PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

# Quark and gluon propagators in quantum chromodynamics

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(Received 24 May 1979)

The quark and gluon propagators of massless quantum chromodynamics are studied in the Landau gauge using renormalization-group techniques in combination with the analytic properties which follow from Lorentz invariance and spectrum conditions. The investigations include the possibility of a spontaneous breakdown of global color symmetry with mass generation for quarks and gluons. The explicit asymptotic behavior of the propagators for large momenta is determined along *all directions* in the cut complex plane. It is found that the propagators cannot be entire functions except possibly in the infrared-stable limit. The existence of branch cuts drawn to infinity is verified. Projected propagators are introduced where contributions from negative-norm states are omitted. The original as well as the projected propagators satisfy unsubtracted spectral representations. The positivity condition of the transverse projected weight function leads to a restriction for the anomalous dimension of the gluon field, which implies a *lower bound* for the number of flavors.

## I. INTRODUCTION

The small-distance behavior of the quark-gluon system can be described by the asymptotic expansions of Green's functions in the limit of vanishing coupling ($g\rightarrow +0$). Due to asymptotic freedom, the theory is well defined in this limit.[1] On the other hand, we know that the long-distance behavior cannot be obtained without contributions which are not seen in the asymptotic expansion. It is expected that the long-distance structure of the theory gives rise to the complete or partial confinement of quarks and the screening of color charges.

In the confined phase of the field theory, only color-singlet systems should be associated with asymptotic fields. Consequently there are no such fields in colored channels. The question then arises as to what is the expected structure of the Green's functions in these channels. In particular, we are interested in the quark and gluon propagators and related vertex functions. *A priori*, we do not need to have the usual analyticity properties in colored channels, as long as these are preserved in color-singlet amplitudes. However, in quantum chromodynamics[2] it is assumed that quarks and gluons can be associated with local quantum fields, and hence we can analyze colored Green's functions within the framework of relativistic field theories.

In our work the model of massless quantum chromodynamics will be studied with a Euclidean normalization mass as the only dimensional parameter. Though there are no intrinsic masses in this model, it is possible that spontaneous symmetry breaking via a dynamical Higgs mechanism[3] leads to mass gaps[4] for gluons and/or quarks.[5] A gluon mass gap may be related to the screening of color charges and the confinement of quarks.[6] In this paper we consider quark and gluon propagators, and in a sequel we discuss the effective coupling and the quark-gluon vertex function.

In a covariant gauge, particularly the Landau gauge, we use Lorentz covariance and some minimal spectral properties in order to obtain cut-plane analyticity for the structure functions of the propagators. With the help of these analytic properties, together with the consequences of the renormalization group,[7] we derive the explicit asymptotic behavior of the propagator functions in *all directions* of the complex plane, as well as along the positive real axis. For all structure functions, we find sufficient boundedness for the existence of unsubtracted Lehmann representations. We also obtain superconvergence conditions, which have important implications for the theory.

As a first application of our results, we show that the propagators cannot be entire functions except at an infrared-stable fixed point. Even in the case of complete confinement, there must be *colored cuts* drawn to infinity.[8]

Furthermore, we find that the asymptotic behavior of the transverse gluon propagator and of its discontinuity is critically dependent upon the sign of the anomalous dimension of the gluon field. We introduce covariant, projected propagators, where all negative-norm states are omitted, and assume that these projected propagators approach their free-field values in the weak-coupling limit. Thus we obtain certain conditions which lead to a restriction for the anomalous dimension of the gluon propagator. This restriction implies a *lower* bound for the number of flavors.

The plan of this paper is as follows: The normalization conditions are introduced in Sec. II, followed by a discussion of the renormalization group in the presence of global symmetry breaking and for projected propagators in Sec. III. Section IV contains the asymptotic properties in the Euclidean region and Sec. V the spectral representations for full and projected propagators as well as a discussion of mass gaps. In Sec. VI, a detailed proof is given of the asymptotic behavior in all directions of the complex plane and along the positive real axis. Our conclusions are found in the final section.

## II. NORMALIZATION CONDITIONS

The formal power-series expansions of the time-ordered Green's functions are uniquely determined by imposing certain normalization conditions on the propagators and the vertex functions.[9] These conditions serve to define a finite renormalized coupling constant and to normalize the field operators. Apart from some general requirements, it is a matter of convenience how the specific form of the normalization conditions is chosen. In this section the conditions for the propagators will be formulated which define the normalization of the quark and gluon field operators. The precise definition of the renormalized coupling constant will not be relevant for the discussions in this paper.

For the propagators of the gluon and quark fields in the Landau gauge, structure functions are introduced by

$$-iD'^{ab}_{F\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+i0}\right)D^{ab}(k^2)\,,\quad a,b=1,2,\ldots,8 \tag{2.1}$$

$$-iS'^{jl}_F(k)=A^{jl}(k^2)+\gamma\cdot k\,B^{jl}(k^2)\,,\quad j,l=1,2,3\,. \tag{2.2}$$

$D'^{ab}_F$ and $S'^{jl}_F$ denote the Fourier transforms of $\langle TA^a_\mu(x)A^b_\nu(y)\rangle$ and $\langle T\psi^j(x)\overline{\psi}^l(y)\rangle$, where flavor indices have been suppressed. It will not be assumed that the propagators are multiples of the unit matrices $\delta_{ab}$ or $\delta_{jl}$, respectively, since we want to include the possibility that the global color symmetry is spontaneously broken. We propose to normalize the color components of the quark and gluon field operators separately by imposing the conditions

$$-k^2D^{aa}(k^2)=1\,,\quad a=1,2,\ldots,8 \tag{2.3}$$

and

$$-k^2B^{jj}(k^2)=1\,,\quad j=1,2,3 \tag{2.4}$$

at the Euclidean normalization point

$$k^2=\kappa^2<0\,.$$

Similar conditions are imposed on the ghost propagator.

