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Color-singlet condensation in quantum chromodynamics and flux squeezing

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The nonperturbative condensation of the operator $G_{\mu\nu}{}^2$ in quantum chromodynamics is discussed using the renormalization-group technique. It is shown that magnetic condensation, $\langle G_{\mu\nu}{}^2 \rangle > 0$, leads to a new vacuum which has lower energy than the perturbative vacuum. From this fact it is concluded that Green's functions calculated in the normal vacuum have tachyonic singularities. By assuming the gauge-invariant local expansion of the effective action it is shown that the condensed vacuum has the property of a vanishing dielectric constant. If the color electric field is applied by introducing heavy quarks at infinity, the condensation is partly broken and there is a consistent solution in which an infinite tube of the color electric flux is formed. Arguments used rely heavily on the instability of the normal vacuum and on the negative character of the β function. An attempt at the mean-field-type approximation is made. Comparison with the previous phenomenological approach is also given.

I. INTRODUCTION

The central problem in studying the low-energy spectra of quantum chromodynamics (QCD)[1] is certainly to determine the ground state. Many people suspect that the normal perturbative ground state may not be a true one.[2]

In this paper we discuss one of the dynamical aspects of pure QCD, excluding Higgs particles, which seems to play an important role in determining the true vacuum. (Quarks are introduced as external color sources.) The same problem has previously been discussed in an intuitive approximate way. Our starting point is a very simple observation that the two-body force between normal massless gluons is attractive in the color-singlet channel, and gluons can form a color-singlet bound state which is necessarily a tachyonic one because gluons are massless. This means that the normal vacuum sits on the maximum, instead of minimum, of the potential corresponding to this bound state. The problem was first studied by a variational approach in terms of the Cooper pair,[3] which is the nonrelativistic analog of the tachyon, and then discussed[4] in terms of a tachyon by solving the Bethe-Salpeter (BS) equation in the ladder approximation. Both led to the same qualitative picture that such a bound state is formed for arbitrarily small coupling constant, i.e., the critical coupling constant is zero. However, these approaches rely on gauge-noninvariant approximations.

It is now clear that the local gauge invariance of the vacuum of QCD is an essential ingredient of the theory. In order to discuss the color-singlet condensation phenomenon gauge invariantly and nonperturbatively, we choose in this paper the operator $\hat{G}_{\mu\nu}{}^2(x)$ or $\int d^4x\, \hat{G}_{\mu\nu}{}^2(x)$ and discuss its nonperturbative condensation. Here $\hat{G}^a_{\mu\nu}$ is the usual Yang-Mills field-strength tensor. We have used the fact that any field can be an interpolating field of the bound state as long as it has the same quantum number as the state we want to discuss. Thus the source term J coupled to $\hat{G}_{\mu\nu}{}^2$ is introduced and J^a_μ coupled to the gluon field $\hat{A}^a_\mu$ are also introduced to discuss the situation where quarks are present in the condensed vacuum. The introduction of J or J^a_μ does not spoil the renormalizability of the theory so that the nonperturbative condensed solution, if it exists, should be a solution of the renormalization-group equation (RGE). The assumption taken in this paper is that the RGE has a nontrivial solution, specifically (23) below is assumed to have a finite solution. Then the analysis of Sec. II shows that the magnetic-type condensation of $\hat{G}_{\mu\nu}{}^2$, i.e., $\Delta\phi \equiv \frac{1}{4}\langle :\hat{G}_{\mu\nu}{}^2: \rangle > 0$, leads to the vacuum which has lower energy than the normal vacuum. It occurs for arbitrarily small coupling constant. The reason we believe in the existence of a nontrivial solution of (23) is twofold. One is due to the results of the ladder approximation[4] where the physical elements leading to the condensation is the attractive force in the color-singlet channel. The other is due to the general statement that any zero-mass theory which has the property of asymptotic freedom shows a nonperturbative condensation phenomenon. There is no proof of this statement but there is also no example which contradicts it. As an example we discuss in Appendix A the condensation of the Lagrangian in $\lambda\phi^4$ theory. In the discussion in Sec. II the problem of operator mixing is neglected. It has been discussed by several authors[5,6] with the results that we can ignore the mixing when only the physical quantities are discussed.

Section III is devoted to the proof of the general statement that if some composite operator shows nonperturbative condensation then any Green's function calculcated in the normal vacuum has tachyonic singularities in the channel which has the same quantum number as the above composite operator. According to this theorem, the Green's functions of QCD if calculated in the normal vacuum have tachyonic, i.e., spacelike, singularities in the color-singlet channel. They will become complex for spacelike momentum. The imaginary part at zero momentum is related to the decay probability of the normal vacuum.

Now the true vacuum is filled with gluons which condense nonperturbatively forming a color-singlet composite state. The normal gluons cannot be in the asymptotic states. The problem is to determine the "color electrostatic" property of the vacuum, which is discussed in Sec. IV. The condensation of $\hat{G}_{\mu\nu}{}^2$ does not violate the local gauge invariance of the vacuum so that the effective action is expected to have a gauge-invariant local expansion (54) below. We have in mind the situation that an infinitely heavy quark and antiquark with definite color index are introduced at infinity so that the static Abelian constant color electric field is chosen for the argument of the effective potential. Then we see that the applied color electric field E breaks the condensation so that $\Delta\phi$ becomes a function of $G\equiv\frac{1}{2}E^2$. It is also seen that the dielectric constant ϵ of the vacuum diminishes as the condensation $\Delta\phi$ increases and ϵ vanishes as $\Delta\phi$ takes the vacuum value $\Delta\phi=\Delta\phi_c$. In deriving these results the sign of $\Delta\phi$ ($\Delta\phi>0$) and of the β function ($\beta<0$) play important roles. The stationarity condition, that is, the sourceless condition $J_\mu=0$, is satisfied by the normal solution $G_{\mu\nu}=0$ and by $\epsilon=0$. The former solution cannot represent the condensed solution because the tachyonic singularities are present in the Green's functions owing to the results of Sec. III (see also Appendix B). The perfect diaelectric property $\epsilon=0$ leads to a tube-like solution for the color electric flux. We also attempt to discuss the behavior of the dielectric constant by mean-field approximation

Section IV is devoted to a discussion of the connection between the present approach and the previous phenomenological one.[7] We get qualitatively the same picture of the stable vacuum and the mechanism of flux squeezing. In the phenomenological approach the condition $\epsilon=0$ emerges as a stability condition of the vacuum. In the present approach we are forced to take the solution $\epsilon=0$ because the other solution $G_{\mu\nu}=0$ corresponds to the unstable normal solution.

The picture of the hadronic bound state we get in our paper is similar to the one discussed by Callan, Dashen, and Gross.[8] But their instanton density is replaced here by a more general quantity, the condensation $\langle:\hat{G}_{\mu\nu}{}^2:\rangle$. The antishielding property of the vacuum is due to the more general fact that the β function is negative. The precise mechanism of the condensation need not be specified. According to the picture discussed in Sec. III, however, the bag constant of the MIT bag model[9] is supplied by the condensation energy density of the tachyonic bound state.

In Sec. VI discussions are presented on several points which seem to be crucial to the present investigations. We look for the vacuum solution satisfying $J=J_\mu^a=0$ by taking a particular field configuration as an argument of the effective action, i.e., we probe the vacuum by the external field of the specific configuration. For a discussion of more complicated x-dependent (gluonium-like) solutions, we need a more general effective action which is not the subject of the present paper.

II. NONPERTURBATIVE CONDENSATION OF $\hat{G}_{\mu\nu}{}^2$

For the purpose of discussing the nonperturbative translationally invariant condensation of $\hat{G}_{\mu\nu}{}^2$, the constant source J is introduced as

$$e^{i\Omega W[J]}\equiv\int\exp\left[-\tfrac{1}{4}i(1+J)\int\hat{G}_{\mu\nu}{}^2(x)d^4x\right][d\hat{A}]\,, \tag{1}$$

where $\hat{G}^a_{\mu\nu}=\partial_\mu\hat{A}^a_\nu-\partial_\nu\hat{A}^a_\mu+gf^{abc}\hat{A}^b_\mu\hat{A}^c_\nu$, $\hat{G}_{\mu\nu}{}^2\equiv\hat{G}^a_{\mu\nu}\hat{G}^a_{\mu\nu}$, and Ω is the space-time volume. The internal group is assumed to be SU(N) with the structure constants f^{abc}. Throughout the paper, except in Sec. III, the caret is used for the field operators or for the fields which are integrated out in the functional formalism. The fields without the caret are c-number quantities. It is known[10] that given a Lagrangian of the form

$$\hat{\mathfrak{L}}_{J,g}(x)=-\tfrac{1}{4}(1+J)\hat{G}_{\mu\nu}{}^2(x)\,, \tag{2}$$

we need a $\delta^4(0)$ term in the functional integrand in order to reproduce the correct perturbation series. Thus (1) is modified to

$$e^{i\Omega W[J]}=\int\exp\left[i\int\hat{\mathfrak{L}}_{J,g}(x)d^4x+\tfrac{1}{2}\delta^4(0)\Omega\ln(1+J)\right][d\hat{A}]\,. \tag{3}$$

The gauge we choose in this paper is the axial gauge or the Lorentz-type background gauge in which $Z_1=Z_3$ holds. In the latter gauge ghost fields must be introduced. We suppress these gauge terms for simplicity because they do not affect our discussions below.

A. Defining g_J

By the change of integration variables $(1+J)^{1/2}\hat{A}^a_\mu \to \hat{A}^a_\mu$, we rewrite (1) as

$$e^{i\Omega W[J]}=\int \exp\left[i\int \hat{\mathcal{L}}_{0,g_J}(x)d^4x\right][d\hat{A}], \quad (4)$$

where

$$\hat{\mathcal{L}}_{0,g_J}(x)=-\tfrac{1}{4}(\partial_\mu \hat{A}^a_\nu-\partial_\nu\hat{A}^a_\mu+g_Jf^{abc}\hat{A}^b_\mu\hat{A}^c_\nu)^2(x), \quad (5)$$

$$g_J=g/(1+J)^{1/2}. \quad (6)$$

Such a transformation has been used by Kluberg-Stern and Zuber[5] in their discussion of the insertion of $\hat{L}\equiv\int\hat{\mathcal{L}}(x)d^4x$. Now instead of making the theory finite governed by the Lagrangian of (2), we can make the equivalent theory (4) finite. So the renormalized coupling constant is introduced as usual:

$$g^r_J=[Z(g^r_J,\Lambda/\mu)]^{1/2}g_J=\sqrt{Z}\,g/(1+J)^{1/2}. \quad (7)$$

We define g^r as

$$g^r\equiv g^r_{J=0}=[Z(g^r,\Lambda/\mu)]^{1/2}g. \quad (8)$$

Here $\sqrt{Z}=Z_1^{-1}Z_3^{3/2}=\sqrt{Z_3}$ in the usual notation and Λ is the cutoff and μ the subtraction point. It is not necessary to give the precise renormalization scheme to fix Z.

In order to discuss the expectation value of $\hat{G}_{\mu\nu}{}^2$, d/dJ is applied to (1) but this produces extra infinities due to the hard character of $\hat{G}_{\mu\nu}{}^2$. The source J should thus be renormalized as

$$J=J^rZ_G(g^r,J^r,\Lambda/\mu). \quad (9)$$

In general Z_G depends on J^r, which is easily seen perturbatively (see below). From (7) and (8), bare quantities are eliminated to given

$$1+JZ_G(g,J,\Lambda/\mu)=\frac{g^2}{g_J{}^2}\frac{Z(g_J,\Lambda/\mu)}{Z(g,\Lambda/\mu)}, \quad (10)$$

where we have suppressed the superscript r because only the renormalized quantities are used from now on. Z_G is chosen in such a way that g_J does not have Λ/μ dependence. Now in order to see the perturbative structure we expand as

$$\begin{aligned} g_J{}^2&=\frac{g^2}{1+J}+f_1(J)g^4+f_2(J)g^6+\cdots,\\ Z&=1+Z^{(1)}g^2+Z^{(2)}g^4+\cdots,\\ Z_G&=1+Z_G^{(1)}g^2+Z_G^{(2)}g^4+\cdots. \end{aligned} \quad (11)$$

From (10) it is seen that

$$\begin{aligned} Z_G^{(1)}&=-Z^{(1)}-f_1(J)(1+J)^2/J,\\ Z_G^{(2)}&=\frac{1+J}{J}\left[-(1+J)f_2(J)+(1+J)^2f_1^2(J)+(1+J)f_1(J)Z^{(1)}+\left(1-\frac{1}{(1+J)^2}\right)Z^{(2)}+\frac{J}{1+J}Z^{(1)^2}\right], \end{aligned} \quad (12)$$

and so on. Thus Z_G can be chosen to be J independent up to the order g^2. But for higher orders Z_G contains J-dependent infinities. Also higher-order terms of $g_J{}^2$, $f_i(J)$ $(i\geq 1)$, depend on the renormalization prescription of $\hat{L}$. There is, however, a natural choice of renormalization conditions which make all $f_i(J)$ $(i\geq 1)$ vanish. It is a generalization of the scheme discussed by Kluberg-Stern and Zuber,[5] who discussed one insertion of $\hat{L}$.

