heisenberg and Schrödinger pictures coincide.

The bare vacuum, defined by $\psi(x)|0\rangle=0$, can immediately be seen to be a right and left eigenstate of H with eigenvalue zero. If we choose a particular value of the bare intercept, the magic value, as

$$\Delta_0=\Delta_{0M}=-r_0^2/\lambda_0\,, \tag{2.3}$$

we can see by inspection that the states

$$|\phi\rangle=\exp\left[\frac{2i\Delta_{0M}}{r_0}\int d^Dx\,\psi^\dagger(x)\right]|0\rangle \tag{2.4}$$

and

$$\langle\tilde{\phi}|=\langle 0|\exp\left[\frac{2i\Delta_{0M}}{r_0}\int d^Dx\,\psi(x)\right] \tag{2.5}$$

are right and left eigenstates with eigenvalue zero. Since the ground state is then degenerate, the magic value of Δ_0 corresponds to the ordered phase of RFT. It is more convenient to define the states

$$|1\rangle=iN(|\phi\rangle-|0\rangle) \tag{2.6}$$

and

$$\langle\tilde{1}|=iN(\langle\tilde{\phi}|-\langle 0|) \tag{2.7}$$

as our second right and left ground states so that they satisfy the orthonormality relation

$$\langle\tilde{\alpha}|\beta\rangle=\delta_{\alpha\beta},\quad \alpha=0,1,\quad \beta=0,1\,, \tag{2.8}$$

where

$$N=\left[1-\exp\left(-\frac{4\Delta_{0M}^2}{r_0^2}V\right)\right]^{-1/2} \tag{2.9}$$

and $V=\int d^Dx$ is the total volume of impact-parameter space.

Next we make a similarity transformation on H by means of the operator

$$S=\exp\left[-\frac{2i\Delta_{0M}}{r_0}\int d^Dx\,\psi(x)\right]. \tag{2.10}$$

Keeping $\Delta_0=\Delta_{0M}$ results in transforming H into a positive semidefinite Hermitian operator H':

$$H'=S^{-1}HS=\int d^Dx\left[\alpha_0'\vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi-\frac{r_0^2}{4\Delta_{0M}}\psi^\dagger\times\left(\psi^\dagger+\frac{2i\Delta_{0M}}{r_0}\right)\left(\psi-\frac{2i\Delta_{0M}}{r_0}\right)\psi\right]. \tag{2.11}$$

Since H and H' are related by a similarity transformation they must have the same spectrum. We also know that there are no states with energy lower than zero since H' is positive semidefinite. This illustrates an advantage of including the four-Pomeron coupling in that we have the option of working with a Hermitian version of the theory, which is the course we will follow in this paper.

We now introduce a cubic lattice of spacing a, in impact-parameter space. The creation and annihilation operators at lattice site i are given by

$$\psi_i^\dagger=a^{-D/2}\int_{\Delta V_i}d^Dx\,\psi^\dagger(x) \tag{2.12}$$

and

$$\psi_i=a^{-D/2}\int_{\Delta V_i}d^Dx\,\psi(x)\,, \tag{2.13}$$

so that they satisfy the commutation relation

$$[\psi_i,\psi_j^\dagger]=\delta_{ij}\,. \tag{2.14}$$

The Hamiltonian H' of (2.11) then becomes

$$H' = \sum_j H'_{0j} + H'_g, \tag{2.15}$$

where the single-site Hamiltonian is

$$H'_{0j} = -\frac{r_0^2 a^{-D}}{4\Delta_{0M}} \psi_j^\dagger \left(\psi_j^\dagger + \frac{2i\Delta_{0M}}{r_0} a^{D/2}\right)\left(\psi_j - \frac{2i\Delta_{0M}}{r_0} a^{D/2}\right)\psi_j \tag{2.16}$$

or

$$H'_{0j} = -\Delta_{0M}\psi_j^\dagger\psi_j + \tfrac{1}{2} i r_0 a^{-D/2}(\psi_j^{\dagger 2}\psi_j - \psi_j^\dagger\psi_j^2) + \tfrac{1}{4}\lambda_0 a^{-D}\psi_j^{\dagger 2}\psi_j^2 \tag{2.17}$$

and the gradient term becomes

$$H'_g = \frac{\alpha'_0}{a^2}\sum_{\langle ij\rangle} (\psi_i^\dagger - \psi_j^\dagger)(\psi_i - \psi_j), \tag{2.18}$$

where $\langle ij\rangle$ indicates a sum over nearest neighbors. The slope of the Pomeron trajectory α'_0 is believed to be small[9] so we treat the gradient term as a perturbation on the single-site term. A standard perturbative expansion in the lattice spacing shows that the single-site eigenvalues are

$$E_n = \tfrac{1}{4}\lambda_0 a^{-D} n(n-1) + O(a^{-D+2}) \tag{2.19}$$

which show that only the lowest eigenvalues remain finite and degenerate as the lattice spacing goes to zero. This is the motivation of the spin-model approximation of truncating the Hilbert space to the two lowest states at each site. Equation (2.19) illustrates another advantage of including the four-Pomeron coupling. In the model with only a three-Pomeron coupling the two lowest states are well separated from the rest only when $(\Delta_0^2/r_0^2)a^D \gg 1$,[6,7] so the zero-lattice-spacing limit is not allowed.

Next we must determine the two lowest single-site eigenstates. By inspection we see that the ordinary vacuum $|0\rangle_j$ and

$$\exp\left(\frac{2i\Delta_{0M}}{r_0} a^{D/2}\psi_j^\dagger\right)|0\rangle_j$$

have eigenvalue zero. However, to satisfy the biorthonormality relation

$$_i\langle \alpha | \beta\rangle_j = \delta_{ij}\delta_{\alpha\beta}, \tag{2.20}$$

we choose

$$|\bar{1}\rangle_j = N'_0\left[\exp\left(\frac{2i\Delta_{0M}}{r_0} a^{D/2}\psi_j^\dagger\right) - 1\right]|0\rangle_j, \tag{2.21}$$

where the normalization constant is

$$N'_0 = \left[\exp\left(\frac{4\Delta_{0M}^2}{r_0^2} a^D\right) - 1\right]^{-1/2}. \tag{2.22}$$