Due to the indefinite metric of the state space, propagators may become negative for some $k^2<0$. At such points it is not possible to normalize a field operator in the proposed way. In order to avoid difficulties of this kind, we introduce projected field operators

$$A^{+a}_\mu=pA^a_\mu p\,,\quad \psi^{+j}=p\psi^j p\,, \tag{2.5}$$

where $p$ denotes the projection operator on a subspace $\mathcal{H}^+$ with positive- (semi-)definite metric.[10] The decomposition of a space with indefinite metric into subspaces of positive- and negative-definite metric is not unique. If the asymptotic fields form a complete system and the gluons are massive, the choice of the subspace $\mathcal{H}^+$ is obvious. A Lorentz- and gauge-invariant $\mathcal{H}^+$ is then constructed by applying polynomials of all those incoming fields to the vacuum state which describe physical particles. If colored particles are confined, corresponding asymptotic fields do not exist and asymptotic completeness does not hold. But it should still be possible in this case to define a subspace $\mathcal{H}^+$ of states with non-negative norm excluding ghost or unphysical Goldstone modes. Suitable definitions of $\mathcal{H}^+$ will be discussed in a separate paper.

Although the projected field operators need not be local, the vacuum expectation values of their commutator or anticommutators vanish at spacelike distances:

$$\begin{aligned}&\langle[A^{+a}_\mu(x)\,,\,A^{+b}_\nu(y)]\rangle=0\,,\\&\langle\{\psi^{+j}(x)\,,\,\overline{\psi}^{+l}(y)\}\rangle=0\,,\ \text{ if } (x-y)^2<0\,.\end{aligned} \tag{2.6}$$

As a consequence, the propagators are Lorentz-invariant distributions. The propagator of $A^{+a}$ need not be transverse in the Landau gauge. In order to eliminate the longitudinal contribution, we form the field operator

$$A^{+a}_{\mu\nu}=\partial_\mu A^{+a}_\nu-\partial_\nu A^{+a}_\mu\,. \tag{2.7}$$

The Fourier transform $G^+_{\mu\nu,\rho\lambda}$ of its propagator

$$\langle TA^{+a}_{\mu\nu}(x)A^{+b}_{\rho\lambda}(y)\rangle \tag{2.8}$$

only involves the transverse structure function $D^{+ab}$:

$$\begin{aligned}-iG^{+ab}_{\mu\nu,\rho\lambda}(k)=(&k_\mu k_\rho g_{\nu\lambda}-k_\mu k_\lambda g_{\nu\rho}\\&-k_\nu k_\rho g_{\mu\lambda}+k_\nu k_\lambda g_{\mu\rho})D^{+ab}(k^2)\,.\end{aligned} \tag{2.9}$$

For the Fourier transform of the projected quark propagator $\langle T\psi^{+j}(x)\overline{\psi}^{+l}(y)\rangle$, we introduce structure functions by

$$-iS'^{+jl}_F(k)=A^{+jl}(k^2)+\gamma\cdot k\,B^{+jl}(k^2)\,. \tag{2.10}$$

In Sec. V it will be shown that $D^{+aa}$ and $B^{+jj}$ satisfy unsubtracted Lehmann representations with nonnegative weight functions. Since these structure functions are positive for Euclidean momenta, the gluon and quark fields may be normalized by the conditions.

$$-k^2 D^{+aa}(k^2)=1\,, \tag{2.11}$$

$$-k^2 B^{+jj}(k^2)=1\,,\ \text{at } k^2=\kappa^2 \tag{2.12}$$

for any negative value of $\kappa^2$. This suggests replacing Eqs. (2.3) and (2.4) by the modified normalization conditions (2.11) and (2.12). With the new conditions, the normalization of the field operators may be possible for parameter values $g$ and $\kappa^2$ at which the conventional normalization fails.

In perturbation theory, coupling parameter and normalization mass uniquely determine the Green's functions in the Landau gauge of quantum chromodynamics as formal power series.[9] Independent of perturbation theory, this uniqueness property need not hold. We will, however, postulate existence and uniqueness of field operators for sufficiently small values of $g$ below an appropriate bound. Within this domain, it is assumed that the parameter values $g$ and $\kappa^2$ uniquely determine the field operators

$$A_\mu^a(x,g,\kappa^2)\,,\quad \psi^j(x,g,\kappa^2)\,,\quad C^a(x,g,\kappa^2)\,, \tag{2.13}$$

and their time-ordered functions. These quantities are determined by Eqs. (2.11) and (2.12) or possibly by Eqs. (2.3) and (2.4), the normalization conditions of the ghost propagators, and the defining equation of the coupling constant.

## III. RENORMALIZATION GROUP

In this section we first briefly summarize some familiar aspects of the renormalization group[11] in the way we prefer to formulate it. Then we adapt the methods to the situation where the global color symmetry is broken and we discuss the applications to the projected propagators.