We define n insertion of renormalized $\hat{L}$ into the propagator by

$$\pi^{(n)}(xy)^{ab}_{\mu\nu}\equiv(D_{J=0}{}^{-1})(xx')^{aa'}_{\mu\sigma}\left[\left(\frac{d}{dJ}\right)^nD_J(x'y')^{a'b'}_{\sigma\rho}\right]_{J=0}(D_{J=0}{}^{-1})(y'y)^{b'b}_{\rho\nu}, \quad (13)$$

where $D_J(xy)^{ab}_{\mu\nu}$ is the propagator $\langle\hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\rangle$ of the theory governed by $\hat{\mathcal{L}}_{J,g}$ of (2) where $1+J$ is replaced by $1+JZ_G$. New divergences appear for each n so we can impose a renormalization condition for (13). In Fourier space $\pi^{(n)ab}_{\mu\nu}$ has the form

$$\pi^{(n)}(q)^{ab}_{\mu\nu}=\pi^{(n)}(q^2)q^2\delta^{ab}g_{\mu\nu}+\text{gauge term}.$$

We choose

$$\pi^{(n)}(q^2=\mu^2)=(-1)^nn!. \quad (14)$$

The factor $(-1)^nn!$ is taken from the tree contributions. [For $n=0$, (14) is the renormalization condition determining Z_3.]

Now in an equivalent theory governed by $\hat{\mathcal{L}}_{J=0,g_J}$ of (5), the propagator $D_{g_J}(xy)^{ab}_{\mu\nu}=\langle\hat{A}^a_{\mu J}(x)\hat{A}^b_{\nu J}(y)\rangle$ is defined in the same way as (13) and (14) with $n=0$. Here $\hat{A}^a_{\mu J}$ is the renormalized field and contains a factor $(1+JZ_G)^{1/2}/\sqrt{Z}$. Thus $g_J\hat{A}^a_{\mu J}=g\hat{A}^a_\mu$ and hence the inverse propagator $D_J{}^{-1}(x,y)^{ab}_{\mu\nu}$ for finite J is related to $D_{g_J}{}^{-1}(xy)^{ab}_{\mu\nu}$ by $D_J{}^{-1}=D_{g_J}{}^{-1}g^2/g_J{}^2$. With the above renormalization condition for $D_{g_J}{}^{-1}$, and writing the Fourier transform of $D_J{}^{-1}(x,y)^{ab}_{\mu\nu}$ as $\pi_J(q^2)q^2\delta^{ab}g_{\mu\nu}+$ gauge term we have

$$\pi_J(q^2=\mu^2)=g^2/g_J{}^2. \quad (15)$$

It is easy to see that (14) requires $d^n\pi_J(q^2=\mu^2)/d^nJ|_{J=0}$ to be unity for $n=0$, 1 and zero for $n>1$. This leads to $f_i=0$ $(i\geq 1)$, i.e.,

$$g_J{}^2 = \frac{g^2}{1+J}. \tag{16}$$

Z_G is fixed by (12) order by order setting $f_i = 0$ in the equation.

As has been stated, if we change the renormalization scheme of $\hat{L}$, then f_i will be changed but the lowest-order relation of g_J in (11), i.e., (16), is unchanged. Thus if we restrict ourselves to small g then the relation (16) is renormalization prescription invariant.

If we apply d/dJ to (10) and set $J=0$, the result of Ref. 5 is reproduced,

$$Z_G(g, 0, \Lambda/\mu) = 1 - (g/2Z)\partial Z/\partial g, \tag{17}$$

and hence the anomalous dimension of $\hat{L}$ is given by

$$\gamma_G(g) = (1/Z_G)\mu d Z_G/d\mu = -gd(\beta/g)/dg, \tag{18}$$

where

$$\beta(g) = \mu dg/d\mu = b_0 g^3 + b_1 g^5 + \cdots. \tag{19}$$

In the above discussion, the problem of operator mixing is not discussed with the hope that such a mixing does not affect the physical quantities to be discussed in the following.

B. Nonperturbative condensation of $\hat{G}_{\mu\nu}{}^2$

Having obtained a finite g_J, we discuss in this subsection the condensation of $\hat{L} = \int \hat{\mathfrak{L}}(x) d^4x$. For this purpose we need renormalized $W[J]$, which is a sum of vacuum graphs in the presence of the source J. They are quartically divergent in perturbation theory. In familiar examples where the source couples to soft operators such as the scalar boson field ϕ, these quartic divergences can be subtracted by taking the difference of $W[J]$ and $W[J=0]$. In this case the J-independent subtraction makes $\Delta W[J] \equiv W[J] - W[J=0]$ finite: We can apply the usual renormalization scheme to $\Delta W[J]$ if the theory is renormalizable at all.

In our case, however, J couples to the hard operator $\hat{G}_{\mu\nu}{}^2$ and J-independent subtraction does not work. In order to discuss this problem, we temporarily introduce the source term $J^a_\mu(x)$ coupled to the gauge field $\hat{A}^a_\mu$ and consider $W[J, J^a_\mu]$. For fixed J, $V[J, A^a_\mu]$ is defined by the Legendre transform

$$V[J,A^a_\mu] = -W[J, J^a_\mu] + \int J^a_\mu(x) \frac{\delta W[J, J^a_\mu]}{\delta J^a_\mu(x)} d^4x, \tag{20}$$

$$\frac{\delta W}{\delta J^a_\mu(x)} = A^a_\mu(x).$$

To render V finite, we subtract $V_{\text{pert}}[J, A^a_\mu = 0]$ from $V[J, A^a_\mu]$ where $V_{\text{pert}}[J, A^a_\mu = 0]$ is the energy of the perturbative vacuum in the presence of J. It is a sum of one particle irreducible (1PI) vacuum graphs calculated perturbatively and hence it is quartically divergent. The difference ΔV,

$$\Delta V[J, A^a_\mu] = V[J, A^a_\mu] - V_{\text{pert}}[J, A^a_\mu = 0], \tag{21}$$

when expanded around $A^a_\mu = 0$, can be made finite by the usual renormalization scheme at least in perturbation theory so it satisfies the RGE. We perform the usual renormalization after the scale change $(1+J)^{1/2} A_\mu = A_{\mu J}$ and $g/(1+J)^{1/2} = g_J$ so that ΔV is a function of g_J, $A_{\mu J}$, and μ and satisfies

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g_J) \frac{\partial}{\partial g_J} - \gamma(g_J) \int A^a_{\mu J}(x) \frac{\delta}{\delta A^a_{\mu J}(x)} d^4x\right) \Delta V = 0. \tag{22}$$

Now we can easily see that $A^a_{\mu J}\, \delta/\delta A^a_{\mu J} = A^a_\mu \delta/\delta A^a_\mu$ [see (60) below]. Thus at the stationary point $[\delta\Delta V/\delta A^a_\mu(x) = J^a_\mu(x) = 0]$, provided a stationary point other than the perturbative vacuum state exists, ΔV is given by a nontrivial solution of

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g_J) \frac{\partial}{\partial g_J}\right) \Delta V(g_J, \mu) = 0, \tag{23}$$

where

$$\Omega \Delta V(g_J, \mu) \equiv \Delta V[J, A^a_\mu] \,|_{\delta\Delta V/\delta A^a_\mu = 0}.$$

The reason we have introduced J^a_μ or A^a_μ is twofold. The first is, as explained above, to define the quantity which is finite after renormalization at least in the perturbative sense. The second is that we want to introduce quarks in Sec. IV. In that case the equation to be solved is changed into the form $\delta\Delta V/\delta A^a_\mu = j^a_\mu$.

Equation (23) can be written down directly if we apply the argument given by Gross and Neveu[11] that any physical quantity should be independent of the renormalization point. Our $\Delta V(g_J, \mu)$ is the difference of the energy density of nonperturbative (if it exists) and perturbative vacuum so it is expected to be a physical quantity. In this paper we assume that a nontrivial solution to (23) exists. The reason we believe this has been given in the Introduction.

Now we define $\Delta\phi$ and $\Delta\phi_c$ by

$$\Delta\phi = \frac{\partial \Delta V(g_J, \mu)}{\partial J}, \quad \Delta\phi_c = \Delta\phi\,|_{J=0}, \tag{24}$$

which is the difference of the expectation value of $-(1/\Omega)\hat{L} = (1/\Omega)\int \hat{G}_{\mu\nu}{}^2(x) d^4x$ measured in the nonperturbative vacuum state and the perturbative vacuum state. In this sense we write

$$\Delta\phi = \frac{1}{\Omega}\left\langle \tfrac{1}{4}{:}\int \hat{G}_{\mu\nu}{}^2(x)d^4x{:}\right\rangle = \langle \tfrac{1}{4}{:}\hat{G}_{\mu\nu}{}^2(x){:}\rangle \ . \tag{25}$$

From (16), $\partial/\partial g_J = -2(g^2/g_J{}^3)\partial/\partial J$, so that by (23),

$$\mu\frac{\partial}{\partial\mu}\Delta V(g,\mu) = 4\Delta V(g,\mu) = 2\frac{\beta(g)}{g}\Delta\phi_c\ , \tag{26}$$

where the fact that $\Delta V \propto \mu^4$ has been used. We restrict ourselves for small g where we know that β is negative, $b_0 < 0$.

From (26) we reach the following conclusion: Nonperturbative magnetic condensating of $\hat{G}_{\mu\nu}{}^2$ leads to a nonperturbative vacuum which has lower energy than the perturbative one.

By magnetic we mean $\Delta\phi_c > 0$, i.e., $\Delta\phi_c = \langle \frac{1}{2}{:}\hat{H}^2 - \hat{E}^2{:}\rangle > 0$. The sign of $\Delta\phi_c$ will play an important role when quarks are introduced in Sec. IV and it agrees with the sign obtained by Shifman, Veinstein, and Zakharov[12] from the analysis of their sum rules. The RGE gives a definite relation (26) between the order parameter $\Delta\phi$ and the energy density ΔV, which is the case because the order parameter is the Lagrangian itself. This is not the case for other order parameters.

ΔV or $\Delta\phi$ is complex in general reflecting the decay of the vacuum of higher energy. For this case we take the real part of (26). $\Delta\phi$ or ΔV behaves as $e^{2/b_0 g^2}$ as $g\to 0$. *The condensation occurs for arbitrarily small coupling so that the critical coupling constant g_c is zero.* $\Delta\phi$ satisfies the correct RGE as is seen by applying $\mu\partial/\partial\mu + \beta(g)\partial/\partial g$ to (26),

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_G(g)\right)\Delta\phi = 0\,, \tag{26'}$$

with $\gamma_G(g)$ given by (18).

We now define the effective potential $\Delta V(\Delta\phi)$ of $\Delta\phi$ by the Legendre transform of $\Delta V(g_J,\mu)$. It is a difference of energy densities as a function of the difference of $\langle\frac{1}{4}\hat{G}_{\mu\nu}{}^2\rangle$. It is easy to get $\Delta V(\Delta\phi) = \Delta V(g_J,\mu) - J\partial\Delta V/\partial J$, with $\Delta\phi = \partial\Delta V(g_J,\mu)/\partial J$. In the magnetic region $\Delta\phi > 0$, it takes the form

$$\Delta V(\Delta\phi) = \frac{b_0 g^2}{2}\Delta\phi\left[C - \ln\left(\frac{-b_0 g^2}{\mu^4}\Delta\phi\right) + \frac{2}{b_0 g^2}\right],$$

with C some finite constant. In Fig. 1, we plot $\Delta V(\Delta\phi)$.

In Appendix A, we discuss O(N) $\lambda\phi^4$ theory with negative renormalized λ in the large-N limit.[13] To illustrate the method we have used, that is, to absorb J dependence into the coupling constant, the condensation of the Lagrangian is discussed.

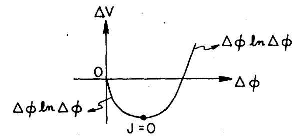

FIG. 1. The effective potential $\Delta V(\Delta\phi)$.

There it is seen that the Lagrangian indeed condenses with the "magnetic" sign $\langle -{:}\mathcal{L}{:}\rangle > 0$ in agreement with our result of QCD. We have also examined the Gross-Neveu model,[11] the two-dimensional massless four-Fermi interaction, in the large-N limit. This model again shows the nonperturbative condensation of the Lagrangian with magnetic sign.

Any type of condensation is surely an infrared effect. In the pairing approach,[3,4] we can see explicitly that it is a dynamical effect of infrared gluons. In the present formal approach this point is not clear. However, the masslessness of the gluons plays an essential role in the present discussions too: It makes the RGE simple (homogeneous) and solvable by an elementary integration.

III. THE TACHYONIC SINGULARITY

The purpose of this section is to show that the nonperturbative condensation of any composite operator implies the existence of tachyonic (spacelike) singularities in the relevant channel of the Green's functions calculated in the normal vacuum.

In Sec. II, the condensation of $\hat{G}_{\mu\nu}{}^2$ has been discussed. It involves the color-singlet $J^P = 0^+$ composite operators in $\hat{A}^a_\mu$ up to fourth order. Naively we expect the appearance of 0^+ tachyonic singularities in the color-singlet channel of the Green's functions if we calculate them in the normal vacuum. This problem has been discussed by Kugo[14] in the ladder approximation for a specific type of interaction. Generalizing his arguments, we discuss the problem to all orders for any type of interaction.