If we then define

$$\psi_j \equiv \frac{2i\Delta_{0M}}{r_0} a^{D/2}\sigma_j, \tag{2.23}$$

then the Hamiltonian H' on this subspace of states reduces to

$$H'_g = \alpha'_0 \frac{4\Delta_{0M}^2}{r_0^2} a^{D/2}\sum_{\langle ij\rangle} (\sigma_i^\dagger - \sigma_j^\dagger)(\sigma_i - \sigma_j), \tag{2.24}$$

where

$$\sigma_i = \begin{bmatrix} 0 & N'_0 \\ 0 & 1 \end{bmatrix}. \tag{2.25}$$

By treating H'_g perturbatively at $\Delta_0 = \Delta_{0M}$, we have been led to a problem in degenerate perturbation theory because $|0\rangle_j$ and $|1\rangle_j$ are degenerate. To proceed we specialize to $D = 1$. This is necessary since we can only solve the spin model in one dimension and at $\Delta_0 = \Delta_{0M}$ using the present simple techniques. These conditions are certainly not general ones, or physical ones. However, these conditions do allow us to proceed quite far in the analysis of the ordered phase of RFT. We proceed by making a change of basis via the transformation matrices

$$U = \begin{bmatrix} 1 & N'_0(1+N_0'^2)^{-1/2} \\ 0 & (1+N_0'^2)^{-1/2} \end{bmatrix} \tag{2.26}$$

and

$$U^{-1} = \begin{bmatrix} 1 & -N'_0 \\ 0 & (1+N_0'^2)^{1/2} \end{bmatrix}, \tag{2.27}$$

which satisfy

$$UU^{-1} = U^{-1}U = 1, \tag{2.28}$$

so that we will be able to diagonalize the Hamiltonian on our truncated Hibert space. The single-site eigenstates are now

$$\begin{aligned} |\tilde{0}\rangle_i &= |0\rangle_i, \\ |\tilde{\phi}\rangle_i &= N'_0(1+N_0'^2)^{-1/2}|0\rangle_i + (1+N_0'^2)^{-1/2}|\bar{1}\rangle_i \equiv |\phi\rangle_i, \\ {}_i\langle\tilde{0}| &= {}_i\langle 0| - N'_0\, {}_i\langle\bar{1}|, \\ {}_i\langle\tilde{\phi}| &= (1+N_0'^2)^{1/2}\, {}_i\langle\bar{1}|, \end{aligned} \tag{2.29}$$

and we see that the right and left eigenstates are no longer adjoints of each other. The creation and annihilation operators in H'_g become

$$\tilde{\sigma}_j = U^{-1}\sigma_j U = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.30}$$

and

$$\tilde{\sigma}_j^\dagger = U^{-1}\sigma_j^\dagger U = \begin{bmatrix} -N_0'^2 & -N'_0(1+N_0'^2)^{1/2} \\ N'_0(1+N_0'^2)^{1/2} & 1+N_0'^2 \end{bmatrix}. \tag{2.31}$$

We now introduce box states which are defined by

$$|l, r\rangle = |\overline{l, r}\rangle - \left[\frac{N_0'}{(1+N_0'^2)^{1/2}}\right]^{r-l}|0\rangle, \qquad (2.32)$$

where

$$|\overline{l, r}\rangle = \prod_{i=-\infty}^{l-1}|0\rangle_i \prod_{j=l}^{r-1}|\phi\rangle_j \prod_{k=r}^{\infty}|0\rangle_k \qquad (2.33)$$

and

$$|0\rangle = \prod_{i=-\infty}^{\infty}|0\rangle_i, \qquad (2.34)$$

where l (r) labels the left (right) end of the box. In the first term, the sites from $-\infty$ to $l-1$ and r to ∞ are in the state $|0\rangle_i$. The sites from l to $r-1$ are occupied by the state $|\phi\rangle_i$. Box states are seen to be relevant to the calculation of the propagator because $\tilde{\sigma}_l^+$ creates a box state from the bare vacuum:

$$\tilde{\sigma}_l^+|0\rangle = N_0'(1+N_0'^2)^{1/2}|l, l+1\rangle. \qquad (2.35)$$

In addition, the Hamiltonian H_g' maps box states into box states:

$$H_g'|l, r\rangle = \frac{\alpha_0' 4\Delta_{0M}^2}{a r_0^2}\{(4N_0'^2+2)|l, r\rangle - N_0'(1+N_0'^2)^{1/2} \times[|l-1, r\rangle + |l+1, r\rangle + |l, r-1\rangle + |l, r+1\rangle]\}. \qquad (2.36)$$

We may diagonalize H_g' on the subspace of box states by forming the following linear combinations:

$$|\chi_B(\tau, \theta)\rangle = \sum_{l=-\infty}^{\infty}\sum_{r=l}^{\infty} e^{i\theta(l+r)/2}\sin\tau(r-l)|l, r\rangle, \quad 0 \le \theta < 2\pi, \quad 0 \le \tau < \pi. \qquad (2.37)$$

These states form a complete set since we may write

$$|l, r\rangle = \int_0^{2\pi}\frac{d\theta}{\pi}\int_0^{\pi}\frac{d\tau}{\pi} e^{-i\theta(l+r)/2}\sin\tau(r-l)|\chi_B(\tau, \theta)\rangle. \qquad (2.38)$$

Letting H_g' operate on the linear combination of box states of (2.37), which we will hereafter also call box states, we find

$$H_g'|\chi_B(\tau, \theta)\rangle = (\bar{a} - 4\bar{b}\cos\tfrac{1}{2}\theta\cos\tau)|\chi_B(\tau, \theta)\rangle, \qquad (2.39)$$

where

$$\bar{a} = \frac{8\alpha_0'\Delta_{0M}^2}{a r_0^2}(2N_0'^2+1), \quad \bar{b} = \frac{4\alpha_0'\Delta_{0M}^2}{a r_0^2}N_0'(1+N_0'^2)^{1/2}. \qquad (2.40)$$

The minimum energy for the box states occurs when $\cos\frac{1}{2}\theta\cos\tau = 1$ and in the continuum limit is

$$\delta_0 \equiv \lim_{a\to 0}(\bar{a} - 4\bar{b}) = \frac{4\alpha_0'\Delta_{0M}^4}{r_0^4}. \qquad (2.41)$$

The normalization of the box states is calculated directly from the definition given by (2.37) and is

$$\langle\chi_B(\tau', \theta')|\chi_B(\tau, \theta)\rangle = \pi^2 f(\tau, \theta)\delta(\theta-\theta')\delta(\tau-\tau'), \qquad (2.42)$$

where

$$f(\tau, \theta) = \left[\frac{1}{1-e^{A-i\theta/2-i\tau}} + \frac{1}{1-e^{A+i\theta/2+i\tau}} - 1\right] \times\left[\frac{1}{1-e^{A+i\theta/2-i\tau}} + \frac{1}{1-e^{A-i\theta/2+i\tau}} - 1\right] \qquad (2.43)$$

and

$$A \equiv -\frac{2\Delta_{0M}^2}{r_0^2}a. \qquad (2.44)$$

We see that at $\Delta_0 = \Delta_{0M}$ the Reggeon quantum spin model has emerged naturally as the way to find the correct linear combinations of the degenerate perturbation theory for H_g'. In addition to the box states, Bronzan and Sugar[9] have also identified linear combinations of "kink" states that also diagonalize H_g'. (These states consist of $|0\rangle_i$ states to the left of lattice site i and $|\phi\rangle_i$ to the right of site i.) These states do not enter into the calculation of the propagator so we will not discuss them further.

