Non-self-dual static gauge fields

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We exhibit exact non-self-dual static solutions to the SU(2) Yang-Mills field equations by solving the equation $\nabla^2 V + \lambda V^3 = 0$ using cylindrical and spherical coordinates. The resulting gauge fields are complex and have singularities. For the cylindrically symmetric solution, we convert it into a real gauge field coupled to the Higgs field in the limit in which the self-interaction potential of the Higgs field vanishes.

I. INTRODUCTION

Recently there have been many attempts to construct static solutions to the classical Yang-Mills (YM) field equations.[1] However, most of the explicit solutions obtained are self-dual or self-anti-dual,[2-7] and non-self-dual static solutions are scarce as they are harder to come by.[8-11] Non-self-dual solutions may contain features of the YM theory which are not exhibited by the self-dual solutions,[12] for example merons[13] are non-self-dual solutions and have noninteger topological numbers. Furthermore, the self-duality condition can in fact be linearized whereas non-self-dual solutions result from solving nonlinear differential equations. Finally self-dual solutions are necessarily sourceless but non-self-dual fields do not automatically satisfy the sourceless YM equations.

In Ref. 5, families of self-dual static solutions are presented by using the ansatz

$$A_0^a(x) = \pm \frac{i}{g}\partial^a \ln V, \quad A_j^a(x) = \frac{1}{g}\epsilon_{jab}\partial^b \ln V, \quad \partial_\mu A^{a\mu} = 0, \tag{1}$$

where V is a function of the spatial coordinates x^i only. If one defines $\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma_k$, $\sigma^{i0} = \pm\frac{1}{2}i\sigma^i$ where σ^i are the Pauli matrices, expression (1) can be written as

$$A_\mu(x) = g\frac{\sigma^a}{2i}A_\mu^a(x) = i\sigma_{\mu\nu}\partial^\nu \ln\psi(x) \tag{2}$$

with $\psi(x) = V$, which is the Corrigan-Fairlie-'t Hooft-Wilczek ansatz[14] in Minkowski space. On substituting the ansatz (2) into the YM equations

$$\partial_\nu F^{\mu\nu} + [A_\nu, F^{\mu\nu}] = 0, \quad F_{\mu\nu} = g\frac{\sigma^a}{2i}F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3}$$

where the signature of the metric $g_{\mu\nu}$ is $(-+++)$, a nonlinear equation for the function V follows:

$$\nabla^2 V + \lambda V^3 = 0. \tag{4}$$

Here λ is a constant which can be positive or negative. The vanishing of λ renders Eq. (4) the self-duality condition. To solve Eq. (4) is not easy unless one imposes certain symmetry requirements. In Ref. 11 we write V as a function of $u = p_i x^i + q$, with p^i being a constant vector and q an arbitrary constant, and transform Eq. (4) into a differential equation of one variable. The general solutions for V are the Jacobi elliptic functions and $V = u^{-1}$. These solutions lead to gauge fields which are periodic in u and have constant energy densities. They are singular on parallel planes. Furthermore, the solution $V = u^{-1}$ possesses zero energy density in spite of the fact that it corresponds to the gauge fields which are non-self-dual. This means that although the self-duality condition always yields gauge fields with vanishing energy density, the converse need not be true.[15] In this paper we wish to present other solutions of Eq. (4) by employing cylindrical and spherical coordinates for the Laplacian ∇^2. These are separately carried out in Secs. II and III. In Sec. IV the cylindrically symmetric gauge fields are coupled minimally to the Higgs triplet and the resulting configuration seems to correspond to a string of non-Abelian "electric" sources lying on the z axis. We end with brief remarks in Sec. V.