The Fourier components of any real boson field are denoted by $\hat{\phi}_i(p)$. Here the index i represents all the attributes of the field except for the momentum (Lorentz or internal-group indices) and the Wick rotation in momentum space is assumed. We base our arguments on the effective action for composite operators up to fourth order derived by De Dominicis and Martin[15] and investigate its eigenspectrum around the stationary solution. It is known that the stationary equations are Schwinger-Dyson (SD) equations and the eigenvalue equations are BS equations. We proceed step by step and discuss first the case of the two-body operator and then proceed to higher operators.

A. Two-body operator

Consider $\langle\hat{\phi}_i(p)\hat{\phi}_j(q)\rangle = G_{ij}(p,q)$ and write the effective action Γ for G (Ref. 15),

$$\Gamma(G) = -\tfrac{1}{2}\,\mathrm{Tr}\ln G^{-1}G_0 - \tfrac{1}{2}\mathrm{Tr}\,G\,G_0^{-1} + \Gamma^{(2)}(G)\,, \tag{27}$$

where we have suppressed all the indices so that Tr is over indices (i,j) and the momentum. The indices are recovered whenever necessary. $\Gamma^{(2)}$ in (27) is the two-particle-irreducible vacuum graph with G for the internal line. G_0 is the free propagator. The stationary condition is the SD equation for the propagator,

$$G^{-1} - G_0^{-1} = 2\delta\Gamma^{(2)}(G)/\delta G\,. \tag{28}$$

Note that $\delta\Gamma^{(2)}/\delta G$ represents the complete proper, i.e., 1PI, self-energy part. The solution of (28) is written as G_s. In order to discuss the stability of the solution G_s, we write $G = G_s + \delta G$ and keep the term up to the second order in δG,

$$\Gamma(G) \simeq \Gamma(G_s) + \tfrac{1}{2}\,\delta G M \delta G\,, \tag{29}$$

$$M = -G_s^{-1}G_s^{-1} + K^{(2)},\quad K^{(2)} \equiv \left.\frac{\delta^2\Gamma^{(2)}}{\delta G\delta G}\right|_{G=G_s}. \tag{30}$$

For the discussion of the eigenspectrum of M, note that M is already diagonal in the total momentum P due to the translational invariance of the vacuum. Explicitly

$$\delta G M \delta G = \sum_{\substack{P,p,q,\\ ijmn}} \delta G_{ij}(\tfrac{1}{2}P+p, \tfrac{1}{2}P-p) M_{ij,mn}(P,p,q) \times \delta G_{mn}(\tfrac{1}{2}P+q, \tfrac{1}{2}P-q)\,.$$

In order to diagonalize in the relative momenta and in the indices (i,j), observe that $K^{(2)}$ is nothing but the BS kernel: It is the complete two-particle-irreducible connected four-point Green's function with the internal line G_s. In the usual BS equation we discuss the spectrum of the coupling constant rather than the spectrum of the total momentum (energy). So we introduce, following Kugo,[14] a coupling constant λ as a measure of the magnitude of the kernel $K^{(2)}$,

$$K^{(2)} = \lambda\tilde{K}^{(2)},\quad \tilde{K}^{(2)} = \frac{1}{\lambda}K^{(2)}\,. \tag{31}$$

λ is assumed to be positive to give an attractive force. Now M in (30) can be diagonalized by the BS equation

$$G_s^{-1}G_s^{-1}\chi_n = \lambda_n(P^2)\tilde{K}^{(2)}\chi_n\,. \tag{32}$$

The normalization of χ_n and the orthogonality relations of χ_n or of $\hat{\chi}_n \equiv G_s^{-1}G_s^{-1}\chi_n$ are

$$\chi_n^\dagger G_s^{-1}G_s^{-1}\chi_m = \hat{\chi}_n^\dagger G_s G_s \hat{\chi}_m = [\lambda/\lambda_n(P^2)]\,\delta_{nm}\,. \tag{33}$$

$\hat{\chi}_n$ satisfies

$$\hat{\chi}_n = \lambda_n(P^2)\tilde{K}^{(2)}G_s G_s\hat{\chi}_n\,, \tag{34}$$

$$\hat{\chi}_n^\dagger \tilde{K}^{(2)-1}\hat{\chi}_m = \lambda\delta_{nm}\,. \tag{35}$$

The above normalization has been chosen because, in the massless theory in which we are interested, it is $\hat{\chi}$ that has a finite value as $P_\mu \to 0$. Thus $\lambda_n(P^2)G_sG_s \to O(1)$ as $P_\mu \to 0$.

Expanding δG in normal modes

$$\delta G_{ij}(\tfrac{1}{2}P+p, \tfrac{1}{2}P-p) = \sum A_n(P)\,\chi_{n,ij}(\tfrac{1}{2}P+p, \tfrac{1}{2}P-p) \times \frac{\lambda_n(P^2)}{\lambda}\,, \tag{36}$$

or

$$\delta G = \sum_n A_n(P) G_s G_s \hat{\chi}_n \frac{\lambda_n(P^2)}{\lambda}\,,$$

we get from (34) or (35),

$$\tfrac{1}{2}\delta G M \delta G = -\tfrac{1}{2}\sum_n A_n^\dagger(P)\,\frac{\lambda_n(P^2)-\lambda}{\lambda}\,A_n(P)\,. \tag{37}$$

The effective potential V, representing the vacuum energy density, is the negative of the $P_\mu = 0$ mode of the effective action so

$$V = -\left.\Gamma(G_s)\right|_{p=0} + \tfrac{1}{2}\sum A_n^\dagger(0)\,\frac{\lambda_n(0)-\lambda}{\lambda}\,A_n(0)\,. \tag{38}$$

The factor $\lambda_n(P^2)/\lambda$ has been extracted in (36) so that $A_n(P)$ is $O(1)$ as $P_\mu \to 0$. For the solution G_s to represent the stable solution, $\lambda_n(0)$ should satisfy

$$\lambda_n(0) - \lambda \geq 0 \quad (\text{for all } n)\,, \tag{39}$$

which is guaranteed if the lowest solutions $\lambda_{n=0}$ satisfies

$$\lambda_{n=0}(0) - \lambda \geq 0\,. \tag{40}$$

The relation between (40) and the presence or absence of the tachyonic singularities has been discussed in Ref. 14. Here we consider the following problem which is our concern in this section. If we know that some two-body operator shows nonperturbative condensation, can we conclude the existence of the tachyonic spectrum in $\lambda_n(P^2)$ if we take the normal perturbative solution of the SD equations?

Now we assume that a two-body operator $\hat{O}^{(2)} = \sum C_{ij}\hat{\phi}_i\hat{\phi}_j$ shows the nonperturbative condensation. Here $\hat{O}^{(2)}$ can be local or nonlocal. The effective potential $V(O^{(2)})$ for $O^{(2)} = \langle\hat{O}^{(2)}\rangle = \sum C_{ij}G_{ij}$ can be constructed from that of G_{ij} and the stationary value $O_s^{(2)}$ is given by $O_s^{(2)} = \sum C_{ij}(G_{ij})_s$. We take the normal solution for G_s so that $O_s^{(2)}$ does not realize a minimum of $V(O^{(2)})$. Suppose (39) is satisfied for all n. $V(O^{(2)})$ is obtained from the effective potential $V(G)$ if the G's are restricted in a particular direction in G space specified by C_{ij}. So $O_s^{(2)}$ is a minimum solution of $V(O^{(2)})$ because (39) tells us that in G space $V(G)$ does not decrease in any direction around the solution G_s. We conclude that

$$\lambda_{n=0}(0) - \lambda < 0 . \tag{41}$$

In the next subsection the condensation of the composite operator $\hat{O}^{(4)}$ of up to fourth order in the field is discussed. There we get the same condition (41) for the spectrum of the normal vacuum if $\hat{O}^{(4)}$ condenses nonperturbatively. Taking this result in advance, we now discuss the case of QCD. The fact that (41) leads to the existence of the tachyonic bound state has been shown in Ref. 14 for the massive theory. Here we will see that it is also the case for QCD.

From Sec. II we know that $\hat{G}_{\mu\nu}{}^2$ condenses for arbitrarily small coupling λ. (We identify λ in this subsection with g^2 of QCD.) It means

$$\lambda_{n=0}(0) = 0 . \tag{42}$$

The only assumption we need to prove the statement of this section is that as the coupling constant is increased the binding energy of the bound state is increased. There is no rigorous proof of this statement, but any physically sensible solution to the BS equation is known to enjoy this property. Then as λ_n is increased P^2 moves toward the spacelike region and from (41) we conclude that the trajectory $\lambda_{n=0}(P^2)$ becomes tachyonic for $\lambda_n > 0$. In Fig. 2(a) schematic form of $\lambda_n(P^2)$ is given. Setting $\lambda_n = \lambda$ and $n=0$ in (32) it is seen that for arbitrarily small coupling constant a techyonic bound state is formed. It has the quantum numbers of the color singlet and $J^P = 0^+$ which produces a spacelike pole in the normal Green's functions in the color-singlet channel. In general it also has a branch point due to, for example, the contribution of the graph shown in Fig. 3.

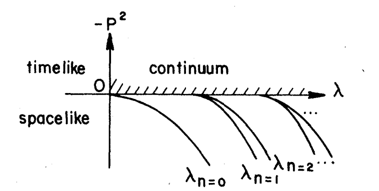

FIG. 2. The typical behavior of the trajectory $\lambda_n(P^2)$.

Thus we conclude that any Green's function evaluated in the normal vacuum has an imaginary part if all the external momenta are set equal to zero. The asymptotic state of a normal gluon does not exist because it forms a color-singlet tachyonic bound state and condenses in the vacuum simply because it is energetically favorable.

We mention here the result of the ladder calculations,[4] where the solution to the color-singlet tachyonic bound state has been explicitly given. In this approximation the equation $\lambda = \lambda_{n=0}(P^2)$ is shown to give $P^2 = \Lambda^2 e^{-c/\lambda}$ ($c > 0$) for small P^2, where Λ is the cutoff and λ is identified with g^2. The tachyon bound state exists for arbitrarily small coupling constant. It agrees with the conclusion of this subsection and of Sec. II based on nonperturbative arguments.

B. Inclusion of three- and four-body operators

We define, following De Dominicis and Martin,[15] the effective action $\Gamma(G, C^{(3)}, C^{(4)})$ where

$$\langle \phi_i(p)\phi_j(q)\phi_k(r)\rangle = G_{ii'}(p,p')G_{j,j'}(q,q')G_{k,k'}(r,r')C^{(3)}_{i'j'k'}(p',q',r') , \tag{43}$$

$$\langle \phi_i(p)\phi_j(q)\phi_k(r)\phi_l(s)\rangle = G_{ij}(p,q)G_{kl}(r,s) + \text{perm} + G_{ii'}(p,p')G_{jj'}(q,q')G_{kk'}(r,r')G_{ll'}(s,s')C^{(4)}_{i'j'k'l'}(p',q',r',s') . \tag{44}$$

$C^{(3)}$ and $C^{(4)}$ are the connected part of the Green's functions with external legs deleted. Γ has the form as shown in Ref. 15,

$$\Gamma(G, C^{(3)}, C^{(4)}) = -\tfrac{1}{2}\operatorname{Tr}\ln G^{-1}G_0 - \tfrac{1}{2}\operatorname{Tr}GG_0{}^{-1} - \frac{1}{3!}C_0^{(3)}GGGC^{(3)} - \frac{1}{2}\frac{1}{3!}C^{(3)}GGGC^{(3)} - \frac{1}{4!}C_0^{(4)}GGGC^{(4)} - \frac{1}{2}\frac{1}{4!}C^{(4)}GGGC^{(4)} + \Gamma^{(4)}(G, C^{(3)}, C^{(4)}) , \tag{45}$$

where $C_0^{(3)}$ and $C_0^{(4)}$ are the bare three- and four-particle vertices, respectively. $\Gamma^{(4)}$ represents essentially the four-particle-irreducible vacuum graph with internal line replaced by G and three- and four-particle vertices by $C^{(3)}$ and $C^{(4)}$, respectively. $\Gamma^{(4)}$ contains extra diagrams to avoid multiple counting of the vacuum graph but we need not specify $\Gamma^{(4)}$ explicitly. The stationary re-

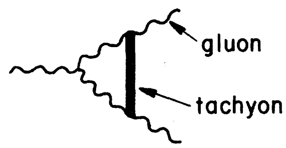

FIG. 3. An example of the diagram which gives rise to the branch point in Green's function in the spacelike region of the momentum. The wavy line represents a gluon and the solid line a color-singlet tachyon bound state.

quirements $\delta\Gamma^{(4)}/\delta G=\delta\Gamma^{(4)}/\delta C^{(3)}=\delta\Gamma^{(4)}/\delta C^{(4)}=0$ reproduce SD equations for G, $C^{(3)}$, and $C^{(4)}$, the solution of which we denote by G_s, $C_s^{(3)}$, and $C_s^{(4)}$. Writing $G=G_s+\delta G$, $C^{(3)}=C_s^{(3)}+\delta C^{(3)}$, $C^{(4)}=C_s^{(4)}+\delta C^{(4)}$ and defining $C_i=(G,C^{(3)},C^{(4)})$, $\delta C_i=(\delta G,\delta C^{(3)},\delta C^{(4)})$, and $C_{is}=(G_s,C_s^{(3)},C_s^{(4)})$ with $i=1,2,3$, Γ becomes

$$\Gamma(C_i)\simeq\Gamma(C_{is})+\tfrac{1}{2}\delta C_i M^{ij}\delta C_j\,. \tag{46}$$