The renormalization group of quantum chromodynamics is defined as the group of transformations

$$\begin{gathered} A_\mu^a \to \sqrt{z_a}\,A_\mu^a\,,\quad \psi^j \to \sqrt{\zeta_j}\,\psi^j\,,\quad C^a \to \sqrt{\tau_a}\,C^a\,,\\ z_a, \zeta_j, \tau_a > 0\,, \end{gathered} \tag{3.1}$$

which multiply the fields by finite positive factors. Such transformations are called equivalence transformations; they only change the normalization of the field operators. Parameter values $g,\kappa^2$ and $g',\kappa'^2$ are called equivalent:

$$g,\kappa^2 \sim g',\kappa'^2\,, \tag{3.2}$$

if the corresponding field operators are related by an equivalence transformation (3.1), namely

$$\begin{aligned} A_\mu^a(x,g',\kappa'^2)&=\sqrt{z_a}\,A_\mu^a(x,g,\kappa^2)\,,\\ \psi^j(x,g',\kappa'^2)&=\sqrt{\zeta_j}\,\psi^j(x,g,\kappa^2)\,,\\ C^a(x,g',\kappa'^2)&=\sqrt{\zeta_a}\,C^a(x,g,\kappa^2)\,. \end{aligned} \tag{3.3}$$

The effective coupling or invariant charge

$$Q=Q(k^2,g,\kappa^2)=Q\left(\frac{k^2}{\kappa^2},g\right) \tag{3.4}$$

is usually defined in terms of time-ordered functions as a dimensionless invariant of the renormalization group which satisfies

$$Q(k^2,g,\kappa^2)=g \text{ at } k^2=\kappa^2\,. \tag{3.5}$$

In the following, we quote some general properties of the invariant charge which are independent of the chosen definition for the renormalized coupling constant.

In the Landau gauge, invariance under the renormalization group means that

$$Q(k^2,g',\kappa'^2)=Q(k^2,g,\kappa^2) \tag{3.6}$$

holds for equivalent parameter pairs (3.2). Equations (3.5) and (3.6) yield

$$g'=Q(\kappa'^2,g,\kappa^2)=Q\left(\frac{\kappa'^2}{\kappa^2},g\right) \tag{3.7}$$

as the value of $g'$. From Eqs. (3.6) and (3.7), the relation

$$u\,\frac{\partial Q^2}{\partial u}=\beta(g^2)\,\frac{\partial Q^2}{\partial g^2}\,,\quad u=\frac{k^2}{\kappa^2}\,, \tag{3.8}$$

follows. With Eq. (3.5), this differential equation can be solved by

$$u=\exp\left[\int_{g^2}^{Q^2} dx\,\beta^{-1}(x)\right] \tag{3.9}$$

for sufficiently small values of $g$ and $Q$. Equation (3.9) is only valid in a domain where $Q$ is monotonic. The function $Q$ may have extrema which are integrable singularities of $\beta^{-1}$. See Ref. 12 for the modification of Eq. (3.9) in this case.

The first two coefficients of the formal power series

$$\beta(g^2)=g^4(\beta_0+\beta_1 g^2+\cdots) \tag{3.10}$$

are independent of the chosen normalization conditions.[13] In the Landau gauge, their values are[14]

$$\begin{aligned} \beta_0&=-\frac{1}{16\pi^2}\left(11-\tfrac{2}{3}N_f\right)\,,\\ \beta_1&=\frac{1}{2}\,\frac{1}{(8\pi^2)^2}\left(\tfrac{19}{3}N_f-51\right)\,, \end{aligned} \tag{3.11}$$

with $N_f$ being the number of flavors. Here we have assumed that the gauge group is SU(3) and that the quarks are in the fundamental (triplet) representation. For $N_f \leq 16$, the sign of $\beta_0$ is negative. This is the case of asymptotic freedom to which the discussion of this paper will be restricted.

In the absence of integrable singularities of $\beta^{-1}$, the relation (3.9) is valid in the domain

$$0 < g < g_\infty , \tag{3.12}$$

where $g_\infty$ is the smallest, positive zero of $\beta$ at which $\beta^{-1}$ is not integrable. If there is no such infrared stable zero of $\beta$, we set $g_\infty = \infty$, with $\beta^{-1}$ being nonintegrable at infinity. Then Eq. (3.9) is valid for all positive values of the coupling constant. We have

$$g_\infty = \lim_{u \to +0} Q(u,g) = \lim_{k^2 \to -0} Q(k^2, g, \kappa^2) , \tag{3.13}$$

with the limit being independent of $g$. In particular

$$\lim_{u \to +0} Q(u,g) = \infty \text{ if } g_\infty = \infty , \tag{3.14}$$

with $\beta^{-1}$ being nonintegrable at infinity.

With Eqs. (3.9) and (3.10), it can be shown that the product $Q^2 \ln u$ approaches the finite positive limit $\beta_0^{-1}$ for $u \to \infty$. Hence

$$\ln \frac{k^2}{\kappa^2} Q^2\left(\frac{k^2}{\kappa^2}, g\right) \simeq |\beta_0^{-1}| , \quad \text{for } k^2 \to -\infty \tag{3.15}$$

represents the leading asymptotic behavior of the effective coupling for large Euclidean momenta. Here and in the work that follows, the symbol $\simeq$ indicates that the ratio of two functions approaches unity in the limit.