II. CYLINDRICAL SOLUTIONS

Using the cylindrical coordinates (ρ, ϕ, z), Eq. (4) admits the solution

$$V = \pm(-\lambda)^{-1/2}\frac{1}{\rho}, \quad \rho^2 = (x^1)^2 + (x^2)^2, \tag{5}$$

which leads to the perfectly cylindrically symmetric gauge fields

$$A_\mu = -i\sigma_{\mu A}n^A/\rho, \quad n^A = x^A/\rho. \tag{6}$$

Here the capital Latin indices $A, B, \ldots$ take on values 1 and 2 only. For the ansatz (2), the field strength can be expressed as

$$F_{\mu\nu} = (i/V^2)[\sigma_{\nu\alpha}(V\partial_\mu\partial^\alpha V - 2\partial_\mu V\partial^\alpha V) - \sigma_{\mu\nu}\partial^\alpha V\partial_\alpha V + \sigma_{\mu\alpha}(2\partial_\nu V\partial^\alpha V - V\partial_\nu\partial^\alpha V)] \tag{7}$$

and the energy density for the gauge field is

$$(g^2/\lambda)\theta_{00} = \tfrac{1}{2}\lambda V^4 + \partial_i V \partial^i V . \tag{8}$$

The electric and magnetic fields associated with the solution (5) are easily computed to give

$$E_A = \pm\frac{1}{2\rho^2} n_A n_B \sigma^B, \quad E_3 = \pm\frac{1}{2\rho^2}\sigma_3 \tag{9a}$$

and

$$B_A = \frac{i}{2\rho^2}\epsilon_{AB3}\epsilon^{3CD} n_C n_D n^B ,$$
$$B_3 = \frac{i}{2\rho^2} n_3 n_A \sigma^A . \tag{9b}$$

The fields are singular along the z axis, indicating the presence of sources lying along a straight line. From Eqs. (8) and (5), we find that the energy density is real and is given by

$$\theta_{00} = -\frac{1}{2g^2\rho^4}, \tag{10}$$

which increases rapidly when far away from the line of sources. The total energy diverges due to the singularity at $\rho = 0$.

By trial, we are able to modify solution (5) to the form

$$V = (A/\rho)E(p\phi), \tag{11}$$

which are solutions of Eq. (4) and where A, p are nonvanishing arbitrary constants. Here $E(p\phi)$ is a function of the coordinate ϕ only and satisfies the relation

$$p^2 E'' + E + \lambda A^2 E^3 = 0 \tag{12}$$

or

$$p^2 E'^2 + E^2 + \tfrac{1}{2}\lambda A^2 E^4 = p^2 c(k), \tag{13}$$

where $E' = dE/d(p\phi)$, etc. This indicates that $E(p\phi)$ are the Jacobi elliptic functions with parameter k such that $0 \leq k \leq 1$ and $c(k)$ is a constant. The gauge field as calculated from the solution (11) is

$$A_\mu = i\sigma_{\mu A}\left[\frac{pE'}{E}(\delta_2^A x^1 - \delta_1^A x^2) - x^A\right]\Big/\rho^2,$$

which has a singularity along the z axis as well as those arising from the elliptic functions. The field strengths corresponding to solution (11) are respectively

$$E_1 = \pm\frac{1}{2E^2\rho^2}[\sigma_1 S - 2p^2 c(k) x^2(\sigma_1 x^2 - \sigma_2 x^1)/\rho^2],$$
$$E_2 = \pm\frac{1}{2E^2\rho^2}[\sigma_2 S - 2p^2 c(k) x^1(\sigma_2 x^1 - \sigma_1 x^2)/\rho^2], \tag{14a}$$
$$E_3 = \pm\frac{1}{2E^2\rho^2} S\sigma_3$$

and

$$B_A = \frac{i}{E^2\rho^2}[p^2 c(k) x^A \sigma_B x^B/\rho^2 - \tfrac{1}{2}\sigma_A S],$$
$$B_3 = \frac{i}{E^2\rho^2}\sigma_3(p^2 c(k) - \tfrac{1}{2}S), \tag{14b}$$

where we write S for

$$S \equiv p^2 c(k) - \tfrac{1}{2}\lambda A^2 E^4(p\phi).$$

Comparing with solution (5), the elliptic functions, which depend on the azimuthal angle ϕ only, in a sense provided structure to the electric and magnetic fields. This means that the source associated with the singularity for solution (5) is simpler in structure than that associated with the singularity of solution (11). The energy density as computed from Eq. (8) is given by

$$\theta_{00} = \lambda\left(\frac{Ap}{g\rho^2}\right)^2 c(k). \tag{15}$$

Depending on which of the twelve Jacobi elliptic function is used and also on the value of the parameter k, the constant $c(k)$ in expression (15) can be negative or positive.