Explicitly for example

$$M^{23}=\delta^2\,\Gamma/\delta C^{(3)}\delta C^{(4)}\,,$$

$$\delta C_2 M^{23}\delta C_3=\delta C^{(3)}_{ijk}(P,pq)M_{ijk,lmnv}(P,pq,rst)\times\delta C^{(4)}_{lmnv}(P,rst)\,.$$

We have used the translational invariance of the vacuum and so (46) is already diagonal in P. For the diagonalization of (46), the following coupled 3×3 BS equations are solved:

$$\hat{G}\chi_n=\lambda_n\tilde{K}\chi_n\,, \tag{47}$$

where $\hat{G}$ is diagonal and $\hat{G}_{11}=G_s^{-1}G_s^{-1}$, $\hat{G}_{22}=G_sG_sG_s$, and $\hat{G}_{33}=G_sG_sG_sG_s$. $\tilde{K}$ is given by

$$\tilde{K}_{ij}=(1/\lambda)\delta^2\Gamma^{(4)}/\delta C_i\delta C_j\,\big|_{C_i=C_{is}}$$

and $\chi_n=(\chi_n^{(1)},\chi_n^{(2)},\chi_n^{(3)})$. We have introduced $\tilde{K}$ which is related to M by $M=-\hat{G}+K$, $\tilde{K}=(1/\lambda)K$. The kernel K or $\tilde{K}$ has the property of four-particle irreducibility. (It is not irreducible in the sense of Faddeev.[16]) The orthogonality relations are

$$\chi_n^\dagger\hat{G}\chi_m=[\lambda/\lambda_n(P^2)]\,\delta_{nm}\,. \tag{48}$$

As in the two-body case we expand δC_i in χ_n,

$$\delta C_i=\sum_n A_n(P)\chi_n^{(i)}\lambda_n(P^2)/\lambda\,. \tag{49}$$

Then (46) becomes

$$\Gamma(C_i)=\Gamma(C_{is})+\tfrac{1}{2}\sum_n A_n^\dagger\left(\frac{\lambda-\lambda_n}{\lambda}\right)A_n\,.$$

For the effective potential V it becomes

$$V(C_i)=V(C_{is})+\tfrac{1}{2}\sum_n A_n^\dagger\left(\frac{\lambda_n(P^2=0)-\lambda}{\lambda}\right)A_n\,, \tag{50}$$

so that the stability of the solution C_{is} requires (39).

Now suppose we know that a composite operator $\hat{O}^{(4)}$ involving up to fourth order of the field $\hat{\phi}_i$ shows the nonperturbative condensation. In terms of $\hat{\phi}_i$, $\hat{O}^{(4)}$ can be written as

$$\hat{O}^{(4)}=\sum a_{ij}\hat{\phi}_i\hat{\phi}_j+\sum a_{ijk}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k+\sum a_{ijkl}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k\hat{\phi}_l\,. \tag{51}$$

The effective potential $V(O^{(4)})$ of $O^{(4)}=\langle\hat{O}^{(4)}\rangle$, which is a function of C_i by (51), can be obtained by $V(C_i)$ of (50). In particular, the stationary point $O_s^{(4)}$ of $V(O^{(4)})$ is determined by that of $V(C_i)$. The small oscillation of $O^{(4)}$ around $O^{(4)}=O_s^{(4)}$ can be written as a linear combination of the small oscillation of C_i around $C_i=C_{is}$. Thus if (39) holds $O^{(4)}=O_s^{(4)}$ is a minimum point (at least locally) of $V(O_s^{(4)})$. Taking $O_s^{(4)}$ as a normal solution leads to the contradiction so that (41) must hold.

IV. THE PRESENCE OF COLOR ELECTRIC FIELD

A. The local expansion

Up to now we have discussed the situation where color sources J_μ^a are absent. In Sec. II, we have seen that for $J_\mu^a=0$ there are two kinds of vacuums ($J=0$) satisfying $\Delta\phi=0$ or $\Delta\phi=\Delta\phi_c>0$. The former solution corresponds to the normal perturbative vacuum which also satisfies $A_\mu^a=0$. [Recall the definition of $\Delta\phi$ given in (24).] In this section the color electromagnetic properties of the condensed vacuum satisfying $\Delta\phi=\Delta\phi_c$ are discussed. The condensed nonperturbative vacuum is filled with gluons forming a color-singlet composite state so that it will have a color electromagnetic property substantially different from the normal vacuum. For the strong color electric field (near the source, i.e., quark) we know from perturbation theory that the vacuum has an antishielding property because of asymptotic freedom. For the small electric field (away from the quark), nonperturbative condensation of $\Delta\phi$ is expected to play an important role.

In the presence of a color source, we need the effective action $\Gamma(\phi,A_\mu)$ of two variables ϕ and A_μ, which is defined in the following way. (We neglect for the moment the renormalization problem and also use ϕ instead of $\Delta\phi$ for simplicity.) We introduce $W(J,J_\mu)$ by

$$e^{iW(J,J_\mu)}=\int\exp\left[-\tfrac{1}{4}i\int d^4x(1+J(x))\,\hat{G}_{\mu\nu}{}^2(x)+i\int d^4x\,J_\mu^a(x)\hat{A}_\mu^a(x)\right][d\hat{A}]\,, \tag{52}$$

and $\Gamma(\phi, A_\mu)$ by

$$\Gamma(\phi, A_\mu) = W(J, J_\mu) - \int d^4x\, J(x) \frac{\delta W}{\delta J(x)} - \int d^4x\, J^a_\mu(x) \frac{\delta W}{\delta J^a_\mu(x)}, \quad \phi(x) = -\frac{\delta W}{\delta J(x)}, \quad A^a_\mu(x) = \frac{\delta W}{\delta J^a_\mu(x)}. \tag{53}$$

Because we are interested in long-range phenomena, we need the expansion of Γ suitable for the study of the soft region. For this purpose Γ is first expanded around $\phi = A^a_\mu = 0$, the coefficients being the Green's functions which are 1PI in the field A_μ. In order to obtain a relevant series, these Green's functions are expanded around zero momentum and then terms with the same number of powers of momenta are summed up. Each term of this expansion suffers from infrared divergences in perturbation theory so that only the sum has a meaning. The situation is the same as Coleman and Weinberg's[17] discussions on the massless $\lambda\phi^4$ theory. As was pointed out by them, zero-momentum expansion yields a local expansion in x space.

The gauge we choose in this section is the background Lorentz gauge[18]

$$D^{ab}_\mu \hat{A}'^b_\mu \equiv (\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu)\hat{A}^{b\prime}_\mu = 0,$$

where $\hat{A}'_\mu$ is the quantum part of the gauge field. In this gauge (with corresponding ghost interaction of course) Γ still has a local gauge invariance and we get a gauge-invariant local expansion of Γ,

$$\Gamma(\phi, A_\mu) = \int d^4x\, \Gamma^{(0,0)}(\phi(x), K_i(x)) + \int d^4x\, \partial_\mu\phi\partial_\mu\phi(x)\,\Gamma^{(1,0)}(\phi(x), K_i(x)) + \int d^4x\, D_\mu G_{\mu\nu} D_{\mu'} G_{\mu'\nu}(x)\,\Gamma^{(0,1)}(\phi(x), K_i(x)) + \cdots, \tag{54}$$

where the $K_i(x)$'s are independent local invariants formed by A^a_μ: $K_i(x) = (G_{\mu\nu}{}^2(x), (G_{\mu\nu}\tilde{G}_{\mu\nu})^2, \cdots)$. For SU(2) the number of K_i is known to be nine[19] but we do not need their explicit form. In Sec. VI we discuss the validity of the expansion (54).

In the presence of quarks, the rule of calculating the effective action tells us that we should solve

$$\frac{\delta\Gamma}{\delta\phi(x)} = 0, \quad \frac{\delta\Gamma}{\delta A^a_\mu(x)} = j^a_\mu(x), \tag{55}$$

where $j^a_\mu(x)$ represents the quark source, which we take to be

$$j^a_\mu(x) = \delta_{a\bar{a}}\delta_{\mu 0}\rho(\vec{x}). \tag{56}$$

This means that quarks are assumed to be infinitely heavy and their direction in color space is specified by $\delta_{a\bar{a}}$. Specifically we put the quark and antiquark at infinity, i.e., $\rho(\vec{x}) = e\delta^2(x) \times[\delta(z-a) - \delta(z+a)]$ with $a \to \infty$. Here e represents the charge of the quark. In this case, Eq. (55) has a solution with $A^a_\mu = \delta_{\mu 0}\delta_{a\bar{a}} A$, so that $G(x) \equiv -\frac{1}{4}G_{\mu\nu}{}^2 = \frac{1}{2}E^2 = \frac{1}{2}(\vec{\nabla}A)^2$ is nonzero while all other K_i's vanish.

In the following the first term $\Gamma^{(0,0)}$ is discussed with the above Abelian configuration of A^a_μ. We neglect other terms of (54) having higher derivatives, which is justified *a posteriori*; as we will see in the following, $\Gamma^{(0,0)}$ gives us a flux-tube solution. In the limit of an infinite vortex ($a \to \infty$), E becomes $(E_x, E_y, E_z) = (0, 0, E)$ with E constant throughout the whole space. For this solution the terms $\Gamma^{(i,j)}$ with $j \geq 1$ vanish and the terms $\Gamma^{(i,0)}$ with $i \geq 1$ contribute to the surface energy of the flux tube and give nonzero thickness to the skin region. To the extent that we neglect surface energy our solution becomes exact in the limit of an infinite tube.

The above procedure of taking only the term $\Gamma^{(0,0)}$ can be reinterpreted as follows. We just calculate the effective potential $\Gamma(\phi, A^a_\mu)$ with constant ϕ and with static Abelian form for A^a_μ which gives constant electric field $\vec{E} = \vec{\nabla}A$. This involves no approximation and is calculated independent of the quarks. Now the quark and antiquark are introduced at spatial infinity. Then the flux configuration can fully be discussed in terms of the above $\Gamma^{(0,0)}$, apart from the contribution of the surface energy.

The infinite flux tube is of course unrealistic because of $q\bar{q}$ pair creation which we neglect in this paper. However, as for the gluon pair creation its effects are already included in the effective action Γ so we can discuss them in terms of the c number A^a_μ. By taking Abelian configuration of A^a_μ, as is the case in this paper, it seems that we are missing the solution in which the color flux of the source is shielded by the gluonic color charge. As has been discussed in Ref. 20 the shielding charges due to gluons are gauge dependent and they can be gauged away: There is a gauge choice in which there is no gluonic charge. The elimination of the gluonic charge has been studied in explicit examples in Ref. 20. In this gauge, the quarks are the only sources of the color. The color content of the quark system is classified *in this gauge*, and we look for the Abelian solution *in this gauge*.

There is still another complication due to the non-Abelian character of quarks. This can easily be taken into account by changing $\delta_{a\bar{a}}$ in (56) and in the solution $A_\mu^a = \delta_{\mu 0}\delta_{a\bar{a}} A$ into $\lambda^a/2$ where λ^a is the Gell-Mann matrix in the case of SU(3). The only change in the final results is to replace the square of the quark charge e^2 into $e^2\sum_a(\lambda^a/2)^2 = \frac{4}{3}e^2$. In this case the above shielding problem becomes irrelevant.

As the quark and antiquark approach each other many terms in (54) begin to contribute and in the extremely opposite case, i.e., near the quark, the expansion (54) becomes a bad one because terms with more derivatives become more important than terms with fewer derivatives.

B. The color electric property of the condensed vacuum

In order to discuss $\Gamma^{(0,0)}$, we can take the source J of ϕ to be x independent, i.e., $J(x) = J$. Then as has been discussed in Sec. II, the J dependence can be absorbed into the coupling constant g *and* the field A_μ^a. The quantity we discuss in the following is just $V(J, A_\mu^a)$ of (20) or (21) in the static Abelian configuration of A_μ^a which gives a constant electric field. It is more convenient to work in the J representation.

To be explicit we discuss here the process of absorbing J. Let J^0, A_μ^0, g^0 be unrenormalized quantities. The J^0 dependence can be absorbed by the change $(1+J^0)^{1/2}A_\mu^{a0} = A_{\mu J}^{a0}$ and $g^0/(1+J^0)^{1/2} = g_J^0$. In the background gauge, the renormalized J, $A_{\mu J}^a$, and g_J are defined by

$$A_{\mu J}^a = [Z(g_J, \Lambda/\mu)]^{-1/2} A_{\mu J}^{a0}$$

$$= (\sqrt{Z})^{-1}[1 + JZ_G(g, J, \Lambda/\mu)]^{1/2} A_\mu^{a0}, \qquad (57)$$

where g_J is defined by (10). Thus

$$G_{\mu\nu J}{}^2 \equiv (\partial_\mu A_{\nu J}^a - \partial_\nu A_{\mu J}^a + g_J f^{abc} A_{\mu J}^b A_{\nu J}^c)^2$$

$$= \frac{1 + JZ_G}{Z(g_J, \Lambda/\mu)} (\partial_\mu A_\nu^{a0} - \partial_\nu A_\mu^{a0} + g^0 f^{abc} A_\mu^{b0} A_\nu^{c0})^2,$$

$$g_J A_{\mu J}^a = g^0 A_\mu^{a0}$$

and hence

$$g_J{}^2 G_J = g^2 G \qquad (58)$$

or

$$G_J = (1+J)G. \qquad (59)$$

More generally,

$$A_{\mu J}^a = (1+J)^{1/2} A_\mu^a. \qquad (60)$$

We have defined $G_J = -\frac{1}{4} G_{\mu\nu J}{}^2$ and $G = G_{J=0}$. ΔV of (21) is now $\Delta V = \Delta V(g_J, G_J, \mu)$. Note that for fixed J, $\Delta V(g_J, G_J, \mu)$ is the generating functional of the 1PI Green's functions evaluated at zero momenta.