We will now expand the Hilbert space to include the higher single-site states that we have truncated to this point. The higher single-site states for the Hamiltonian H_{0j}' of (2.17) can be determined by developing a perturbation expansion in the lattice spacing a. The first state above the two degenerate ground states is given explicitly as

$$\begin{aligned}|\bar{2}\rangle_j = &-i\left(1 - \frac{5}{2}\frac{\Delta_{0M}^2}{r_0^2}a + \frac{283}{200}\frac{\Delta_{0M}^4}{r_0^4}a^2\right)|2\rangle_j \\ &+\left(3^{1/2}\frac{\Delta_{0M}}{r_0}a^{1/2} - \tfrac{19}{10}\times 3^{1/2}\frac{\Delta_{0M}^3}{r_0^3}a^{3/2}\right)|3\rangle_j \\ &+\left(2^{1/2}\frac{\Delta_{0M}}{r_0}a^{1/2} - 2^{-1/2}\frac{\Delta_{0M}^3}{r_0^3}a^{3/2}\right)|1\rangle_j \\ &+\left(\tfrac{6}{5}\times 3^{1/2}\frac{\Delta_{0M}^2}{r_0^2}a - \tfrac{53}{75}\times 3^{1/2}\frac{\Delta_{0M}^4}{r_0^4}a^2\right)|4\rangle_j \\ &-\tfrac{8}{15}\times 15^{1/2}\frac{\Delta_{0M}^3}{r_0^3}a^{3/2}|5\rangle_j - i\tfrac{4}{7}\times 10^{1/2}\frac{\Delta_{0M}^4}{r_0^4}a^2|6\rangle_j.\end{aligned} \qquad (2.45)$$

The states $|M\rangle_j$ are harmonic oscillator eigenstates.

Having determined the higher single-site states, we may now determine the matrix representations of the creation an annihilation operators to order a^2 in the complete Hilbert space. However, to find the leading order effect, it is necessary to retain only the first higher single-site state $|\bar{2}\rangle_j$.

(We will explicitly demonstrate this fact in Sec. III after developing our perturbative formalism for including the higher single-site states.) Therefore, we truncate our Hilbert space to the lowest three states and we find

$$\sigma_i=\begin{pmatrix}0 & N_0' & \sigma_{02}\\ 0 & 1 & \sigma_{12}\\ 0 & 0 & \frac{1}{2}\end{pmatrix}, \tag{2.46}$$

where

$$\begin{aligned}\sigma_{02}&=-i2^{-1/2}+i2^{-3/2}\frac{\Delta_{0M}^2}{r_0^2}a\,,\\ \sigma_{12}&=-i2^{-1/2}\frac{r_0}{\Delta_{0M}}a^{1/2}+i2^{-3/2}\frac{\Delta_{0M}}{r_0}a^{1/2}\\ &\quad -i\frac{109}{1200}\times 2^{1/2}\frac{\Delta_{0M}^3}{r_0^3}a^{3/2}\,.\end{aligned} \tag{2.47}$$

We must transform to a nonadjoint basis to calculate the effect of the Hamiltonian on the subspace of box states. The simple generalization of the transformation matrices to

$$U=\begin{pmatrix}1 & N_0'(1+N_0'^2)^{-1/2} & 0\\ 0 & (1+N_0'^2)^{-1/2} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.48}$$

and

$$U^{-1}=\begin{pmatrix}1 & -N_0' & 0\\ 0 & (1+N_0'^2)^{1/2} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.49}$$

allow the creation and annihilation operators to be represented by

$$\tilde\sigma_j=U^{-1}\sigma_jU=\begin{pmatrix}0 & 0 & \tilde\sigma_{02}\\ 0 & 1 & \tilde\sigma_{12}\\ 0 & 0 & \frac{1}{2}\end{pmatrix} \tag{2.50}$$

and

$$\tilde\sigma_j^+=U^{-1}\sigma_j^+U=\begin{pmatrix}-N_0'^2 & -N_0'(1+N_0'^2)^{1/2} & 0\\ N_0'(1+N_0'^2)^{1/2} & 1+N_0'^2 & 0\\ \tilde\sigma_{20}^+ & \tilde\sigma_{21}^+ & \frac{1}{2}\end{pmatrix}, \tag{2.51}$$

where

$$\begin{aligned}\tilde\sigma_{02}&=\sigma_{02}-N_0'\sigma_{12}\\ &=-i2^{-3/2}\frac{r_0^2}{\Delta_{0M}^2}a-i2^{-5/2}+i\frac{91}{600}\times 2^{-3/2}\frac{\Delta_{0M}^2}{r_0^2}a\,,\\ \tilde\sigma_{12}&=\sigma_{12}(1+N_0'^2)^{1/2}=\tilde\sigma_{02}^*\,,\\ \tilde\sigma_{20}^+&=\sigma_{02}^*=i2^{-1/2}-i2^{-3/2}\frac{\Delta_{0M}^2}{r_0^2}a\,,\\ \tilde\sigma_{21}^+&=(\sigma_{02}^*N_0'+\sigma_{12}^*)/(1+N_0'^2)^{1/2}=\tilde\sigma_{20}^{+*}\,.\end{aligned} \tag{2.52}$$

With these representations, we find that the Hamiltonian of (2.24) when operating on a box state produces the following result:

$$\begin{aligned}H_g'|\chi_B(\tau,\theta)\rangle&=[\bar a-4\bar b\cos\tfrac{1}{2}\theta\cos\tau]|\chi_B(\tau,\theta)\rangle\\ &\quad -i2^{3/2}\frac{\alpha_0'\Delta_{0M}^2}{ar_0^2}[1+O(a)]\sum_{l=-\infty}^{\infty}\sum_{r=l}^{\infty}e^{i\theta(l+r)/2}\sin\tau(r-l)\\ &\quad\times\{|l,r,-\rangle+|l,r,+\rangle+|l+1,r,-\rangle+|l,r-1,+\rangle\}.\end{aligned} \tag{2.53}$$

The higher-order term, represented by $O(a)$, must be neglected since we would get contributions of competing orders from the next two higher single-site states had we not truncated the Hilbert space to the three lowest states. The state $|l,r,-\rangle$ in (2.53) corresponds to the $|\overline{l,r}\rangle$ state given by (2.33) but with a $|\bar 2\rangle_j$ single-site state at the $l-1$ site. Similarly, the $|l,r,+\rangle$ state has a $|\bar 2\rangle_j$ single-site state at the r site. We also wish to point out that the result of (2.53) does not depend upon the order of applying the transformation and truncating the Hilbert space. Equation (2.53) will be used to determine corrections to the box states in a perturbative formalism which we develop in Sec. III.

III. PERTURBATION THEORY FOR THE REGGEON QUANTUM SPIN MODEL

In this chapter we will develop a perturbative formalism to find the leading correction to the box states, which we treat as a first approximation. This formalism differs from the standard perturbation theory we have applied previously in that the states are characterized by the continuous variables τ and θ. Therefore we have a problem in degenerate perturbation theory in the continuum sector. Our problem is defined by

$$H'|\chi(\tau,\theta)\rangle=E(\tau,\theta)|\chi(\tau,\theta)\rangle\,, \tag{3.1}$$

where

$$H' = H_0' + \lambda H_g' \tag{3.2}$$

as given by (2.15). λ has been included for bookkeeping purposes. We begin by expanding $|\chi(\tau,\theta)\rangle\!\rangle$ and $E(\tau,\theta)$ in a power series in λ.

$$|\chi(\tau,\theta)\rangle\!\rangle = |\chi_B(\tau,\theta)\rangle\!\rangle + \lambda|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + \lambda^2|\chi^{(2)}(\tau,\theta)\rangle\!\rangle + \cdots , \tag{3.3}$$

$$E(\tau,\theta) = E^{(0)}(\tau,\theta) + \lambda E^{(1)}(\tau,\theta) + \lambda^2 E^{(2)}(\tau,\theta) + \cdots . \tag{3.4}$$

Recall from (2.42) that the box states satisfy

$$\langle\!\langle\chi_B(\tau',\theta')|\chi_B(\tau,\theta)\rangle\!\rangle = \pi^2 f(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau'). \tag{3.5}$$

We substitute these expressions into (3.1) and equate equal powers of λ which then yield the following set of equations:

$$H_0'|\chi_B\rangle = E^{(0)}|\chi_B\rangle , \tag{3.6a}$$

$$H_0'|\chi^{(1)}\rangle + H_g'|\chi_B\rangle = E^{(0)}|\chi^{(1)}\rangle + E^{(1)}|\chi_B\rangle , \tag{3.6b}$$

$$H_0'|\chi^{(2)}\rangle + H_g'|\chi^{(1)}\rangle = E^{(0)}|\chi^{(2)}\rangle + E^{(1)}|\chi^{(1)}\rangle + E^{(2)}|\chi_B\rangle , \tag{3.6c}$$

etc. We have temporarily dropped the τ, θ labeling to simplify the notation. We see immediately that $E^{(0)}=0$.