Solution (11) can be simplified if we set $p = 1$. In this case $A = \pm(-2/\lambda)^{1/2}$, and Eq. (12) becomes

$$E'' + E - 2E^3 = 0 .$$

The parameter k for the elliptic functions is now restricted to $k^2 = 0$. With this restriction the functions $E(\phi)$ are either $(\sin\phi)^{-1}$ or $(\cos\phi)^{-1}$ and the constant $c(k)$ takes zero value. For these simplified solutions the electric and magnetic fields are respectively

$$E_i = \pm\frac{E^2}{2\rho^2}\sigma_i$$

and

$$B_i = \pm\frac{i}{2\rho^2}E^2\sigma_i .$$

These expressions look simpler in appearance than Eqs. (9a) and (9b). Actually the function V is now effectively either $V \sim 1/x^1$ or $1/x^2$ and depends on one coordinate only. It is interesting to observe that for this simplified case $c(k)$ vanishes, and that means the energy density by virtue of Eq. (15) also vanishes. Again we have found an example of SU(2) non-self-dual gauge field with zero energy density.[15]

III. SPHERICAL SOLUTIONS

So far the solutions obtained from solving Eq. (4) are real functions. The function V may be complex since the ansatz (1) already provides a complex gauge field even though V is restricted to be real. Using the spherical coordinates $(r,\ \theta,\ \phi)$, the expression

$$V=\frac{1}{r}E(u)\,, \qquad u=k_1\ln r+k_2\ln\csc\theta\pm ik_2\phi\,, \tag{16}$$

gives a family of solutions to Eq. (4), where k_1 and k_2 are real parameters. If $k_2=-k_1$, then $E(u)$ is a simple function

$$E(u)=\pm\left(c-\frac{\lambda u}{k_1}\right)^{-1/2}, \tag{17a}$$

where c is an arbitrary constant. For $k_2\neq\pm k_1$, $E(u)$ satisfies the following differential equation for which we fail to obtain explicit solutions:

$$E''-\frac{E'}{k_1+k_2}+\frac{\lambda E^3}{k_1^{\,2}-k_2^{\,2}}=0\,. \tag{17b}$$

The gauge field as derived from solution (16) is

$$A_\mu=i\sigma_{\mu j}\left(\frac{E'}{E}\partial^j u+\frac{n^j}{r}\right), \tag{18}$$

with $n^j=x^j/r$ being the unit vector and

$$\partial^j u=\frac{k_1}{r}n^j-k_2\cot\theta\partial^j\theta\pm ik_2\partial^j\phi\,, \tag{19a}$$

$$\partial^j\phi=(-\sin\phi,\ \cos\phi,\ 0)/(r\sin\theta)\,, \tag{19b}$$

$$\partial^j\theta=(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)/r\,. \tag{19c}$$

The gauge field has a singularity at the origin and those due to the function E'/E. We have calculated the electric and magnetic fields for the above solutions. Their expressions are lengthy and they vary with distance as $1/r^2$ with modification by the function $E(u)$. The energy density is complex and is given by

$$(g^2/\lambda)\theta_{00}(x)=(\tfrac{1}{2}\lambda E^2+1)\frac{E^2}{r^4}+\left(E'\partial^j u\partial_j u-\frac{2}{r}EE'n^j\partial_j u\right)\frac{1}{r^2}. \tag{20}$$

From the above, we see that solution (16) leads to a gauge field with Coulomb-type behavior modified by the structure function $E(u)$. However, as the energy density is complex we do not believe solution (16) has any physical significance.