Now we are in a position to discuss the color electrostatic properties of the vacuum. Because there is no source for ϕ or $\Delta\phi$, J can be set equal to zero after the calculation. When $J=0$, $\Delta\phi$ and G are not independent, $\Delta\phi = \Delta\phi(G)$. The dielectric constant is defined to be

$$\epsilon(G) = -\left.\frac{\partial \Delta V(g_J, G_J, \mu)}{\partial G}\right|_{J=0}, \qquad (61)$$

so that ϵ becomes a function of $\Delta\phi$. This relation tells us how the condensation $\Delta\phi$ affects the dielectric constant ϵ. To see the relation between $\Delta\phi$ and ϵ, we first note the renormalization-group equation satisfied by $\Delta V(g_J, G_J, \mu)$,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g_J)\frac{\partial}{\partial g_J} - 2\gamma(g_J)G_J\frac{\partial}{\partial G_J}\right)\Delta V = 0. \qquad (62)$$

In the background gauge, it is known that

$$\gamma(g_J) = \frac{\beta(g_J)}{g_J}. \qquad (63)$$

Then

$$\Delta\phi = \frac{\partial\Delta V}{\partial J} \qquad (64)$$

$$= \frac{\partial G_J}{\partial J}\frac{\partial\Delta V}{\partial G_J} + \frac{\partial g_J}{\partial J}\frac{\partial \Delta V}{\partial g_J}$$

$$= \frac{1}{1+J}\left(G_J\frac{\partial\Delta V}{\partial G_J} - \frac{g_J}{2}\frac{\partial\Delta V}{\partial g_J}\right)$$

$$= \frac{1}{1+J}\frac{g_J}{2\beta(g_J)}\mu\frac{\partial}{\partial\mu}\Delta V$$

$$= \frac{4}{1+J}\frac{g_J}{2\beta(g_J)}\left(1 - G_J\frac{\partial}{\partial G_J}\right)\Delta V, \qquad (64')$$

where we have used (62), (63), and the fact that $\Delta V = \mu^4 F(g, G/\mu^4)$ with some function F. Equation (64) is exact in our configuration of the electric field. Putting $J=0$ in (64′) and using (59) and (61),

$$\Delta\phi\,|_{J=0} = \frac{2g}{\beta(g)}\left(1 - G\frac{\partial}{\partial G}\right)\Delta V$$

$$= \frac{2g}{\beta(g)}(\Delta V + G\epsilon) \qquad (65)$$

$$\underset{g\to 0}{\simeq} \frac{2}{b_0 g^2}(\Delta V + G\epsilon). \qquad (65')$$

Equation (65) shows that $\beta\Delta\phi/2g$ is the Legendre transform of ΔV. By differentiating (65) with respect to G,

$$\frac{\partial\Delta\phi}{\partial G} = \frac{2g}{\beta(g)} G \frac{\partial\epsilon}{\partial G} \qquad (66)$$

$$\underset{g\to 0}{\simeq} \frac{2}{b_0 g^2} G\frac{\partial\epsilon}{\partial G}, \qquad (66')$$

at $J=0$. In what follows we need a negative character of $\beta(g)$ which is known to be correct at least for small g.

One can derive a closed equation satisfied by $\Delta V_{J=0}$ which is given in Appendix B. There it is shown that the sourceless (stationary) condition $J_\mu = 0$ is satisfied by perturbative solution $G=0$ or nonperturbative solution $\epsilon=0$. Note that at these values the term $G\epsilon$ in (65) vanishes. On the other hand, from Sec. II we know that at $J_\mu=0$ there are two kinds of vacuums satisfying $\Delta\phi=0$ (normal, perturbative) or $\Delta\phi=\Delta\phi_c>0$ (condensed, nonperturbative). The main purpose of this subsection is to show that the condensed vacuum has the property of $\epsilon=0$.

It is easy to see that the normal solution $G=0$ satisfies (65′) and (66′) perturbatively and that $\Delta\phi\to 0$ as $G\to 0$. Indeed up to the order g^2, ϵ is given by

$$\mathrm{Re}\,\epsilon(G)=1-\tfrac{1}{2}b_0 g^2\ln(G/\mu^4)\,, \tag{67}$$

$$\mathrm{Im}\,\epsilon(G)=-\pi\tfrac{1}{2}b_0 g^2\,, \tag{67′}$$

where Savvidi's renormalization condition[21] $\mathrm{Re}\,\epsilon(G=\mu^4)=1$ is adopted with the subtraction point taken in the electric region $(G>0)$. ΔV is then

$$\begin{aligned}\Delta V &= -\int_0^G \epsilon(G)dG\\ &= -G[1-\tfrac{1}{2}b_0g^2-\tfrac{1}{2}b_0g^2\ln(G/\mu^4)]+\tfrac{1}{2}i\pi b_0 g^2 G\,.\end{aligned} \tag{68}$$

On the other hand, $\Delta\phi=\langle\frac{1}{4}{:}\hat{G}_{\mu\nu}{}^2{:}\rangle$ is given by the tree graph up to this order so that

$$\Delta\phi=-G\,. \tag{69}$$

When the color electric field is applied to the normal vacuum, the response of ϵ, ΔV, and $\Delta\phi$ is given by (67), (67′), (68), and (69). In particular, as G goes to zero $\Delta\phi$ and ΔV vanish as they should. Equations (67), (67′), (68), and (69) are easily seen to satisfy (65′) and (66′).

Now we look for the nonperturbative solution of (65′) or (66′). In doing so the following observation should be made. The perturbative solution of ϵ has an imaginary part corresponding to the fact that a pair of gluons is created out of the vacuum and runs away to infinity because the asymptotic states of gluons do exist in perturbation theory. But if we include nonperturbative effects, gluons cannot be in the asymptotic states as we remarked in the end of Sec. III A. Thus ϵ should not have such an imaginary part for the physical branch of solution. We therefore classify the real solution of (66′), which has two types of solutions depending on the following two situations.

$$\textit{Case I}:\ \frac{\partial\Delta\phi}{\partial G}\neq 0,\ \ \frac{\partial\epsilon}{\partial G}\neq 0\,.$$

In this case G can be eliminated in favor of ϵ,

$$\frac{\partial\Delta\phi}{\partial\epsilon}=\frac{2}{b_0 g^2}\,G(\epsilon)\,. \tag{70}$$

In the presence of the electric field $(G>0)$, $\partial\Delta\phi/\partial\epsilon<0$. This means that as $\Delta\phi$ increases (recall that $\Delta\phi>0$ for the condensed solution) ϵ decreases, which means that the condensation has an antishielding effect. It is simply because $\Delta\phi$ is made up of gluons (in a color-singlet composite state) which we know from perturbation theory possesses the antishielding property. Now from the RGE we know[21] that $\epsilon(g,G)=\epsilon(\bar{g}(t),\mu^4)g^2/\bar{g}^2(t)$, with $d\bar{g}/dt=\bar{\beta}(\bar{g})$, $t=\ln(G/\mu^4)$, and

$$\bar{\beta}=\beta/(4+2\gamma),\quad \gamma(g)=\beta(g)/g\,. \tag{71}$$

Thus for large G, due to the asymptotic freedom we have $\epsilon\sim\ln(G/\mu^4)$. As G diminishes, ϵ decreases while $\Delta\phi$ increases along the trajectory shown in Fig. 4. The vacuum satisfies the sourceless condition: either $G=0$ or $\epsilon=0$. At $G=0$, there is a solution corresponding to the normal vacuum with $\Delta\phi=0$ so that the trajectory passes through the point N in Fig. 4, where ϵ will be complex because of the presence of tachyonic singularities. The question is whether the condensed vacuum satisfying $\Delta\phi=\Delta\phi_c$ is realized by $G=0$ or by $\epsilon=0$. Suppose it satisfies $G=0$ as is shown by C in Fig. 4. This means that at $G=0$ there are two vacuums satisfying $J=0$ so that $\Delta V(g_J, G_J=0,\mu)$ is a two-valued function of J. However, $\Delta V(g_J, G_J=0,\mu)$ satisfies RGE (62), without the last term, i.e. (23), so that

$$\Delta V=A\mu^4\exp\left[-4\int^{g_J}dx/\beta(x)\right]$$

with some constant A. At $J=0$ there is a solution giving $\Delta V=0$. Thus $A=0$ leading to $\Delta V\equiv 0$. Therefore $G=0$ corresponds uniquely to the normal vacuum satisfying $\Delta\phi=0$. The same conclusion is obtained if $\Delta\phi$ is considered as a function of J and G. At $G=0$, $\Delta\phi$ satisfies (26′) with the change $g\to g_J$ from which we get $\Delta\phi\equiv 0$. The discussion in Sec. III therefore shows that if we expand the effective action Γ around $A_\mu^a=0$ (or ΔV around $G=0$) corresponding to the normal vacuum, the tachyonic singularities are present in the Green's functions.

The condensed solution $\Delta\phi=\Delta\phi_c$ thus corresponds to the other solution $\epsilon=0$: The condensed

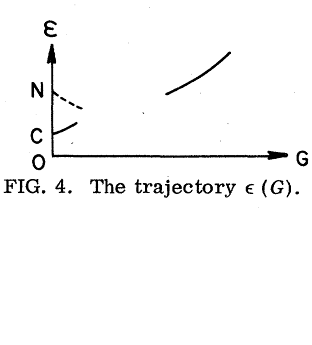

FIG. 4. The trajectory $\epsilon\,(G)$.

vacuum has the property of the perfect "diaelectrics" $\epsilon = 0$.

This leads to the crucial consequence of flux squeezing. The solution $\epsilon = 0$ leads to the relation $\Delta\phi_c = (2/b_0 g^2)\Delta V$ as is required by (26). In Sec. V the same condition ($\epsilon = 0$) is derived by the consistency of the phenomenological Lagrangian or by the stability of the nonperturbative vacuum.

There are two cases for the allowed trajectories of $\epsilon(G)$ as shown in Figs. 5(a) and 5(b). At some G_0, ϵ vanishes $[\epsilon(G_0) = 0]$. It is easy to see that G_0 cannot be magnetic in sign ($G_0 < 0$). This is because $\partial\Delta\phi/\partial\epsilon > 0$ for $G_0 < 0$ so that $\Delta\phi$ cannot take the value $\Delta\phi_c$ at $G = G_0 < 0$. Therefore $G_0 > 0$. More complicated trajectories are possible than those given in Figs. 5(a) and 5(b), but what we need in the following are $\epsilon \sim \ln G$ ($G \sim \infty$) and $\epsilon(G_0) = 0$ at $G_0 > 0$.

We see from (66) that the applied color electric field partly breaks the tachyonic bound states which condense in the vacuum. This becomes clear if we adopt Savvidi's renormalization condition.[21] Then we have $\epsilon(g, G) = g^2/\bar{g}^2(t, g)$ where $\bar{g}$ is governed by $\bar{\beta}$ of (71) and $t = \ln(G/\mu^4)$. Thus $G\,\partial\epsilon/\partial G = -2(g^2/\bar{g}^3)\bar{\beta}(\bar{g})$, which is positive in the region where $\bar{\beta}$ is negative. It follows from (66) that $\partial\Delta\phi/\partial G < 0$, which implies that if the color electric field is increased then $\Delta\phi$ decreases.

$$\textit{Case II}: \ \frac{\partial\Delta\phi}{\partial G} \equiv 0, \ \ \frac{\partial\epsilon}{\partial G} \equiv 0 .$$

In this case $\epsilon(G) = C_1$ where C_1 is a constant independent of G. It is easy to show (see Appendix B) that the equation $\partial\Delta\phi/\partial G \equiv 0$ holds if and only if $C_1 = 0$, in which case $\Delta\phi \equiv \Delta\phi_c$. This solution is not analytically connected with the solution of case I. The fact that $\epsilon \equiv 0$ and $\Delta\phi \equiv \Delta\Phi_c$ can be solutions is physically understood as follows. If $\epsilon \equiv 0$ for any applied field G, the displacement D defined by $D = \epsilon E$ is identically zero. In Appendix B we show that it is D, not E, which acts as an effective source and couples to quantum fluctuations of the gluon fields. So in case $D = 0$, tachyonic bound states which condense in the vacuum are not affected by the applied electric field and $\Delta\phi = \Delta\phi_c$ for any G so that the energy of the vacuum does not change in the presence of G.

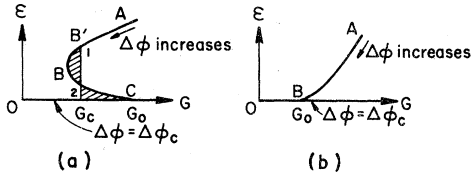

FIG. 5. Two possible trajectories of $\epsilon\,(G)$.

The solution $\epsilon \equiv 0$ is also shown in Fig. 5(a) or 5(b) which passes, of course, through the point $G = 0$. This is also required by the Lorentz invariance of the vacuum state.