Next, we multiply (3.6b) and (3.6c) on the left by $\langle\chi_B(\tau',\theta')|$ and we find

$$E^{(1)}(\tau,\theta) = \frac{\langle\!\langle\chi_B(\tau',\theta')|H_g'|\chi_B(\tau,\theta)\rangle\!\rangle}{\langle\!\langle\chi_B(\tau',\theta')|\chi_B(\tau,\theta)\rangle\!\rangle} \tag{3.7}$$

and

$$\langle\!\langle\chi_B(\tau',\theta')|H_g'|\chi^{(1)}(\tau,\theta)\rangle\!\rangle = E^{(1)}(\tau,\theta)\langle\!\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + E^{(2)}(\tau,\theta)\langle\!\langle\chi_B(\tau',\theta')|\chi_B(\tau,\theta)\rangle\!\rangle . \tag{3.8}$$

We may express $|\chi^{(1)}(\tau,\theta)\rangle\!\rangle$ as an expansion over a complete set of states

$$|\chi^{(1)}(\tau,\theta)\rangle\!\rangle = \sum_n |\chi_n\rangle\langle\chi_n|\chi^{(1)}(\tau,\theta)\rangle\!\rangle , \tag{3.9}$$

where the states $|\chi_n\rangle$ satisfy

$$H_0'|\chi_n\rangle = E_n^{(0)}|\chi_n\rangle \tag{3.10}$$

and are composed of an infinite product of the single-site states, $|\overline{n}\rangle_i$. The projection operator onto this space of states may be split up as follows:

$$\mathcal{P} = \sum_n |\chi_n\rangle\langle\chi_n| = \int_0^{2\pi}\frac{d\theta''}{\pi}\int_0^{\pi}\frac{d\tau''}{\pi}\,\frac{|\chi_B(\tau'',\theta'')\rangle\!\rangle\langle\!\langle\chi_B(\tau'',\theta'')|}{f(\tau,\theta)} + {\sum_m}' |\chi_m\rangle\langle\chi_m| , \tag{3.11}$$

where the prime on the sum indicates the exclusion of box states. Equation (3.11) then allows us to express $|\chi^{(1)}(\tau,\theta)\rangle\!\rangle$ as

$$|\chi^{(1)}(\tau,\theta)\rangle\!\rangle = \int_0^{2\pi}\frac{d\theta''}{\pi} \times\int_0^{\pi}\frac{d\tau''}{\pi}\,\frac{|\chi_B(\tau'',\theta'')\rangle\!\rangle\langle\!\langle\chi_B(\tau'',\theta'')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle}{f(\tau'',\theta'')} + {\sum_m}' |\chi_m\rangle\langle\chi_m|\chi^{(1)}(\tau,\theta)\rangle\!\rangle . \tag{3.12}$$

Next we multiply (3.6b) on the left by $\langle\chi_m|$ and obtain

$$E_m^{(0)}\langle\chi_m|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + \langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle\!\rangle = E^{(1)}(\tau,\theta)\langle\chi_m|\chi_B(\tau,\theta)\rangle\!\rangle . \tag{3.13}$$

Since $\langle\chi_m|$ is orthogonal to $|\chi_B(\tau,\theta)\rangle\!\rangle$, we find

$$\langle\chi_m|\chi^{(1)}(\tau,\theta)\rangle\!\rangle = \frac{\langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle\!\rangle}{-E_m^{(0)}} . \tag{3.14}$$

Now consider Eq. (3.8). We substitute the expansion of $|\chi^{(1)}(\tau,\theta)\rangle\!\rangle$ [Eq. (3.12)] into the left-hand side of (3.8), and using (3.5) we obtain

$$\int_0^{2\pi}\frac{d\theta''}{\pi}\int_0^{\pi}\frac{d\tau''}{\pi}\,\frac{\langle\!\langle\chi_B(\tau',\theta')|H_g'|\chi_B(\tau'',\theta'')\rangle\!\rangle\langle\!\langle\chi_B(\tau'',\theta'')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle}{f(\tau'',\theta'')} + {\sum_m}' \langle\!\langle\chi_B(\tau',\theta')|H_g'|\chi_m\rangle\langle\chi_m|\chi^{(1)}(\tau,\theta)\rangle\!\rangle$$
$$= E^{(1)}(\tau,\theta)\langle\!\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + E^{(2)}(\tau,\theta)\pi^2 f(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau') . \tag{3.15}$$

Using (3.5) and (3.7) we may evaluate the integral. Then using (3.14) in the second term on the left-hand side of (3.15) we find

$$E^{(1)}(\tau',\theta')\langle\!\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + {\sum_m}' \frac{\langle\!\langle\chi_B(\tau',\theta')|H_g'|\chi_m\rangle\langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle\!\rangle}{-E_m^{(0)}}$$
$$= E^{(1)}(\tau,\theta)\langle\!\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle\!\rangle + E^{(2)}(\tau,\theta)\pi^2 f(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau') . \tag{3.16}$$

In our three-level Hilbert space we have shown in (2.53) that H_g' acting on a box state will produce a modified box state in which one of the ends of the box will be excited to a $|\overline{2}\rangle_j$ state. Therefore the only states that will contribute in the sum over m will contain combinations of $|0\rangle_j$ and $|\overline{1}\rangle_j$ states, accompanied by one $|\bar{2}\rangle_j$ state. From (2.19) we see that the energy of these states comes entirely from the $|\bar{2}\rangle_j$ state and is $\lambda_0/2a = -r_0^2/2\Delta_{0M}a$. The remaining sums in (3.16) are then evaluated and we find

$$\sum_m{}' \frac{\langle\chi_B(\tau',\theta')|H_g'|\chi_m\rangle\langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle}{-E_m^{(0)}} = g(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau') + h(\tau,\tau',\theta)\delta(\theta-\theta'), \tag{3.17}$$

where

$$g(\tau,\theta) = -\frac{64\pi^2\alpha_0'^2\Delta_{0M}^5}{ar_0^6}\left[\frac{1}{1-e^{A+i\theta/2+i\tau}} + \frac{1}{1-e^{A+i\theta/2-i\tau}} + \frac{1}{1-e^{A-i\theta/2-i\tau}} + \frac{1}{1-e^{A-i\theta/2+i\tau}} - 2\right] \tag{3.18}$$

and

$$h(\tau,\tau',\theta) = -\frac{64\pi\alpha_0'^2\Delta_{0M}^5}{ar_0^6}\left\{\left[\frac{1}{1-e^{A+i\theta/2+i\tau'}} - \frac{1}{1-e^{A+i\theta/2-i\tau'}}\right]\left[\frac{1}{1-e^{A+i\theta/2+i\tau}} - \frac{1}{1-e^{A+i\theta/2-i\tau}}\right] + \left[\frac{1}{1-e^{A-i\theta/2+i\tau'}} - \frac{1}{1-e^{A-i\theta/2-i\tau'}}\right]\left[\frac{1}{1-e^{A-i\theta/2+i\tau}} - \frac{1}{1-e^{A-i\theta/2-i\tau}}\right]\right\}. \tag{3.19}$$