The formal expansion of $Q^2$ with respect to powers of $g^2$ follows from Eqs. (3.9) and (3.10) in the form

$$Q^2 = g^2 + \beta_0 g^4 \ln \frac{k^2}{\kappa^2} + O(g^6) . \tag{3.16}$$

Under the renormalization-group transformation (3.3), the gluon propagators (2.1), as well as the projected propagators (2.9), are multiplied by a factor $(z_a z_b)^{1/2}$, the corresponding quark propagators by a factor $(\zeta_j \zeta_l)^{1/2}$. For example, the diagonal elements of the gluon propagators transform like

$$\begin{aligned} &D^{aa}(k^2, g', \kappa'^2) = z_a D^{aa}(k^2, g, \kappa^2) , \\ &D^{+aa}(k^2, g', \kappa'^2) = z_a D^{+aa}(k^2, g, \kappa^2) , \end{aligned} \tag{3.17}$$

$$g' = Q\left(\frac{\kappa'^2}{\kappa^2}, g\right) . \tag{3.18}$$

By the conventional normalization condition (2.3), the factor $z_a$ is determined to be

$$z_a^{-1} = R^{aa}\left(\frac{\kappa'^2}{\kappa^2}, g\right) , \tag{3.19}$$

with the dimensionless function

$$R^{aa}\left(\frac{k^2}{\kappa^2}, g\right) = -k^2 D^{aa}(k^2, g, \kappa^2) . \tag{3.20}$$

On the other hand, if the normalization condition (2.11) is imposed, we have

$$z_a^{-1} = R^{+aa}\left(\frac{\kappa'^2}{\kappa^2}, g\right) , \tag{3.21}$$

where

$$R^{+aa}\left(\frac{k^2}{\kappa^2}, g\right) = -k^2 D^{+aa}(k^2, g, \kappa^2) \tag{3.22}$$

depends on $k^2/\kappa^2$ and $g$ only, provided the definition of $\mathcal{H}^+$ does not involve new dimensional parameters.

The differential form of the transformation laws on the propagators are the Callan-Symanzik equations[15]

$$\begin{aligned} &\mathfrak{D}_V^{ab} D^{ab} = 0 , \quad \mathfrak{D}_V^{ab} D^{+ab} = 0 , \\ &\mathfrak{D}_F^{jl} A^{jl} = 0 , \quad \mathfrak{D}_F^{jl} A^{+jl} = 0 , \\ &\mathfrak{D}_F^{jl} B^{jl} = 0 , \quad \mathfrak{D}_F^{jl} B^{+jl} = 0 . \end{aligned} \tag{3.23}$$

In the Landau gauge, the differential operators are

$$\begin{aligned} &\mathfrak{D}_V^{ab} = \kappa^2 \frac{\partial}{\partial \kappa^2} + \beta \frac{\partial}{\partial g^2} + \tfrac{1}{2}(\gamma_V^a + \gamma_V^b) , \\ &\mathfrak{D}_F^{jl} = \kappa^2 \frac{\partial}{\partial \kappa^2} + \beta \frac{\partial}{\partial g^2} + \tfrac{1}{2}(\gamma_F^j + \gamma_F^l) . \end{aligned} \tag{3.24}$$

It should be noted that the differential equations of the original and the projected propagators have identical coefficients $\beta$ and $\gamma_V^a$ or $\gamma_F^j$, respectively, provided the same type of normalization [in our discussion either (2.3) and (2.4) or (2.11) and (2.12)] is used for the field operators.

With the conventional normalization conditions (2.3) and (2.4), the $\gamma$ functions have formal power series with respect to $g^2$. In the Landau gauge considered here, they are given by

$$\gamma_V^a = g^2(\gamma_{V0} + \gamma_{V1} g^2 + \cdots) , \tag{3.25}$$

$$\gamma_F^j = g^4(\gamma_{F1} + \gamma_{F2} g^2 + \cdots) . \tag{3.26}$$

The coefficients, as computed in perturbation theory, are, of course, color independent. For the differences $\gamma_V^a - \gamma_V^b$ $(a \neq b)$, $\gamma_F^j - \gamma_F^l$ $(j \neq l)$, all coefficients in the power-series expansion vanish. The coefficient $\gamma_{V0}$ is[16]

$$\gamma_{V0} = -\frac{1}{16\pi^2}(13 - \tfrac{4}{3} N_f) . \tag{3.27}$$

The value of $\gamma_{V0}$ does not depend on the chosen type of normalization conditions. We discuss this point in some detail for the proposed modification (2.11) and (2.12) of the conventional normalization conditions (2.3) and (2.4). For given parameter values $g$ and $\kappa^2$, the conventionally normalized field operators $A'^a_\mu$, $\psi'^j$ and the field operators $A''^a_\mu$, $\psi''^j$ normalized according to Eqs. (2.11) and (2.12), are related by

$$A'^a_\mu(x,g,\kappa^2)=\sqrt{d_a}\,A''^a_\mu(x,g,\kappa^2)\,,\qquad \psi'^j(x,g,\kappa^2)=\sqrt{\delta_j}\,\psi''^j(x,g,\kappa^2)\,. \tag{3.28}$$

The normalized factors are determined by the values of the un-normalized projected propagators at the normalization point:

$$d_a(g^2)=-k^2D^{+aa}(k^2)\,,\qquad \delta_j(g^2)=-k^2B^{+jj}(k^2),\ k^2=\kappa^2<0\,. \tag{3.29}$$