Note that in the Legendre transformed space, both case I and case II can be expressed as a single solution $\Delta\phi = \Delta\phi(\epsilon)$ which takes the value $\Delta\phi_c$ at $\epsilon = 0$.

Next we discuss the flux configuration for each of the two cases shown in Figs. 5(a) and 5(b). In Figs. 6(a) and 6(b), $\mathcal{L}(G) \equiv -\Delta V(J = 0, G)$ is shown, where we have defined $\mathcal{L} = 0$ for $\Delta\phi = \Delta\phi_c$.

C. The flux configuration

We discuss the case corresponding to Figs. 5(a) and 5(b) separately.

(a) The case where $\epsilon(G)$ behaves like Fig. 5(a) has been discussed by Callan, Dashen, and Gross.[8] We give a brief argument to show that it leads to a tubelike solution of the color electric flux. We are discussing the situation where a static quark and antiquark of charge $\pm e$ are introduced at $z = \pm\infty$. In the axially symmetric (about z-axis) solution, the electric field is directed along the z axis and the magnitude is constant over the whole space because the tangential component of the electric field is continuous. (This is simply because we are discussing the situation where quark and antiquark are separated infinitely apart.)

We calculate the energy per unit length of the tube of the cross section σ and see if there is an optimal value of σ. If σ is infinite, the flux is not squeezed. The Hamiltonian is defined by

$$H = E\,\frac{\partial\mathcal{L}}{\partial E} - \mathcal{L} = ED - \mathcal{L},$$

where $\mathcal{L}(E) \equiv -\Delta V(J = 0, G)$, $G = \frac{1}{2}E^2$, and the displacement is given by $D = \partial\mathcal{L}/\partial E = \epsilon E$. The flux is nonzero only for the region where $\epsilon \neq 0$ so we minimize the following $H\sigma$ under the condition $D\sigma = e$,

$$\begin{aligned} H\sigma &= (ED - \mathcal{L})\sigma \\ &= e(E - \mathcal{L}/D), \end{aligned}$$

where E is a function of D. The optimal value of

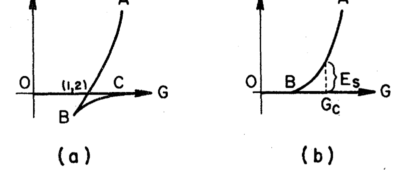

FIG. 6. Two possible forms of $\mathcal{L}(G)$.

σ is given by

$$0=\frac{\partial(H\sigma)}{\partial\sigma}=\frac{\partial D}{\partial\sigma}\frac{\partial(H\sigma)}{\partial D}=-\frac{e}{\sigma^2}\frac{\mathcal{L}}{(\mathcal{L}')^2}=-\frac{\mathcal{L}}{e}.$$

Thus $\mathcal{L}=0$ determining E or G, which is denoted by E_c or G_c. σ is determined through $D\sigma=e$. The equation $\mathcal{L}=0$ is nothing but the Maxwell-equal-area rule as shown in Fig. 5(a). The finite metastable branch (BB') is present. The structure of the tube is shown in Fig. 7. The color electric pressure in region I is balanced by the condensation energy ΔV (or the binding energy of gluons in the color-singlet channel in the terminology of Sec. III) due to the discontinuity $\delta\phi$ in the magnitude of the condensation at the surfaces. The discontinuity $\delta\phi$ is easily seen to be given by

$$\delta\phi=(\Delta\phi)_1-(\Delta\phi)_2=(2/b_0g^2)G_c\epsilon(G_c)<0,$$

where the points 1 and 2 are indicated in Figs. 5(a), 6(a), and 7.

By using (65) $\mathcal{L}$ can be eliminated to give

$$H=\tfrac{1}{2}ED+\frac{\beta(g)}{2g}[\Delta\phi(E)-\Delta\phi_c].$$

Thus the Hamiltonian consists of two positive terms, the usual color electrostatic energy $\frac{1}{2}ED$ and the energy which is required to break the condensation. In the equilibrium configuration of the flux tube these two terms have equal magnitude and the bag constant B can be identified as

$$B=\tfrac{1}{2}E_cD(E_c)=\frac{\beta(g)}{2g}[\Delta\phi(E_c)-\Delta\phi_c].$$

E_c satisfies the RGE

$$(1/E_c)\mu\frac{d}{d\mu}E_c=-\gamma(g),$$

i.e.,

$$E_c=\mu^2\exp\left[-\int^g\frac{2+\gamma(x)}{\beta(x)}dx\right].$$

This is derived by noting that $\mathcal{L}$ satisfies (62) with $J=0$ and has the form $\mathcal{L}(t,g)=\mathcal{L}(t_0,\bar{g}(t,g))(g^2/\bar{g}^2)G$, with $t=\ln(E^2/\mu^4)$. So $\mathcal{L}=0$ is realized for $\bar{g}=c$ with c some numerical constant and hence $\ln(E_c^2/\mu^4)=\int_g^c dx/\bar{\beta}(x)$. One can also derive $(1/D_c)\mu(d/d\mu)D_c=\gamma(g)$. In the limit of a static quark, σ can be shown, using the above relations, to be renormalization point independent, $d\sigma/d\mu=0$. If $E_c=\infty$ then we get an infinitely thin flux tube, a string with $\sigma=0$, which corresponds to infinite binding energy for the gluons.

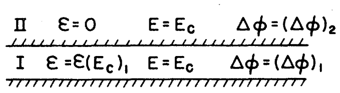

FIG. 7. The structure of the flux tube.

(b) The case given in Fig. 5(b) or Fig. 6(b) is the limit of case (a) where $\epsilon(G_c)\to 0$. Then $\sigma=e/D(G_c)=e/\epsilon(G_c)E_c\to\infty$; the flux is not squeezed. In this case if we include the derivative terms of ϕ in the local expansion (54), which we have neglected so far, they contribute to the surface energy and prevent the flux from spreading out. Let us consider the term $\partial_\mu\phi\partial_\mu\phi$. For small g this can be approximated by $A\int d^4x[\dot{\phi}^2-(\vec{\nabla}\phi)^2]$ with $A=\Gamma^{(1,0)}(\phi=0,A_\mu=0)$. If $A>0$ the energy is not bounded from below so that A should be negative. Neglecting the thickness of the skin, we are led to the problem of minimizing the following $H\sigma$ under the condition $D\sigma=e$,

$$\sigma H=\sigma(E\,\partial\mathcal{L}/\partial E-\mathcal{L})+S\sqrt{\sigma},$$

where $S\sqrt{\sigma}$ represents the surface energy with some positive constant S. It is easy to see that σH is minimized by nonzero D or σ. In the present case flux squeezing is a combined effect of the condensation phenomenon (volume effect) and the surface effect. Surface energy, which is shown as E_s in Fig. 6(b), makes it possible for $\mathcal{L}$ to take a nonzero value inside the flux tube without destroying the mechanical equilibrium. By contrast, in case (a) flux squeezing is caused by the condensation phenomenon alone and the derivative terms $\Gamma^{(i,0)}$ $(i\geq 1)$ provide the thickness to the skin of the flux tube.

D. The mean-field approximation

Up to this point, the arguments were formal. In order to get the trajectory $\epsilon(G)$, we need some approximation which takes into account the effects of condensation. In this subsection an attempt is made to discuss $\epsilon(G)$ by the mean-field-type approximation, thereby in particular, determining which of the two cases given in Figs. 5(a) or 5(b) is realized. We define first the local color electric field in the color dielectric medium, which is done classically.

We make a cavity with the dielectric constant ϵ_0 in the three-dimensional dielectric vacuum which has the dielectric constant ϵ. If the electric field is supplied from the source, the electric field in the dielectric medium is defined to be the one in this cavity. If we take a spherical cavity,[22]

$$A_\mu^{\mathrm{local}}=f(\epsilon,\epsilon_0)A_\mu, \tag{72}$$

with

$$f(\epsilon,\epsilon_0)=\frac{3\epsilon}{2\epsilon+\epsilon_0}. \tag{73}$$

We have denoted by A_μ^{local} the potential in the dielectric medium. Equation (73) depends on the shape and the size of the cavity, but what we need in the following is the fact that $A_\mu^{\rm local}$ vanishes at $\epsilon=0$ and $A=0$ and this is independent of the shape. Now in the above approximation it is the local color electric field constructed from $A_\mu^{\rm local}$ that acts as an effective field in the medium. Our assumption here is that if we calculate the effective potential $\tilde V(A_\mu^{\rm local})$ of $A_\mu^{\rm local}$ neglecting the condensation and then substitute (72) for $A_\mu^{\rm local}$ then it will be a good approximation to the effective potential $V(A_\mu)$, $V(A_\mu)\simeq \tilde V(A_\mu^{\rm local})$. We take the limit of the constant electric field (in the z direction) and because we are discussing the case $J=0$, ϵ is a function of G. Thus we are led to a nonlinear self-consistency relation,

$$\epsilon(G)=-\frac{\partial V(G)}{\partial G}=-\frac{\partial G^{\rm local}}{\partial G}\frac{\partial \tilde V(G^{\rm local})}{\partial G^{\rm local}}\,, \tag{74}$$

where

$$G^{\rm local}=-\tfrac{1}{4}(\partial_\mu A_\nu^{\rm local}-\partial_\nu A_\mu^{\rm local})^2=f^2G\,, \tag{75}$$

$$-\frac{\partial\tilde V}{\partial G^{\rm local}}=\frac{g^2}{\bar g^2(t,g)}\,. \tag{76}$$

Equation (76) has been given by Savvidi.[21] Here

$$t=\ln(G^{\rm local}/\mu^4)\,,$$

$$\int_{g^2}^{\bar g^2} dy/2\tilde\beta(y)=t\,,$$

$$\tilde\beta(y)=g\bar\beta(g)\,,$$

$$y=g^2\,,$$

and $\bar\beta$ is given by (71). The renormalization is performed with respect to the local field. With (72), (73), (75), and (76), we see that (74) has the following two types of solution, corresponding to the two cases in Sec. IV B.

(*i*) $\epsilon(G)\equiv 0$. This corresponds to case II in Sec. IV B.

(*ii*) *The second type*:

$$G\frac{\partial\epsilon}{\partial G}=\left[\frac{\bar g^2(t,g)}{g^2}-\frac{9\epsilon}{(2\epsilon+\epsilon_0)^2}\right]\frac{(2\epsilon+\epsilon_0)^3}{18\epsilon_0}\,. \tag{77}$$

This second solution (77) corresponds to case I in Sec. IV B. To discuss the latter solution we need $\bar g$ which is governed by $\tilde\beta$.[23] Now we know from Sec. III that the effective potential $V(G)$, when expanded around $G=0$, has an imaginary part due to tachyonic singularities in its expansion coefficients. This leads to a condition on $\tilde\beta$ such that $\tilde\beta(y)$ has no zero in the region $0<y\leqslant\infty$ and that $\int^\infty dx/\tilde\beta(x)$ is finite. The reason is that if the above conditions on $\tilde\beta$ do not hold $\partial V/\partial G$, for example, has no singularities in $0<G<\infty$ and is real when G approaches zero.

With the above behavior of $\tilde\beta$, it is easy to analyze the nonlinear equation (77) by the phase-space method. For large G, $\epsilon\sim\ln G$. At some finite $G(=G_c)$, $\bar g^2=\infty$ where $d\epsilon/dG=+\infty$, and ϵ is finite $(\epsilon=\epsilon_c)$, suggesting that the solution realizes the curve of Fig. 5(a). As $\epsilon\to 0$, $\bar g$ approaches the infrared fixed point of $\tilde\beta$. Because $\tilde\beta(y)$ has no zero in the region $0<y\leqslant\infty$,

$$G\frac{\partial\epsilon}{\partial G}\bigg|_{\epsilon=0}$$

cannot be real and positive, thus excluding the case of Fig. 5(b).

The unstable branch, corresponding to (BC) in Fig. 5(a), depends on the behavior of $\tilde\beta(y)$ away from the real positive y axis [if we can use (77) at all in the region where $\bar g^2$ is not real and positive]. In general the branch (BC) will be complex because it is unphysical anyway, i.e., $\mathrm{Re}(\partial\epsilon/\partial G)>0$. The imaginary part integrates automatically to zero along BC, $\int_{\rm BC} dG\,\mathrm{Im}\epsilon(G)=0$. Therefore the Maxwell rule still holds. For the unphysical branch, many cases can occur as shown in Fig. 8. The trajectory (BC_1) corresponds to the case where $\bar g^2\to 0_-$ as $\epsilon\to 0$. Note that $y=0_-$ is an infrared fixed point of $\tilde\beta(y)$. Near the point C_1 the solution behaves like $\partial\epsilon/\partial G=A/\ln\epsilon$ with $A=\epsilon_0^2/(4b_0g^2)$. Different trajectories of Fig. 8 give different sizes of the metastable region but they all lead to the flux-tube solution.

Our mean-field approximation suggests the case shown in Fig. 5(a). We should discuss the accuracy of our approximation which is not given in this paper. The essential requirements to have a tubelike solution in the approximation of neglecting the surface energy term are that the solution $\epsilon\equiv 0$ exists and that $V(G)$ has a singularity at $G=C_3>0$ at which $\epsilon=\partial V/\partial G$ is finite. These facts are independent of the detailed form of $f(\epsilon,\epsilon_0)$ or of $\tilde\beta(y)$ and are determined solely by the fact that $f(0,\epsilon_0)=0$.