Using this result in (3.16) we obtain

$$E^{(1)}(\tau',\theta')\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle + g(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau') + h(\tau,\tau',\theta)\delta(\theta-\theta')$$
$$= E^{(1)}(\tau,\theta)\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle + E^{(2)}(\tau,\theta)\pi^2 f(\tau,\theta)\delta(\theta-\theta')\delta(\tau-\tau'). \tag{3.20}$$

From this we learn that, for $\tau \neq \tau'$,

$$\langle\chi_B(\tau',\theta')|\chi^{(1)}(\tau,\theta)\rangle = \frac{h(\tau,\tau',\theta)\delta(\theta-\theta')}{E^{(1)}(\tau,\theta) - E^{(1)}(\tau',\theta')}, \tag{3.21}$$

and for $\tau = \tau'$, we learn

$$E^{(2)}(\tau,\theta) = \frac{g(\tau,\theta)}{\pi^2 f(\tau,\theta)} \tag{3.22}$$

or

$$E^{(2)}(\tau,\theta) = -\bar{c}\left(\frac{1}{\dfrac{1}{1-e^{A-i\theta/2-i\tau}} + \dfrac{1}{1-e^{A+i\theta/2+i\tau}} - 1} + \frac{1}{\dfrac{1}{1-e^{A-i\theta/2+i\tau}} + \dfrac{1}{1-e^{A+i\theta/2-i\tau}} - 1}\right), \tag{3.23}$$

where

$$\bar{c} \equiv \frac{64\alpha_0'^2\Delta_{0M}^5}{ar_0^6}. \tag{3.24}$$

Inserting (3.14) and (3.21) into (3.12) yields the desired result for the first correction to the box states induced by the next higher single-site state:

$$|\chi^{(1)}(\tau,\theta)\rangle = \int_0^{2\pi}\frac{d\theta'}{\pi}\, P\int_0^{\pi}\frac{d\tau'}{\pi}\,\frac{|\chi_B(\tau',\theta')\rangle h(\tau,\tau',\theta)\delta(\theta-\theta')}{f(\tau',\theta')[E^{(1)}(\tau,\theta)-E^{(1)}(\tau',\theta')]} + \sum_m{}'\frac{|\chi_m\rangle\langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle}{-E_m^{(0)}}, \tag{3.25}$$

where the symbol P indicates the principle value of the τ' integral. The energy $E^{(1)}(\tau,\theta)$ as given by (3.7) is easily determined with (2.53) and (3.5) to be our previous result

$$E^{(1)}(\tau,\theta) = \bar{a} - 4\bar{b}\cos\tfrac{1}{2}\theta\cos\tau. \tag{3.26}$$

Using this result in (3.25) and carrying out the θ' integral, the expression for the correction to the box states reduces to

$$|\chi^{(1)}(\tau,\theta)\rangle = |\chi^{(1a)}(\tau,\theta)\rangle + |\chi^{(1b)}(\tau,\theta)\rangle, \tag{3.27}$$

where

$$|\chi^{(1a)}(\tau,\theta)\rangle = \frac{ar_0^6 P}{16\alpha_0'\Delta_{0M}^2\pi N_0'(1+N_0'^2)^{1/2}\cos\theta/2}\int_0^{\pi}\frac{d\tau'\,|\chi_B\rangle h(\tau,\tau',\theta)}{\pi f(\tau',\theta)(\cos\tau'-\cos\tau)} \tag{3.28}$$

and

$$|\chi^{(1b)}(\tau,\theta)\rangle = \sum_m{}'\frac{|\chi_m\rangle\langle\chi_m|H_g'|\chi_B(\tau,\theta)\rangle}{-E_m^{(0)}}. \tag{3.29}$$

The projection operator may then be expressed as

$$\mathcal{P}=\int_0^{2\pi}\frac{d\theta}{\pi}\int_0^{\pi}\frac{d\tau}{\pi}\frac{1}{f(\tau,\theta)}\{|\chi_B(\tau,\theta)\rangle\langle\chi_B(\tau,\theta)|+|\chi_B(\tau,\theta)\rangle\langle\chi^{(1a)}(\tau,\theta)| + |\chi_B(\tau,\theta)\rangle\langle\chi^{(1b)}(\tau,\theta)|+|\chi^{(1a)}(\tau,\theta)\rangle\langle\chi_B(\tau,\theta)|+|\chi^{(1b)}(\tau,\theta)\rangle\langle\chi_B(\tau,\theta)|\}, \quad (3.30)$$

where we are allowed to keep only the leading terms. This expression, which contains corrections induced by the higher single-site states, will be used in Sec. IV to determine the propagator. The higher single-site states make their entrance in equations like (3.17). This particular equation determines the second-order correction to the energy eigenvalues $E^{(2)}(\tau,\theta)$ via the function $g(\tau,\theta)$ as seen in (3.22). We will now show that the expression for $E^{(2)}(\tau,\theta)$ is finite in the continuum limit.

Consider the first-order energy eigenvalues for the box states $E^{(1)}(\tau,\theta)$ given by (3.26). The continuum limit is then

$$\lim_{a\to 0}E^{(1)}(\tau,\theta)=\frac{8\alpha_0'\Delta_{0M}{}^4}{r_0{}^4}+\frac{4\alpha_0'}{a^2}\left(\frac{\theta^2}{8}+\frac{\tau^2}{2}\right)+\cdots. \quad (3.31)$$

Clearly, the second term above indicates that τ and θ must be of order a in the continuum limit. With this fact in hand, $E^{(2)}(\tau,\theta)$ as given by (3.23) is clearly seen to have a finite continuum limit, the a^{-1} dependence of $\bar{c}$ being canceled by the a^1 dependence of the term within the brackets. Or preferably, from (2.43), (3.18), and (3.22), we may determine that $g(\tau,\theta)$ and $f(\tau,\theta)$ behave as a^{-2} in the continuum limit so that $E^{(2)}(\tau,\theta)$ is again finite. From this point of view it becomes clear that any contributions of higher powers of a in $g(\tau,\theta)$ from the sum over intermediate states in (3.17) will vanish in the continuum limit. If we retain the $|\overline{3}\rangle_j$ state in our Hilbert space, we would have an additional term included in (2.53). It would be identical to the term exhibited there, except for $|\overline{3}\rangle_j$ states occupying the positions where we find the $|\overline{2}\rangle_j$ states, and the coefficient will be of order $a^{-1/2}$ rather than the a^{-1} exhibited. This additional one-half power of a will produce vanishing contributions of order a in the continuum limit of $E^{(2)}(\tau,\theta)$. It is this observation that allows us to neglect contributions from the $|\overline{3}\rangle_j$ and higher states to this order of perturbation theory.