IV. THE SCALAR COUPLED GAUGE FIELD

Wu and Yang[16] have shown that the complex gauge field can be made real if one extends the gauge group to the noncompact group. However, it is also possible to understand the complex static gauge field by converting it into the exact solution for the real SU(2) gauge field coupled minimally to the triplet Higgs field Φ^a when the self-interaction potential of the later vanishes.[7] This can easily be performed by setting[5]

$$gA_i^a=\epsilon_{iab}\partial^b\ln V\,, \qquad gA_0^a=\sinh\gamma\partial^a\ln V\,, \qquad g\Phi^a=\cosh\gamma\partial^a\ln V\,, \tag{21}$$

where γ is a constant. As an example we illustrate for the cylindrically symmetric solution (5). Following the notations of Ref. 17, the Lagrangian density for the YM field coupled to the Higgs field is

$$\mathcal{L}=-\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}-\tfrac{1}{2}\pi^{\mu a}\pi_{\mu a}+\tfrac{1}{2}\mu^2\Phi^2-\tfrac{1}{4}\lambda\Phi^4\,, \tag{22}$$

where

$$\pi^a_\mu=\partial_\mu\Phi^a+g\epsilon_{abc}A^b_\mu\Phi^c\,.$$

For the ansatz (21), the electromagnetic field $\mathfrak{F}_{\mu\nu}$ as defined by 't Hooft can be simplified to

$$\mathfrak{F}_{0i}=-\frac{\sinh\gamma}{g}(\partial_b V\partial^b V)^{-1/2}\left(\partial_i\partial^a V-\frac{1}{V}\partial_i V\partial^a V\right)\partial_a\ln V\,, \qquad \mathfrak{F}_{ij}=-\frac{1}{g}\epsilon_{abc}(\partial^a V)(\partial_i\partial^b V)\partial_j\partial^c V/(\partial^f V\partial_f V)^{3/2}\,. \tag{23}$$

For the solution (5), we obtain for the electric field

$$\xi_A=\mathfrak{F}_{0A}=\frac{\sinh\gamma}{g\rho^2}n_A,\quad \xi_3=0 \tag{24}$$

while the magnetic field vanishes. The solution thus suggests the presence of a string of non-Abelian electric sources lying on the z axis. A cylindrically symmetric solution has been discussed in the literature[18,19] and is described as corresponding to a string of monopoles. However, our solution does not satisfy the required boundary condition since the Higgs field does not tend to a constant asymptotically.

The energy density can be evaluated from the time component of the energy-momentum tensor

$$T_{00}=F^a_{0j}F^a_{0j}+\pi^a_0\pi^a_0-\mathcal{L}\,.$$

After some lengthy but straightforward calculation we obtain

$$T_{00}=\frac{1}{g^2}(2\cosh^2\gamma-1)/\rho^4 \tag{25}$$

and the total energy is infinite. If we allow the constant γ to be complex so that $\cosh^2\gamma=\frac{1}{2}$ then the energy vanishes, but in this case the gauge field A_0^a becomes complex although A_i^a and Φ^a still remain real.

V. REMARKS

(1) Topological arguments permit the existence of monopoles with magnetic charge greater than

one. To search for such monopoles with higher magnetic charges one must consider gauge field configurations departing from spherical symmetry since spherically symmetric monopoles cannot have magnetic charge greater than one unit. The solutions given here are nonspherically symmetric and it does not seem that they will yield finite magnetic charges.

(2) Except for the Hsu and Mac[2] solution, practically all static solutions for the SU(2) gauge field possess singularities. These singularities arise naturally in the solutions and may have significance in the theory of elementary particles. Singularities may or may not require the presence of external sources to sustain them.[20]

(3) If we consider an N-dimensional version of Eq. (4), then solution of the form $V=(\mp\lambda r^2)^{1/2}$, where

$$r^2=(x^1)^2+(x^2)^2+\cdots+(x^N)^2\,,$$

exists only for $N=2$ and $N=4$. We have discussed the case $N=2$ in Sec. II whereas the case $N=4$ corresponds to the meron solution[13] in the four-dimensional Euclidean space. If we restrict λ to be an integer, then $V=1/r$ is a solution for any N-dimensional space.

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