Under the change of normalization as given by Eq. (3.28), the coefficients of the differential equations (3.23), (3.24) transform like

$$\beta''=\beta'\,, \tag{3.30}$$

$$\gamma''^a_V=\gamma'^a_V+\frac{\beta'}{d_a}\frac{dd_a}{dg^2}\,, \tag{3.31}$$

$$\gamma''^j_F=\gamma'^j_F+\frac{\beta'}{\delta_j}\frac{d\delta_j}{dg^2}\,. \tag{3.32}$$

If the projected propagators can be expanded with respect to powers of $g^2$, we have the series

$$\gamma''^a_V=g^2(\gamma''^a_{V0}+\gamma''^a_{V1}g^2+\cdots)\,, \tag{3.33}$$

$$\gamma''^j_F=g^4(\gamma''^j_{F1}+\gamma''^j_{F2}g^2+\cdots)\,. \tag{3.34}$$

Since $\beta$ is of order $g^4$, the first coefficients of $\gamma'^a_V$ and $\gamma''^a_V$ are identical:

$$\gamma''^a_{V0}=\gamma'^a_{V0}\,. \tag{3.35}$$

Thus the $\gamma$ functions of the gluon field have the same leading behavior for $g\to 0$ in both normalizations. This statement holds under more general conditions. If the projected gluon propagator and its first $g^2$ derivative converge for $g^2\to +0$, the transformation (3.31) implies

$$\gamma''^a_V(g^2)=\gamma_{V0}g^2+O(g^4)\,.$$

Next, suppose that the expansion of the projected gluon propagator involves powers of $g^2$ and $\ln g^2$, with leading terms given by

$$G^{+ab}_{\mu\nu,\rho\lambda}(k,g^2)=\delta_{ab}G^{(0)}_{\mu\nu,\rho\lambda}(k)+g^2\ln^n g^2G^{+(1)ab}_{\mu\nu,\rho\lambda}(k)+\cdots\,,\ n=1,2,\ldots \tag{3.36}$$

as is typical for nonrenormalizable formulations.[17] In this case, the $g^2$ derivative of (3.36) becomes logarithmically divergent for $g\to 0$. Then $\gamma'^a_V$ and $\gamma''^a_V$ still have the same behavior for $g\to 0$ up to terms of order $g^4\ln^n g^2$. More generally, we will assume that the projected propagators (2.9) and (2.10) approach their free-field values in the weak-coupling limit[18]

$$\lim_{g\to+0}G^{+ab}_{\mu\nu,\rho\lambda}=\delta_{ab}G^{(0)}_{\mu\nu,\rho\lambda}\,,\quad \lim_{g\to+0}S'^{+jl}_F=\delta_{jl}S_F\,, \tag{3.37}$$

and that their derivatives do not behave stronger than $g^{2(\epsilon-1)}$ for some positive $\epsilon$:

$$\frac{\partial}{\partial g^2}G^{+ab}_{\mu\nu,\rho\lambda}=O(g^{2(\epsilon-1)})\,,\qquad \frac{\partial}{\partial g^2}S'^{+jl}_F=O(g^{2(\epsilon-1)})\,,\ 0<\epsilon\le 1\,. \tag{3.38}$$

These assumptions imply

$$G^{+ab}_{\mu\nu,\rho\lambda}=\delta_{ab}G^{(0)}_{\mu\nu,\rho\lambda}+O(g^{2\epsilon})\,,\qquad S'^{+jl}_F=\delta_{jl}S_F+O(g^{2\epsilon})\,. \tag{3.39}$$

The leading behavior of the $\gamma$ functions in the new normalization is then given by

$$\gamma''^a_V=\gamma_{V0}g^2+O(g^{2(1+\epsilon)})\,,\qquad \gamma''^j_F=O(g^{2(1+\epsilon)})\,,\ 0<\epsilon\le 1\,. \tag{3.40}$$

Again the $\gamma$ functions of the gluon field approach

$$\gamma''^a_V\sim\gamma'^a_V\sim\gamma_{V0}g^2\quad\text{for } g^2\to+0$$

in both normalizations up to terms of order $g^{2(1+\epsilon)}$.

The assumptions (3.37) and (3.38) imply that states of negative norm should not contribute to the transverse propagator of the free gluon field as defined by the limit $g^2\to+0$. Even though gluons may not be observable as free particles, in quantum chromodynamics, because of asymptotic freedom, the $g^2\to+0$ limit is nevertheless physically relevant as a high-momentum limit.[18]

*A priori*, the limiting properties (3.37) and (3.38) in $g^2$ have been assumed for real values of the variable $k^2$. Since the propagators can be defined as Fourier transforms of retarded space-time functions (distributions), these limiting properties can be easily extended to the analytic continuations into the complex $k^2$ plane.

## IV. ASYMPTOTIC PROPERTIES IN THE EUCLIDEAN REGION

As an important ingredient for our later derivation of the asymptotic behavior of the propagators in all directions of the complex $k^2$ plane, we derive in this section the limiting behavior along the negative real axis (Euclidean region). We note that, apart from the coefficient, the asymptotic expressions for the structure functions are independent of the normalization. We can therefore choose a normalization which is convenient for later discussions. This will be the conventional normalization (2.3), (2.4) for the regular propagators, and the normalization (2.11), (2.12) for the projected propagators.