Having established that the physical quantities like the energy, have finite continuum limits and noting that α_0' appears only as a coefficient of H_g', the only dimensionless parameter proportional to α_0' that can be formed is

$$\epsilon=16\frac{\alpha_0'\Delta_{0M}{}^3}{r_0{}^4}. \quad (3.32)$$

We have included the factor 16 for later convenience. This dimensionless parameter must characterize the perturbation expansion we have established, and we can expect to recover the two-level spin-model results by taking the $\epsilon\to 0$ limit. Further comments on ϵ will be presented in Sec. IV.

IV. ANALYSIS OF THE PROPAGATOR

The propagator for the untransformed continuum Hamiltonian of (2.1) is defined by[6,8]

$$G(x,y)=\langle 0|\psi(x)e^{-Hy}\psi^\dagger(0)|0\rangle . \quad (4.1)$$

to transform to the Hermitian version of the theory, we fix the bare intercept gap to its magic value Δ_{0M} and insert $SS^{-1}=1$, given by (2.10), between the factors appearing in (4.1) and obtain

$$G(x,y)=\left\langle\alpha\left|\psi(x)e^{-H'y}\left(\psi^\dagger(0)-\frac{2i\Delta_{0M}}{r_0}a^{1/2}\right)\right|0\right\rangle, \quad (4.2)$$

where we have used

$$S^{-1}\psi(x)S=\psi(x),$$
$$S^{-1}\psi^\dagger(0)S=\psi^\dagger(0)-\frac{2i\Delta_{0M}}{r_0}a^{1/2}, \quad (4.3)$$

and defined

$$\langle\alpha|\equiv\langle 0|S . \quad (4.4)$$

The Hamiltonian H' appearing in (4.2) is the Hermitian operator given by (2.11). Since $e^{-H'y}|0\rangle=|0\rangle$, (4.2) reduces to

$$G(x,y)=\langle\alpha|\psi(x)e^{-H'y}\psi^\dagger(0)|0\rangle . \quad (4.5)$$

Following the procedure of Sec. II we introduce a one-dimensional lattice in impact-parameter space (i.e., $D=1$). This allows us to write the propagator as

$$G(x,y)=(1/a)\langle\alpha|\psi_j e^{-H'y}\psi_0^\dagger|0\rangle, \quad (4.6)$$

where $j=|x|/a$, and the Hamiltonian is now given by the lattice version (2.17). The propagator is expected to be an analytic function of Δ_0 at $\Delta_0=\Delta_{0M}$ so we expect that the singularities generated in the complex angular momentum plane exist for all $\Delta_0<\Delta_{0c}$.

In a conventional Hermitian field theory, the propagator is given by a vacuum expectation value. In our case, however, the bare vacuum bra

$\langle 0|$ is replaced by the state $\langle\alpha|$. Let us consider this state $\langle\alpha|$ further. Taking the lattice version of the operator S, we may write

$$\langle\alpha| = \langle 0| \exp\left(-\frac{2i\Delta_{0M}}{r_0} a^{1/2} \sum_j \psi_j\right) . \tag{4.7}$$

If we truncate the Hilbert space to the two lowest states and use the representation of ψ_j given by (2.23) and (2.25), it is easy to show that

$$\langle\alpha| = \prod_j \left({}_j\langle 0| + \frac{1}{N_0'}\, {}_j\langle \overline{1}| \right) \tag{4.8}$$

or

$$\langle\alpha| = \prod_j \frac{(1+N_0'^2)^{1/2}}{N_0'}\, {}_j\langle\phi| , \tag{4.9}$$

where ${}_j\langle\phi|$ is the normalized state adjoint to the ket $|\phi\rangle_j$ introduced in (2.29). We see that $\langle\alpha|$ contains a linear combination of the two degenerate single-site ground states. The norm of the state is easily found from (4.9) and is

$$\langle\alpha|\alpha\rangle = \exp\left(\frac{2\Delta_{0M}^2}{r_0^2} V\right) . \tag{4.10}$$

(This also follows directly from the continuum theory without truncation.) We see that this state has an infinite norm as the volume of the quantization region goes to infinity. This unusual circumstance in the Hermitian version of the theory reflects, through the similarity transformation, the non-Hermitian character of H. The infinite norm also enables the propagator to contain singularities not associated with the spectrum of the Hermitian Hamiltonian H', and therefore of the similarity transformed non-Hermitian Hamiltonian H as well.

When we include the next single-site state in our Hilbert space, we find, using the representation of σ_j given in (2.46),

$$\langle\alpha| = \prod_j \left(\frac{(1+N_0'^2)^{1/2}}{N_0'}\, {}_j\langle\phi| - 2^{3/2} \frac{\Delta_{0M}^2}{r_0^2} a\, {}_j\langle\overline{2}| \right) . \tag{4.11}$$

We see that $\langle\alpha|$ now contains a contribution from the ${}_j\langle\overline{2}|$ single-site state as well.

Equation (2.23) allows us to write the propagator as

$$G(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \langle\alpha| \sigma_j e^{-H'y} \sigma_0^\dagger |0\rangle . \tag{4.12}$$

We then insert the projection operator (3.30) between the exponential factor and $\sigma_0^\dagger$ and allow the Hamiltonian to operate on these states. The propagator can then be written as

$$G(x,y) = \sum_{i=0}^{4} G_i(x,y) , \tag{4.13}$$

where

$$G_0(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \int_0^{2\pi} \frac{d\theta}{\pi} \int_0^{\pi} \frac{d\tau}{\pi} \frac{\langle\alpha|\sigma_j|\chi_B(\tau,\theta)\rangle}{f(\tau,\theta)} \times e^{-E(\tau,\theta)y} \langle\chi_B(\tau,\theta)|\sigma_0^+|0\rangle , \tag{4.14a}$$

$$G_1(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \int_0^{2\pi} \frac{d\theta}{\pi} \int_0^{\pi} \frac{d\tau}{\pi} \frac{\langle\alpha|\sigma_j|\chi_B(\tau,\theta)\rangle}{f(\tau,\theta)} \times e^{-E(\tau,\theta)y} \langle\chi^{(1a)}|\sigma_0^+|0\rangle , \tag{4.14b}$$

$$G_2(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \int_0^{2\pi} \frac{d\theta}{\pi} \int_0^{\pi} \frac{d\tau}{\pi} \frac{\langle\alpha|\sigma_j|\chi_B(\tau,\theta)\rangle}{f(\tau_1\theta)} \times e^{-E(\tau,\theta)y} \langle\chi^{(1b)}(\tau,\theta)|\sigma_0^+|0\rangle , \tag{4.14c}$$

$$G_3(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \int_0^{2\pi} \frac{d\theta}{\pi} \int_0^{\pi} \frac{d\tau}{\pi} \frac{\langle\alpha|\sigma_j|\chi^{(1a)}(\tau,\theta)\rangle}{f(\tau_1\theta)} \times e^{-E(\tau,\theta)y} \langle\chi_B(\tau,\theta)|\sigma_0^+|0\rangle , \tag{4.14d}$$

$$G_4(x,y) = \frac{4\Delta_{0M}^2}{r_0^2} \int_0^{2\pi} \frac{d\theta}{\pi} \int_0^{\pi} \frac{d\tau}{\pi} \frac{\langle\alpha|\sigma_i|\chi^{(1b)}(\tau,\theta)\rangle}{f(\tau,\theta)} \times e^{-E(\tau,\theta)y} \langle\chi_B(\tau,\theta)|\sigma_0^+|0\rangle \tag{4.14e}$$

and $E(\tau,\theta) = E^{(1)}(\tau,\theta) + E^{(2)}(\tau,\theta)$.