V. COMPARISON WITH THE PREVIOUS APPROACH

We first recapitulate the arguments[7] which lead to the phenomenological Lagrangian proposed by Kogut and Susskind[24] and 't Hooft.[25] Let $\phi(x)$ denote the color-singlet $J^P=0^+$ tachyonic state. It

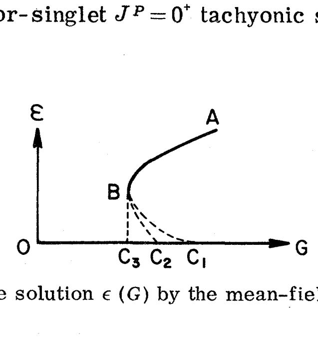

FIG. 8. The solution $\epsilon(G)$ by the mean-field approximation.

can be either a paired bound state or $\langle \frac{1}{4}G_{\mu\nu}{}^2(x)\rangle$. In our previous approach it represents the pairing field in which case it can be seen explicitly that the dominant constituents of $\phi(x)$ are infrared soft gluons. The QCD Lagrangian $\mathcal{L}$ now contains two different dynamical degrees of freedom ϕ and A_μ^a and it is expanded first in A_μ^a, taking the simplest possible terms consistent with the gauge invariance,

$$\mathcal{L}(\hat{\phi},\hat{A}_\mu^a) = -\tfrac{1}{2}\partial_\mu\hat{\phi}\partial_\mu\hat{\phi} - V(\hat{\phi}) - \tfrac{1}{4}\epsilon(\hat{\phi})\hat{G}_{\mu\nu}^{a2}\,. \quad (78)$$

The potential $V(\phi)$ is shown in Fig. 9 and represents the condensation of ϕ. The vacuum satisfies $\langle\hat{\phi}\rangle = \phi_c$, $\langle \hat{A}_\mu^a\rangle = 0$. The sign of ϕ_c cannot be determined here in contrast to the discussion in this paper (Sec. II).

For the condensed vacuum to be stable against the fluctuation around $\phi=\phi_c$, $A_\mu^a=0$, $\epsilon(\phi)$ must satisfy certain conditions. First, if $\epsilon(\phi_c)\neq 0$, the fluctuation of $\hat{A}_\mu^a$, due to the term $-\frac{1}{4}\epsilon(\phi_c)\hat{G}_{\mu\nu}{}^2$, produces tachyon bound states again and condensation proceeds still further. But this is inconsistent so $\epsilon(\phi_c)=0$. In other words, such an effect renormalizes $V(\phi)$ and after the renormalization $\epsilon(\phi_c)$ should vanish. We assume for simplicity

$$\epsilon(\phi) \underset{\phi\sim\phi_c}{\simeq} a\left(\frac{\phi_c-\phi}{\phi_c}\right)^{2\alpha}, \quad (79)$$

with $\alpha, a>0$. Next we consider the small change in ϕ, $\phi=\phi_c+\Delta\phi$. Then the change in the energy density ΔE can be estimated as follows:

$$\Delta E \sim \tfrac{1}{2}V''(\phi_c)(\Delta\phi)^2 - a(\Delta\phi/\phi_c)^{2\alpha}B\,. \quad (80)$$

The term $-\frac{1}{4}\hat{G}_{\mu\nu}{}^2$ contributes to ΔE by an amount $-B$ which is the energy density gained by condensing $\hat{\phi}$ up to the value $\langle\hat{\phi}\rangle=\phi_c$. Note that the term $(\hat{G}_{\mu\nu})^2$ in (78) can be understood as $:\hat{G}_{\mu\nu}{}^2: = \hat{G}_{\mu\nu}{}^2 - \langle 0|\hat{G}_{\mu\nu}{}^2|0\rangle$ without loss of generality. Here $|0\rangle$ represents the normal vacuum. Thus for ΔE to be positive, α should satisfy

$$\alpha \geq 1\,. \quad (81)$$

The condition (81) is also the condition for avoiding double counting, which says that after extracting ϕ, gluon fields A_μ^a should no longer form ϕ. Indeed if (81) is satisfied gluons cannot produce a tachyonic bound state ϕ because its condensation is energetically unfavorable. Now $\mathcal{L}(\phi, A_\mu^a)$ in (78), with (81) understood, is regarded as a c-number effective Lagrangian with the hope that ϕ represents the dominant quantum effects of QCD.

The condition (81) is known[7,24,25] to guarantee the flux-tube solution when quarks are introduced. To compare $\mathcal{L}(\phi, A_\mu^a)$ with the results of this paper we first neglect the term $\partial_\mu\phi\partial_\mu\phi$ and eliminate ϕ by the equation of motion,

$$-\tfrac{1}{4}\epsilon'(\phi)G_{\mu\nu}{}^2 - V'(\phi) = 0\,. \quad (82)$$

Solving (82) to give $\phi=\phi(G)$ and taking the Abelian configuration, $\mathcal{L}$ is expressed by $G \equiv -\frac{1}{4}G_{\mu\nu}{}^2$, which is compared with $\Delta V(G)$ of Sec. IV. We have two types of solutions of (82). One is $\phi\equiv\phi_c$, $\epsilon\equiv 0$, which corresponds to case II of Sec. IV B. The other solution $G=V'(\phi)/\epsilon'(\phi)$ gives the trajectories of case I. If $\alpha>1$, we get the trajectories of Fig. 5(a) with the point C at infinity ($G_0=\infty$). For the case $\alpha=1$, we define

$$\beta \equiv \lim_{\phi\to\phi_c}\frac{d}{d\phi}G = \lim_{\phi\to\phi_c}\frac{d}{d\phi}\frac{V'(\phi)}{\epsilon'(\phi)} = \frac{V'''(\phi_c)\epsilon''(\phi_c) - V''(\phi_c)\epsilon'''(\phi_c)}{2\epsilon''(\phi_c)^2}\,.$$

For $\beta>0$, the case of Fig. 5(a) is realized and for $\beta<0$, Fig. 5(b). See Fig. 10 for various cases.

The condition (81) on α can also be derived by regarding $\mathcal{L}(\phi(G), G)$ as a local approximation to the effective action. Then we know that $\epsilon(\phi(G))$ is complex at $G=0$, which is the case if and only if (81) holds. [As $G\to 0$, ϕ should tend to the normal value zero so that $V'(\phi)/\epsilon'(\phi)$ is required to approach zero as $\phi\to 0$.]

We have neglected the term $\partial_\mu\phi\partial_\mu\phi$. As has been pointed out in Sec. IV C, in the case $\alpha=1$ and $\beta<0$ the tubelike solution is realized by the combined effects of the kinetic term $\partial_\mu\phi\partial_\mu\phi$ and condensation energy.

We have obtained the solution $\epsilon\equiv 0$ by the equation of motion for ϕ which is not the case in Sec. IV. This is due to the approximation taken in (78).

Summarizing, the phenomenological Lagrangian (78) leads to qualitatively the same physical pic-

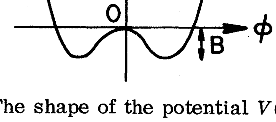

FIG. 9. The shape of the potential $V(\phi)$ of (78).

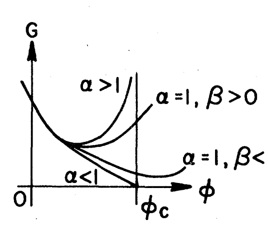

FIG. 10. The relation between $G(=-\frac{1}{4}G_{\mu\nu}{}^2)$ and ϕ given by (82).

ture of the condensed vacuum and its stability conditions as discussed in Sec. IV.

VI. DISCUSSIONS

The assumptions we have made are that (23) has a nontrivial solution and that the effective action Γ has a gauge-invariant local expansion (54). The latter assumption can be verified, of course, perturbatively but it automatically excludes the possibility of a massive phase where gluons acquire equal mass. Naively we expect that, if the condensation is understood as pairing in the color-singlet channel, the gluons become massive in the stable phase. Indeed we have discussed previously the color-singlet condensation in terms of pairing, by studying the formation of the Cooper pair[3] and by performing a variational calculation of the vacuum energy by means of Bogoliubov transformation[3] and we were led to the massive phase. The same problem was discussed covariantly by solving the BS equation for the tachyonic bound state[4] and by adopting the two-loop approximation for the effective potential.[26] The BS equation for the color-octet Goldstone mode has been discussed by Smit.[27] They all led us to the massive phase. But as long as the color-singlet condensation is discussed in terms of pairing, we cannot study the problem gauge invariantly so that it is not known whether the mass thus generated is produced by a dynamical effect of condensation or simply by the reason that we have taken a gauge-noninvariant approximation. This is the reason we have adopted $\hat{G}_{\mu\nu}{}^2$ to measure the color-singlet condensation. With the condensation of $\hat{G}_{\mu\nu}{}^2$ it is rather hard to imagine the mechanism of mass generation. In this way we are led to assume the gauge-invariant local expansion (54), that is, we are looking for the stable ground state in which gluons do not acquire mass after the condensation of $\hat{G}_{\mu\nu}{}^2$. This imposed the condition $\epsilon=0$ on the stable vacuum because the normal solution $G=0$ corresponds to the unstable vacuum. If the massive solution is allowed the solution $G=0$ can be a solution corresponding to the massive stable vacuum. We can clearly see this situation if the term proportional to $(A_\mu^a)^2$ is added to (54) or to (78). Then the condition $\epsilon=0$ does not follow from the stability requirement. Also in the example of $\lambda\phi^4$ discussed in Appendix A we have two phases at $\phi=0$: One is the normal massless vacuum and the other is the massive stable vacuum. The exclusion of the massive phase in QCD is equivalent to the exclusion of the possiblity of the formation of the octet Goldstone bosons which supply longitudinal components to the gluons. We do not yet have the answer to the question whether or not the massive phase is one of the solutions to QCD.

We have also assumed implicitly that ϵ is not negative. This is required in order for the theory to give any sensible answer: If ϵ is negative the energy of the ground state of QCD is not bounded from below so that there is no stable vacuum. There is, however, at the moment no rigorous proof of the non-negativeness of ϵ. The same remark applies to the coefficient $\Gamma^{(1,0)}$ of the $\partial_\mu\phi\partial_\mu\phi$ term in (54).

In this paper only the case of infinite separation of $q\bar{q}$ is discussed. When the separation becomes finite, the approach based on some phenomenological consideration will probably be more effective rather than the approach taking more and more terms of (54) into account.

We have discussed the condensation of $\hat{G}_{\mu\nu}{}^2$ and its physical effects. There are of course infinitely many gauge-invariant operators such as $D_\mu\hat{G}_{\mu\nu}D_{\mu'}\hat{G}_{\mu'\nu}$, etc.[28] or even nonlocal gauge-invariant operators. Our conclusions here will not be modified if these operators are shown to exhibit nontrivial condensations since our arguments are based on the instability of the normal vacuum. We do not know, however, at the moment how we can discuss the condensation of these complicated operators.

However, as pointed out in Ref. 12, the vacuum energy density is proportional to the vacuum expectation value of the trace of the energy-momentum tensor which is determined by that of the operator $\hat{G}_{\mu\nu}{}^2$ [Eq. (26) in our paper]. So one is led to the very important conclusion that *the vacuum energy is determined solely by* $\langle\hat{G}_{\mu\nu}{}^2\rangle$ *and that other complicated operators do not affect it.*

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APPENDIX A: THE CONDENSATION OF THE LAGRANGIAN IN $\lambda\phi^4$ THEORY

As a solvable example we consider the condensation of the Lagrangian in O(N) $\lambda\phi^4$ theory in the large-N limit.[13] Its condensation is rather trivial as we shall see below, but the purpose of this appendix is to show how the method of absorbing J into the coupling constant and field operators works. The Lagrangian is

$$\hat{\mathcal{L}}=-\tfrac{1}{2}\partial_\mu\hat{\phi}^i\partial_\mu\hat{\phi}^i-\tfrac{1}{2}m_0{}^2\hat{\phi}^2-\frac{\lambda_0}{8N}(\hat{\phi}^2)^2\,, \tag{A1}$$

where

$$\hat{\phi}^2=\sum_{i=1}^{N}\hat{\phi}^i\hat{\phi}^i\,.$$

$\hat{\mathcal{L}}$ is the equivalent to

$$\begin{aligned}\hat{\mathcal{L}}(\hat{\phi}^i,\hat{\chi})&=\hat{\mathcal{L}}+\frac{\lambda_0}{8N}\left(\frac{2N}{\lambda_0}\hat{\chi}-\hat{\phi}^2-\frac{2}{\lambda_0}m_0{}^2\right)^2-i\delta^4(0)\ln\lambda_0\\&=-\tfrac{1}{2}\partial_\mu\hat{\phi}^i\partial_\mu\hat{\phi}^i+\frac{N}{2\lambda_0}\hat{\chi}^2-\tfrac{1}{2}\hat{\chi}\hat{\phi}^2-\frac{Nm_0{}^2}{\lambda_0}\hat{\chi}\\&\quad-i\delta^4(0)\ln\lambda_0\,.\end{aligned} \tag{A2}$$

We must keep the $\ln\lambda_0$ term in the following discussions. The effective potential V to leading order in $1/N$ is

$$\begin{aligned}V(\phi,\chi)=&-\tfrac{1}{2}N\chi^2+\tfrac{1}{2}\chi\phi^2+\frac{Nm_0{}^2}{\lambda_0}\chi+i\delta^4(0)\ln\lambda_0\\&+\tfrac{1}{2}N\int\frac{d^4k}{(2\pi)^4}\ln(k^2+\chi)\,.\end{aligned} \tag{A3}$$