Let $R$ be one of the dimensionless functions

$$R^{ab},\ R^{+ab},\ S^{jl},\ S^{+jl},\ T^{jl},\ T^{+jl}$$

defined by

$$\begin{aligned} &R^{ab}=-k^2D^{ab}\,,\quad R^{+ab}=-k^2D^{+ab}\,,\\ &S^{jl}=\sqrt{-k^2}A^{jl}\,,\quad S^{+jl}=\sqrt{-k^2}A^{+jl}\,,\\ &T^{jl}=-k^2B^{jl}\,,\quad T^{+jl}=-k^2B^{+jl}\,. \end{aligned} \tag{4.1}$$

These are functions of the dimensionless variables $k^2/\kappa^2$ and $g$. We write

$$R=R(u,g)\,,\ u=k^2/\kappa^2\,. \tag{4.2}$$

At this point it becomes important to note that the projected functions $R^+$, $S^+$, and $T^+$ also depend upon $u$ and $g$ only, provided we use a definition of $\mathcal{H}^+$ which does not involve new dimensional parameters. For the dimensionless functions $R$, the Callan-Symanzik equation becomes

$$u\frac{\partial R}{\partial u}=\beta\frac{\partial R}{\partial g^2}+\gamma R\,. \tag{4.3}$$

Instead of $u$ we introduce $Q^2$ as a new independent variable by using Eq. (3.9). Regarding $R$ as a function of $g^2$ and $Q^2$, Eq. (4.3) can be converted into an ordinary differential equation with respect to $g^2$ at constant values of $Q^2$. The unique solution is

$$R(u,g)=R(1,Q)\exp\left[\int_{g^2}^{Q^2}dx\,\gamma(x)\beta^{-1}(x)\right], \tag{4.4}$$

valid for sufficiently large $u$ and small $Q^2, g^2$.

From Eq. (4.4) the asymptotic behavior of the propagators follows as well for $g\to+0$ and $k^2\to-\infty$ The discussion is particularly simple for $D^{aa}$, $L^{+aa}$, $B^{jj}$, $B^{+jj}$, since then $R(1,Q)\equiv1$ with the appropriate normalization. In the other cases, the leading behavior of the coefficient $R(1,Q)$ for $g\to+0$ or $u\to\infty$ is determined by the asymptotic form of $R(1,g)$ for small $g$. The behavior of the exponential expression in Eq. (4.4) follows from the behavior of $Q$ and the ratio $\gamma/\beta$. The asymptotic relations (3.10), (3.36), or (3.40) for $\beta$ and $\gamma_F$ imply that the limit

$$\lim_{u\to\infty}\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_F^j(x)\beta^{-1}(x)\right]$$

converges. Since $\gamma_V^a$ is of order $g^2$ for $g\to+0$, the integrand $\gamma_V^a/\beta$ is not integrable at $x=0$. After separating the singular part,

$$\gamma_V^a\beta^{-1}=\gamma_{V0}\beta_0^{-1}x^{-1}+\tau^a\,,$$

the remainder $\tau^a$ is integrable at $x=0$ according to (3.10), (3.25), or (3.40). Hence

$$\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_V^a(x)\beta^{-1}(x)\right]=\left(\frac{Q^2}{g^2}\right)^{\gamma_{V0}/\beta_0}\exp\left[\int_{g^2}^{Q^2}dx\,\tau^a(x)\right],$$

where the exponential converges to a finite limit and the asymptotic form of the coefficient follows from (3.15).

The asymptotic behavior of the normalized structure functions is thus given by

$$R^{aa}=\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_V^{\prime a}(x)\beta^{-1}(x)\right]\simeq C_V^{\prime a}\left(\ln\frac{k^2}{\kappa^2}\right)^{-\gamma_{V0}/\beta_0}, \tag{4.5}$$

$$B^{jj}=\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_F^{\prime j}(x)\beta^{-1}(x)\beta^{-1}(x)\right]\simeq C_F^{\prime j}\,,\ k^2\to-\infty \tag{4.6}$$

with normalization (2.3) and (2.4) and

$$R^{+aa}=\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_V^{\prime\prime a}(x)\right]\simeq C_V^{\prime\prime a}\left(\ln\frac{k^2}{\kappa^2}\right)^{-\gamma_{V0}/\beta_0}, \tag{4.7}$$

$$B^{+jj}=\exp\left[\int_{g^2}^{Q^2}dx\,\gamma_F^{\prime\prime j}(x)\beta^{-1}(x)\right]\simeq C_F^{\prime\prime j}\,,\ k^2\to-\infty \tag{4.8}$$

with normalization (2.11) and (2.12). The coefficients are positive and given by

$$C_V^a=(g^2|\beta_0|)^{-\gamma_{V0}/\beta_0}\exp\left[\int_{g^2}^{0}dx\,\tau^a(x)\right]>0\,, \tag{4.9}$$

$$\tau^a=\tau^{\prime a}\text{ or }\tau^{\prime\prime a}$$

$$C_F^j=\exp\left[\int_{g^2}^{0}dx\,\gamma_F^j(x)\beta^{-1}(x)\right]>0\,, \tag{4.10}$$

$$\gamma_F^j=\gamma_F^{\prime j}\text{ or }\gamma_F^{\prime\prime j}.$$

Finally, some results concerning the behavior of the propagators for $g\to+0$ are listed which will be used later. We first note that

$$\lim_{g\to+0}g^{-2n}G^{ab}_{\mu\nu,\rho\lambda}=0\,,\quad \lim_{g\to+0}g^{-2n}S_F^{\prime jl}=0 \tag{4.11}$$

for any $n>0$ if $a\neq b$, $j\neq l$. Hence, the color off-diagonal elements of the unprojected propagators fall off faster than any power of $g^2$ for $g\to+0$. Similar statements hold for the projected propagators, provided they possess perturbative expansion with color-symmetric coefficients. However, under the assumptions (3.37), (3.38), nothing can be said beyond Eq. (3.39).