After carrying out all sums and integrals introduced in (4.14) by $|\chi_B(\tau,\theta)\rangle$, $|\chi^{(1a)}(\tau,\theta)\rangle$, and $|\chi^{(1b)}(\tau,\theta)\rangle$, and making some simple changes of variables of integration, we have the result that the continuum limit of the propagator to order ϵ is

$$G(x,y) = -\frac{4\Delta_{0M}^2}{r_0^2} I_2 I_1 + (1+\epsilon)[I_1 X_2 - I_2 X_1] - \frac{\epsilon\Delta_{0M}^2}{4r_0^2}\left[1 + 8\frac{\partial}{\partial c}\right][I_1 I_4(c) + I_2 I_3(c)]_{c=1} . \tag{4.15}$$

The integrals appearing in (4.15) may be expressed in terms of the error function

$$\mathrm{Erfc}(u) = \int_u^\infty dx\, e^{-x^2} . \tag{4.16}$$

Specifically, we find

$$I_{1(2)} = \pm \frac{1}{\sqrt{\pi}} \mathrm{Erfc}\left[\mp \frac{|x|}{2(\alpha_0' y)^{1/2}(1+\epsilon)^{1/2}} - \frac{2\Delta_{0M}^2}{r_0^2}(\alpha_0' y)^{1/2}(1+\epsilon)^{1/2}\right], \tag{4.17}$$

$$X_{1(2)} = \frac{1}{2(\alpha_0' y)^{1/2}(1+\epsilon)^{1/2}} \times \exp\left\{-\left[\frac{2\Delta_{0M}^2}{r^2}(\alpha_0' y)^{1/2}(1+\epsilon)^{1/2} \pm \frac{|x|^0}{2(\alpha_0' y)^{1/2}(1+\epsilon)^{1/2}}\right]^2\right\},$$

$$I_{3(4)}(c)=\mp\frac{1}{\sqrt{\pi}}\exp\left(\frac{16\Delta_{0M}{}^4}{r_0{}^4}\alpha_0'y(1+\epsilon)c(c-1)\mp\frac{4\Delta_{0M}{}^2}{r_0{}^2}|x|c\right)\mathrm{Erfc}\left[\mp\frac{|x|}{2(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2}}+\frac{2\Delta_{0M}{}^2}{r_0{}^2}(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2}(2c-1)\right]. \tag{4.18}$$

Then using (4.16) we find we may write the propagator as

$$\begin{aligned}G(x,y)=&\frac{4\Delta_{0M}{}^2}{r_0{}^2\pi}\int_{x_{10}}^{\infty}dx_1e^{-x_1{}^2}\int_{x_{20}}^{\infty}dx_2e^{-x_2{}^2}\left[1+\frac{r_0{}^2(1+\epsilon)^{1/2}(x_1+x_2)}{4\Delta_{0M}{}^2(\alpha_0'y)^{1/2}}\right]\\&-\frac{\Delta_{0M}{}^2}{r_0{}^2}\frac{\epsilon}{4\pi}\left(1+8\frac{\partial}{\partial c}\right)\exp\left[\frac{16\Delta_{0M}{}^4}{r_0{}^4}\alpha_0'y(1+\epsilon)c(c-1)\right]\\&\times\left[\exp\left(\frac{4\Delta_{0M}{}^2}{r_0{}^2}xc\right)\int_{x_{10}}^{\infty}dx_1e^{-x_1{}^2}\int_{x_{30}}^{\infty}dx_3\,e^{-x_3{}^2}+\exp\left(-\frac{4\Delta_{0M}{}^2}{r_0{}^2}xc\right)\int_{x_{20}}^{\infty}dx_2\,e^{-x_2{}^2}\int_{x_{40}}^{\infty}dx_4\,e^{-x_4{}^2}\right]\Bigg|_{c=1},\end{aligned} \tag{4.19}$$

where

$$x_{10(20)}=\mp\frac{x}{2(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2}}-\frac{2\Delta_{0M}{}^2}{r_0{}^2}(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2} \tag{4.20}$$

and

$$x_{30(40)}=\pm\frac{x}{2(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2}}+\frac{2\Delta_{0M}{}^2}{r_0{}^2}(\alpha_0'y)^{1/2}(1+\epsilon)^{1/2}(2c-1)\,. \tag{4.21}$$

Note that we have been able to drop the absolute value signs, since $G(x,y)$ is symmetric under the exchange $|x|\to-|x|$.

The first point of interest in (4.19) is that if we let ϵ go to zero our expression reduces to that found by Bronzan and Sugar[8] in their two-level spin-model approximation. This reveals the fact that

$$\epsilon=\frac{16\alpha_0'\Delta_{0M}{}^3}{r_0{}^4} \tag{4.22}$$

must truly be a small parameter. Therefore the ordered phase corresponds to the strong-coupling regime of RFT, and the technique we have established is a strong-coupling expansion. In obtaining the expression for the propagator, (4.15), we encountered a number of divergent geometric series, which we rendered finite by allowing τ and θ to take on imaginary values. We believe the correspondence with the earlier results of the Reggeon quantum spin model in the $\epsilon\to 0$ limit reveals the validity of this technique for handling the divergent geometric series.

The propagator in the angular momentum plane is given by

$$G(k,E)=\int_{-\infty}^{\infty}dx\,e^{ikx}\int_0^{\infty}dy\,e^{yE}G(x,y)\,. \tag{4.23}$$