The mass and the coupling-constant renormalization are introduced as

$$\frac{m^2}{\lambda}=\frac{m_0{}^2}{\lambda_0}+\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\,, \tag{A4}$$

$$\frac{1}{\lambda}=\frac{1}{\lambda_0}+\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mu^2)^2}\,, \tag{A5}$$

corresponding to the renormalization conditions

$$\partial V/\partial\chi\,|_{\phi=0,\chi=0}=Nm^2/\lambda\,, \tag{A6}$$

$$\partial^2V/\partial\chi^2\,|_{\phi=0,\chi=\mu}=-N/\lambda\,, \tag{A7}$$

respectively. Then $V(\phi,\chi)$ becomes

$$\begin{aligned}V(\phi,\chi)=&-\tfrac{1}{2}\frac{N}{\lambda}\chi^2+\tfrac{1}{2}\phi^2\chi+\frac{Nm^2}{\lambda}\chi\\&+\frac{N}{64\pi^2}\chi^2(\ln(\chi/\mu^2)-\tfrac{3}{2})-i\delta^4(0)\ln\lambda_0\,.\end{aligned} \tag{A8}$$

In the following we consider only the case $m=0$ because in that case calculations can be done explicitly. We take also $\lambda<0$ because the theory is asymptotically free for this choice and the attractive force is present among ϕ so that we expect a dynamical rearrangement of the vacuum. [For a discussion on the sign of renormalized λ in the O(N) model see Ref. 13.] Indeed it is known[13] that the absolute minimum of $V(\phi,\chi)$ is realized by

$$\phi^i=0,\quad\chi=\chi_c=\mu^2\exp\left(1+\frac{32\pi^2}{\lambda}\right), \tag{A9}$$

which shows the pair condensation for the true vacuum. We expect that the condensation of the Lagrangian occurs at the same time because $\mathcal{L}$ contains the term χ^2. To see that this is indeed the case, the source J is introduced and $\mathcal{L}$ is changed to $(1+J)\mathcal{L}$. Then we renormalize J according to $J\to JZ_{\mathcal{L}}$. The relation (10) becomes

$$1+JZ_{\mathcal{L}}(\lambda,J,\Lambda/\mu)=\frac{\lambda Z(\lambda_J,\Lambda/\mu)}{\lambda_J Z(\lambda,\Lambda/\mu)}\,. \tag{A10}$$

From (A5)

$$Z(\lambda,\Lambda/\mu)=1-\tfrac{1}{2}\lambda K(\Lambda/\mu) \tag{A11}$$

with

$$K\left(\frac{\Lambda}{\mu}\right)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mu^2)^2}\,,$$

so

$$1+JZ_{\mathcal{L}}=\frac{1/\lambda_J-\tfrac{1}{2}K(\Lambda/\mu)}{1/\lambda-\tfrac{1}{2}K(\Lambda/\mu)}\,. \tag{A12}$$

Because the $1/N$ limit picks up one-loop graphs, the arguments following (12) suggest that the J independent renormalization of $\mathcal{L}$ may work. Indeed from (A12),

$$\begin{aligned}\lambda_J&=\frac{\lambda}{1+J}\,,\\Z_{\mathcal{L}}{}^{-1}&=1-\tfrac{1}{2}\lambda K(\Lambda/\mu)\,.\end{aligned} \tag{A13}$$

The anomalous dimension of $\mathcal{L}$ is given by

$$\begin{aligned}\gamma_{\mathcal{L}}(\lambda)&\equiv Z_{\mathcal{L}}{}^{-1}\mu dZ_{\mathcal{L}}/d\mu\\&=-\frac{\lambda}{16\pi^2}>0\,.\end{aligned} \tag{A14}$$

Now we discuss the condensation of $\mathcal{L}$. In the $1/N$ limit, the subtraction term corresponding to $V_{\rm pert}[J,A_\mu^a=0]$ in (21) is given by $V(J,\phi=0,\chi=0)$. ΔV is obtained by (A8) with the replacement $\lambda\to\lambda_J$ and $\phi\to\phi_J=(1+J)^{1/2}\phi$ while χ (and $m_0{}^2$) are independent of J. Then the expectation value of $:\hat{\mathcal{L}}:$ is

$$\begin{aligned}\Delta\phi\equiv\langle-:\hat{\mathcal{L}}:\rangle_J&=\frac{\partial\Delta V(\phi_J,\chi,\lambda_J,\mu)}{\partial J}\\&=-\frac{N}{2}\chi^2\frac{1}{\lambda}+\chi\phi^2\,,\end{aligned} \tag{A15}$$

where (A13) has been used. (We use the symbol $\Delta\phi$ and ϕ for different quantities.) There is no J dependence in (A15). This is because in the large-N limit $\Delta\phi$, χ, and ϕ are not independent because essentially only one-loop graphs are included. The true vacuum satisfies (A9) so that $\Delta\phi$ takes the value

$$\Delta\phi=-\frac{N}{2}\chi_c{}^2\frac{1}{\lambda}\,, \tag{A15$'$}$$

which is positive ("magnetic") as in the QCD case. But in the $\lambda\phi^4$ case (A15) shows that if we define the connected part by $-\langle :\hat{\mathcal{L}}:\rangle_c \equiv \Delta\phi + (N/2\lambda)\chi^2 - \frac{1}{2}\chi\phi^2$ then $\langle :\hat{\mathcal{L}}:\rangle_c = 0$ which means that the genuine condensation of the Lagrangian does not occur in the large-N limit. With (A9) and (A14) $\Delta\phi$ is shown to satisfy

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma_{\mathcal{L}}(\lambda)\right)\Delta\phi = 0\,,$$

where we have used

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda) = \frac{\lambda^2}{16\pi^2} \tag{A16}$$

derived from (A11). The effective potential of $\Delta\phi$ can be constructed if we restrict ourselves to the case $\phi = 0$ which satisfies $\partial\Delta V(\phi_J, \chi, \lambda_J)/\partial\phi = 0$. Then (A15) is used to calculate

$$\Delta V(\Delta\phi) = \Delta V(\chi, \lambda_J) - J\partial\Delta V/\Delta J = \Delta\phi\left[1 + \frac{\lambda}{64\pi^2} - \frac{\lambda}{64\pi^2}\ln\left(\frac{-2\lambda\Delta\phi}{N\mu^4}\right)\right]. \tag{A17}$$

$\Delta V(\Delta\phi)$ of (A17) shows the similar behavior as given in Fig. 1. The minimum value of ΔV is

$$\Delta V|_{\partial\Delta V/\partial\Delta\phi = 0} = -\frac{N}{128\pi^2}\mu^4\exp\left(2 + \frac{64\pi^2}{\lambda}\right), \tag{A18}$$

which satisfies (26) at the minimum point

$$4\Delta V = \frac{\beta(\lambda)}{\lambda}\Delta\phi\,. \tag{A19}$$

ΔV and $\Delta\phi$ are real in the large-N limit and the difference of the factor 2 between (26) and (A19) comes from the difference of (16) and (A13).

The Gross-Neveu model[11] can be discussed exactly parallel with the $\lambda\phi^4$ case and the Lagrangian condenses with "magnetic" sign: $\langle :\hat{\mathcal{L}}:\rangle < 0$. Both theories show tachyonic bound-state poles (not cuts in the large-N limit) if we take the normal vacuum. These are examples of the theorem proved in Sec. III.

APPENDIX B: A CLOSED EQUATION FOR ΔV

The total effective action $\Gamma(A_\mu^a)$ is known to satisfy the following equation (J is set equal to zero):

$$\Gamma(A^a) = \frac{1}{i}\ln\int \exp\left[i\int \hat{\mathcal{L}}(x)d^4x - i\int d^4x\frac{\delta\Gamma}{\delta A_\mu^a(x)}\hat{A}_\mu^a(x)\right][d\hat{A}] + \int d^4x A_\mu^a(x)\frac{\delta\Gamma}{\delta A_\mu^a(x)}\,. \tag{B1}$$

We choose the background Lorentz gauge[18] $D_\mu\hat{A}_\mu = 0$ with $D_\mu^{ab} = \delta^{ab}\partial_\mu + igf^{abc}A_\mu^c$, so we insert the factor $\Delta(\hat{A}', A)\delta(D_\mu\hat{A}_\mu')$ where $\hat{A}_\mu' = \hat{A}_\mu - A_\mu$ and $\Delta(\hat{A}, A) = \det M^{ab}$, $M^{ab} = (D_\mu D_\mu)^{ab} - igD_\mu^{ab'}f^{b'db}\hat{A}'^d_\mu$. In the static Abelian configuration, $A_\mu^a = \delta_{a\bar{a}}\delta_{\mu 0}A_\mu$, the local expansion of Γ reads

$$\Gamma(A_\mu^a) = -\int d^4x\, V(G(x)) + \int d^4x\, \partial_\mu G_{\mu\nu}(x)\partial_{\mu'}G_{\mu'\nu}(x)\Gamma^{(1)}(G(x)) + \cdots\,, \tag{B2}$$

with $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G(x) \equiv -\frac{1}{4}G_{\mu\nu}{}^2$. We insert (B2) into (B1) and use the relation

$$\int d^4x\frac{\partial V}{\partial A_\mu(x)}\hat{A}_\mu^{\bar{a}}(x) = \frac{1}{2}\int d^4x\, G_{\mu\nu}(x)\frac{\partial V}{\partial G(x)}\hat{G}_{\mu\nu}^{\bar{a}}(x)$$

with $\hat{G}_{\mu\nu}^{\bar{a}} = \partial_\mu\hat{A}_\nu^{\bar{a}} - \partial_\nu A_\mu^{\bar{a}}$. To single out V we take the configuration where $G_{\mu\nu}(x)$ has only one component G_{0z} and $-G_{0z} \equiv E$ is a constant over all space. In this limit we can derive

$$\Omega V(G) = i\ln\int \exp\left[-\tfrac{1}{4}i\int \hat{G}_{\mu\nu}^{a2}(x)d^4x - \frac{i}{2}G_{\mu\nu}\frac{\partial V}{\partial G}\int d^4x\,\hat{G}_{\mu\nu}^{\bar{a}}(x)\right][d\hat{A}] + 2\Omega G\frac{\partial V}{\partial G} \tag{B3}$$

$$= i\ln\int \exp\left[i\int \mathcal{L}(\hat{A}_\mu^a + A_\mu^a)d^4x - \frac{i}{2}G_{\mu\nu}\frac{\partial V}{\partial G}\int d^4x\,\hat{G}_{\mu\nu}^{\bar{a}}(x)\right][d\hat{A}]\,. \tag{B3$'$}$$

Insertion of the term $\Delta(\hat{A}, A)\delta(D_\mu\hat{A}_\mu)$ is understood. We cannot set $\int d^4x\,\hat{G}_{\mu\nu}^{\bar{a}}(x) = 0$ because of the zero-mass nature of the gluon. Indeed if (B3) is evaluated perturbatively, it has infrared divergences.

Equation (B3) is an analog of the equation satisfied by the effective potential $V(\phi(x))$, not the effective action, in $\lambda\phi^4$ theory, for example, where the equation satisfied by V is derived by taking the limit of constant ϕ. Now we observe the following two points.

(1) Equation (B3) says that the real source of the electric field which couples to the fluctuation of the gluon field is $G_{\mu\nu}\partial V/\partial G$ (not $G_{\mu\nu}$), that is, it is the displacement $D\equiv\epsilon E$, not E, which plays the role of the effective field. The dielectric constant ϵ depends on how much $\Delta\phi$ condenses in the vacuum.

(2) To show $c_1=0$ in case II of Sec. IV B, we write the expression for $\Delta\phi$. Now from $W(J,J_\mu)$ of (52) with constant J, we define

$$V(J,A_\mu^a)=-W(J,J_\mu^a)+\int J_\mu^a(x)\frac{\partial W}{\partial J_\mu^a(x)}d^4x$$

with

$$A_\mu^a=A_\mu^a(J,J_\mu)=\frac{\partial W}{\partial J_\mu^a(x)}\,.$$

Then

$$\Omega\Delta\phi\big|_{J=0}=\frac{\partial V(J,A_\mu^a)}{\partial J}\bigg|_{J=0}=-\frac{\partial W(J,J_\mu^a)}{\partial J}\bigg|_{J=0},$$

where J_μ is expressed as a function of A_μ and J. So we get, in the limit of the constant field,

$$\Delta\phi\big|_{J=0}=\frac{1/\Omega\int\hat{L}\exp(i\int\hat{\mathfrak{L}}d^4x-\frac{1}{2}c_1 iG_{\mu\nu}\int d^4x\,\hat{G}_{\mu\nu}^a)[d\hat{A}]}{\int\exp(i\int\hat{\mathfrak{L}}d^4x-\frac{1}{2}c_1 iG_{\mu\nu}\int d^4x\,\hat{G}_{\mu\nu}^a)[d\hat{A}]},$$

where we have suppressed gauge terms, and $\hat{L}=\int d^4x\,\hat{\mathfrak{L}}(x)$. It is clear that if $c_1\neq 0$, $\Delta\phi$ cannot be $G(=-\frac{1}{4}G_{\mu\nu}{}^2)$ independent so that $c_1=0$.

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