The asymptotic behavior of the color diagonal structure functions for $g\to+0$ is found to be

$$D^{aa}(k^2,g,\kappa^2)=-k^{-2}\left[1+\gamma_{V0}g^2\ln\frac{k^2}{\kappa^2}+O(g^4)\right],$$

$$B^{jj}(k^2,g,\kappa^2)=-k^{-2}\left[1+\gamma'_{F1}g^4\ln\frac{k^2}{\kappa^2}+O(g^6)\right], \qquad (4.12)$$

$$A^{jj}(k^2,g,\kappa^2)=(-k^{-2})^{1/2}[S_jg^2+O(g^4)]$$

[with normalization (2.3) and (2.4)];

$$D^{+aa}(k^2,g,\kappa^2)=-k^{-2}\left[1+\gamma_{V0}g^2\ln\frac{k^2}{\kappa^2}+O(g^{2(1+\epsilon)})\right],$$

$$B^{+jj}(k^2,g,\kappa^2)=-k^{-2}[1+O(g^{2(1+\epsilon)})], \qquad (4.13)$$

$$A^{+jj}(k^2,g,\kappa^2)=(-k^{-2})^{1/2}O(g^{2\epsilon}),\ 0<\epsilon\leq 1$$

[with normalization (2.11) and (2.12)].

By analytic continuation, with the help of the retarded representations of the propagators, these relations hold also in the cut $k^2$ plane. In the Minkowski region, they are valid in the sense of distributions.

## V. SPECTRAL REPRESENTATIONS

For the structure functions of the propagators, we can obtain the Lehmann representations[19]

$$D^{ab}(k^2)=\int_0^\infty dk'^2\,\frac{\rho^{ab}(k'^2)}{k'^2-k^2}, \qquad (5.1)$$

$$A^{jl}(k^2)=\int_0^\infty dk'^2\,\frac{\sqrt{k'^2}\rho_1^{jl}(k'^2)-\rho_2^{jl}(k'^2)}{k'^2-k^2}, \qquad (5.2)$$

$$B^{jl}(k^2)=\int_0^\infty dk'^2\,\frac{\rho_1^{jl}(k'^2)}{k'^2-k^2}. \qquad (5.3)$$

The representations follow from Lorentz invariance and spectrum conditions, provided the functions have the appropriate boundedness properties. It will be shown in the following section that the behavior of the propagator functions for large momenta in all directions of the complex $k^2$ plane is controlled by asymptotic freedom so that the representations (5.1)–(5.3) are indeed valid.

Invariance under the renormalization group implies that the position $\lambda^2$ of a propagator singularity is either fixed at the origin or has the following dependence on the coupling constant[20]:

$$\lambda^2(g,\kappa^2)=\kappa^2c_0\exp\left[\int_{g^2}^{g_0^2}dx\,\beta^{-1}(x)\right]. \qquad (5.4)$$

Here $c_0$ is a numerical constant which, for a given singularity, depends only on the chosen reference $g_0$ of the coupling constant. Since we have $\beta_0<0$, the singularities (5.4) move towards the origin for $g^2\to+0$ according to[21]

$$\lambda^2=\kappa^2c_0f(g^2,g_0^2)\left(\frac{g^2}{g_0^2}\right)^{\beta_1/\beta_0^2}e^{1/\beta_0g^2}, \qquad (5.5)$$

$$\lim_{g^2\to+0}f(g^2,g_0^2)>0.$$

If expanded with respect to powers of $g^2$, the function $\lambda^2$ has vanishing coefficients because of the essential singularity. On the other hand, we have

$$|\lambda^2|\to\infty \text{ for } g\to g_\infty, \qquad (5.6)$$

where $g_\infty$ is the first nonintegrable singularity of $\beta^{-1}(g^2)$. Hence all singularities, which are not placed at the origin, move towards infinity in the infrared-stable limit, or in the limit $g\to\infty$ if there is no infrared-stable point.

The relation (5.4) is of particular interest if applied to the first singularity $M^2{}_{jl}$ of the quark propagator on the positive real axis $k^2\geq 0$. This singularity is either placed at the origin

$$M^2{}_{jl}=0,$$

or it has a positive value with the $g$ dependence

$$M^2{}_{jl}(g,\kappa^2)=\kappa^2c_{F0}^{jl}\exp\left[\int_{g^2}^{g_0^2}dx\,\beta^{-1}(x)\right],\ c_{F0}^{jl}>0. \qquad (5.7)$$