Inserting (4.19) into (4.23) gives us

$$\begin{aligned}G(k,E)=&\frac{2}{\alpha_0'k^2+\left\{\left[4\delta_0+\alpha_0'k^2-\frac{2E}{(1+\epsilon)}\right]^{1/2}-2\delta_0\right\}^2}\left\{1-\frac{2\delta_0{}^{1/2}\epsilon}{(1+\epsilon)\left[4\delta_0+\alpha_0'k^2-\frac{2E}{(1+\epsilon)}\right]^{1/2}}\right\}\\&+\frac{\epsilon}{6}\frac{\Delta_{0M}{}^2}{r_0{}^2}\left\{\frac{1}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2+\frac{8ik\Delta_{0M}{}^2}{r_0{}^2}\right]^{1/2}\left[\frac{-2E}{(1+\epsilon)}+2k^2\alpha_0'+\frac{8ik\alpha_0'\Delta_{0M}{}^2}{r_0{}^2}\right]}\right.\\&+\frac{1}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2-\frac{8ik\Delta_{0M}{}^2}{r_0{}^2}\right]^{1/2}\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2-\frac{8ik\alpha_0'\Delta_{0M}{}^2}{r_0{}^2}\right]}\\&-\frac{4\left(\frac{8ik\Delta_{0M}{}^2}{r_0{}^2}-\frac{32\Delta_{0M}{}^4}{r_0{}^4}\right)}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2+\frac{8ik\Delta_{0M}{}^2}{r_0{}^2}\right]^{3/2}\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2+\frac{8ik\alpha_0'\Delta_{0M}{}^2}{r_0{}^2}\right]}\end{aligned}$$

$$+\frac{4\left(\frac{8ik\Delta_{0M}^2}{r_0^2}+\frac{32\Delta_{0M}^4}{r_0^4}\right)}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2-\frac{8ik\Delta_{0M}^2}{r_0^2}\right]^{3/2}\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2-\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}\right]}$$

$$-\frac{8\left(\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}-\frac{32\Delta_{0M}^4}{r_0^4}\right)}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2+\frac{8ik\Delta_{0M}^2}{r_0^2}\right]^{1/2}\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2+\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}\right]^2}$$

$$+\frac{8\left(\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}+\frac{32\Delta_{0M}^4}{r_0^4}\right)}{\left[\frac{-2E}{\alpha_0'(1+\epsilon)}+k^2-\frac{8ik\Delta_{0M}^2}{r_0^2}\right]^{1/2}\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2-\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}\right]^2}$$

$$-\frac{64\alpha_0'\frac{\Delta_{0M}^2}{r_0^2}}{\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2+\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}\right]^2}-\frac{64\alpha_0'\frac{\Delta_{0M}^2}{r_0^2}}{\left[\frac{-2E}{(1+\epsilon)}+2\alpha_0'k^2-\frac{8ik\alpha_0'\Delta_{0M}^2}{r_0^2}\right]^2}\Bigg\}. \qquad (4.24)$$

From this expression, we see that the propagator has poles on the trajectories

$$E=\left[\alpha_0'k^2\pm 4i\frac{\alpha_0'\Delta_{0M}^2}{r_0^2}k\right](1+\epsilon) \qquad (4.25)$$

or

$$J=1+\left[\alpha_0't\mp 4i\frac{\alpha_0'\Delta_{0M}^2}{r_0^2}\sqrt{-t}\right](1+\epsilon), \qquad (4.26)$$

and cuts on the trajectories

$$E=\left[2\delta_0+\frac{\alpha_0'k^2}{2}\right](1+\epsilon) \qquad (4.27)$$

or

$$J=1-\left[2\delta_0-\frac{\alpha_0't}{2}\right](1+\epsilon), \qquad (4.28)$$

and

$$E=\left[\frac{\alpha_0'k^2}{2}\pm 4i\frac{\alpha_0'\Delta_{0M}^2}{r_0^2}k\right](1+\epsilon) \qquad (4.29)$$

or

$$J=1+\frac{\alpha_0't}{2}(1+\epsilon)\pm 4i\frac{\alpha_0'\Delta_{0M}^2}{r_0^2}\sqrt{-t}\,(1+\epsilon). \qquad (4.30)$$

The new features here are finding a shift in the trajectories of (4.26) and (4.28) which appear in the two-level spin model and the appearance of the Regge-Mandelstam moving cut (4.30) which is absent in the two-level spin model. Reggeon unitarity requires the appearance of these Regge-Mandelstam singularities on the trajectories

$$J_n=1+\frac{\alpha_0't}{n}\pm 4i\frac{\alpha_0'\Delta_{0M}^2}{r_0^2}\sqrt{-t}, \qquad (4.31)$$

owing to the convolution of n Regge poles. Comparing (4.30) to (4.31), after absorbing the $1+\epsilon$ factor into a renormalization of α_0', shows that the effect of including the order ϵ term has been to produce a cut corresponding to the convolution of two Regge poles. Therefore we may make the important inference that it is now possible for RFT to satisfy Reggeon unitarity in the ordered phase.

Owing to the collapse of the poles, (4.25), to a double pole in $G(k,E)$ when $t=0$, we find that

$$\sigma_T\propto \log s, \qquad (4.32)$$

which shows that the Reggeon quantum spin model saturates the Froissart bound in one transverse dimension. This result is due to the infinite norm state $\langle\alpha|$, which gives rise to the poles of (4.25). These singularities are not associated with the spectrum of the Hermitian operator H' and can be understood to be a feature through the similarity transformation of the non-Hermitian character of RFT.

V. SUMMARY AND CONCLUSION

The ordered phase of RFT has been examined in the framework of the Reggeon quantum spin model including the four-Pomeron coupling.[8] We employed the Hamiltonian formulation and have shown that the inclusion of the four-Pomeron coupling allows a similarity transformation to a Hermitian version of the theory when Δ_0 is fixed at its magic value. By working with a Hermitian version of the theory, we were able to isolate the unusual characteristics that arise due to the non-Hermiticity of RFT. In particular, we found that the left vacuum state that appears in a conventional Hermitian field theory, was replaced by the infinite norm state $\langle\alpha|$. The physical consequence of this was the saturation of the Froissart bound.

The gradient term of the Hamiltonian can be treated perturbatively by introducing a lattice in impact-parameter space, while keeping the rapidity variable continuous. The spin-model approximation of truncating the Hilbert space to the two lowest states at each lattice site, is then motivated by inspecting the single-site eigenvalues which show that only the two degenerate single-site ground state energies remain finite as the lattice spacing goes to zero. The spin model also arises naturally in determining the correct linear combinations required to diagonalize the gradient term of the Hamiltonian, namely the box states. We then developed a perturbative formalism for including the higher single-site states which determined the leading corrections to the box states. We were then able to show that this formalism yields finite results in the zero-lattice-spacing limit. This allowed us to identify the small parameter ϵ that characterizes the perturbation expansion, and relate the spin-model results for the propagator directly to the continuum RFT. The main result of these calculations was to show that the next higher single-site state led to the production of the two-Reggeon cut in the angular momentum plane. The absence of this cut in the two-level model created doubts on the validity of RFT in the ordered phase, since these Regge-Mandelstam singularities are required if RFT is to obey the Reggeon unitarity relations.

In arriving at these results we encountered a number of divergent geometric series. We rendered these finite by allowing the variables τ and θ to take on imaginary values. We were able to confirm the validity of this technique by showing that the presupposed condition of recovering the two-level spin-model results did occur in the $\epsilon \to 0$ limit.

We conclude with a word of caution. Our results indicate that the parameters of RFT can be chosen arbitrarily without violating any s-channel or t-channel constraints. However, we have assumed that these parameters are smooth functions of the Reggeon momentum k_i. A recent study[10] of the Regge behavior of spontaneously broken non-Abelian gauge theories showed that the sum of leading logarithms in the Regge limit ($s\to\infty$, t-fixed) satisfies Reggeon unitarity but violates the Froissart bound. The source of conflict lies in the fact that the four-Reggeon coupling is singular, not smooth, as Reggeon momentum $k_i \to 0$ in non-Abelian gauge theories. Therefore the assumption of smooth Reggeon parameters may very well be incorrect, if the correct underlying dynamics of strongly interacting particles is described by spontaneously broken non-Abelian gauge theories.

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