In the first case, the quark mass remains zero; in the second case a nonvanishing mass is developed even though the model does not involve an intrinsic mass apart from the normalization parameter $\kappa^2$. Mass generation for quarks is often regarded as being connected to a spontaneous breakdown of chiral symmetry with the pseudoscalar mesons as Goldstone particles.[22] The quantity $M^2{}_{jl}$ has an essential singularity at $g^2=0$; all coefficients of the power series vanish in agreement with the fact that the propagator cut starts at zero in each order of perturbation theory. $M^2{}_{jl}$ may either be a pole or the starting point of a cut. The possibility that the structure functions have finite values or vanish at the branch point $M^2{}_{jl}$ is also included. $M^2{}_{jl}$ may be taken as the lower limit of the integrals in the Lehmann representations (5.2) and (5.3). If the global color symmetry is preserved, the matrix $M^2{}_{jl}$ is diagonal:

$$M^2{}_{jl}=c_{jl}M^2.$$

Here $M^2$ may be interpreted as square of the quark mass since it is the lowest value in the spectrum of $P_\mu P^\mu$ for states of quark number one. If the global color symmetry is broken, we may define mass parameters $M_j^2$ by the lowest value in the spectrum of $P_\mu P^\mu$ in the subspace of all vectors $\Phi$ with

$$(\Omega,\psi^j(x)\Phi)\neq 0.$$

$M^2{}_{jl}$ is then the larger of the two values $M_j^2$ and $M_l^2$.

In cases of mass generation, there is no singularity of the quark propagator at the origin, and the starting point of the cut moves towards infinity in the limit $g \to g_\infty$. Hence a quark propagator with mass gap must approach an entire function in this limit. This will be discussed in more detail in a sequel to this paper.

As in quantum electrodynamics, the behavior of the Fermi propagator near $M_j^2$ is strongly gauge dependent. Because of contributions from negative-norm states, the strength of the singularity is not relevant for the physical interpretation. It may, for instance, not be used as a criterion for confinement. It is therefore suggested to drop contributions from negative-norm states to the weight functions by forming the projected propagator (2.7). A pole of the projected propagator at $p^2 = M_j^2$ should indicate that a particle of mass $M_j$ can be observed. On the other hand, a sufficiently reduced singularity or vanishing of the propagator at the branch point $M_j^2$ would indicate confinement.

In the spectral representations

$$A^{+jl}(k^2) = \int_{M^2_{jl}}^{\infty} dk'^2 \, \frac{\sqrt{k'^2}\,\rho_1^{+jl}(k'^2) - \rho_2^{+jl}(k'^2)}{k'^2 - k^2}\,, \tag{5.8}$$

$$B^{+jl}(k^2) = \int_{M^2_{jl}}^{\infty} dk'^2 \, \frac{\rho_1^{+jl}(k'^2)}{k'^2 - k^2}\,, \tag{5.9}$$

of the projected propagator, the diagonal elements of the weight functions now satisfy the conditions[19]

$$\rho_1^{+jj} \geq 0\,, \quad 0 \leq \rho_2^{+jj} \leq 2\sqrt{k'^2}\,\rho_1^{+jj} \tag{5.10}$$

implied by the positive-definite metric of $\mathcal{H}^+$. Below, the corresponding positivity condition will be derived for the less familiar case of the projected vector propagator. In the following section it will be proved that the spectral representations (5.8) and (5.9) are convergent even though contributions from negative-norm states to the weight functions have been omitted. This is a general feature of asymptotically free models where the asymptotic behavior for large momenta is controlled by the behavior for small values of the coupling constant. In case of asymptotic freedom, the indefinite metric of the state space is thus not essential for the convergence of spectral representations.

For the vector mesons, the phenomenon of mass generation cannot be discussed as easily as for quarks, since zero-mass modes are likely to contribute to a cut of the transverse propagator starting at $k^2 = 0$. However, it is known from the perturbative treatment of models with spontaneous symmetry breaking by Higgs fields that the vector propagator acquires a pole of nonvanishing mass, while singularities corresponding to zero-mass modes cancel each other in the unitarity equations of the $S$ matrix. A similar mechanism of dynamical symmetry breaking has been proposed in various papers.[5] To our knowledge, it cannot be excluded that stable massive vector-meson states exist in quantum chromodynamics which decouple from the zero-mass states in the $S$ matrix. It is possible then that global invariance under the color group is spontaneously broken, but this does not necessarily have to be so.[23] Further, it is conceivable that the phenomenon of mass generation occurs with massive, confined, or screened vector mesons. Then a branch point instead of a pole is expected for the transverse propagator. In general, we define a gluon mass parameter $m_a^2$ by the lowest value in the spectrum $P_\mu P^\mu$ in the subspace of all vectors $\Phi$ with

$$(\Omega, A_\mu^a(x)\Phi) \neq 0\,, \quad \Phi \in \mathcal{H}^+\,.$$

The first singularity $m^2_{ab}$ of the transverse part $D^{+ab}$ of the propagator (2.6) on the positive real axis appears at the larger of the two values $m_a^2$ and $m_b^2$. $D^{+ab}$ satisfies the Lehmann representation

$$D^{+ab}(k^2) = \int_{M^2_{ab}}^{\infty} dk'^2 \, \frac{\rho^{+ab}(k'^2)}{k'^2 - k^2}\,. \tag{5.11}$$

We now show that the positive metric of $\mathcal{H}^+$ implies that the diagonal elements $\rho^{+aa}$ of the weight function are non-negative. Ignoring color indices, we form vectors

$$\Phi = \int dk\, c^{\mu\nu}(k)(k^2)^{-1/2}\tilde{A}^*_{\mu\nu}(k)\,|0\rangle$$

using test functions $c^{\mu\nu}$ which may be chosen antisymmetric

$$c^{\nu\mu} = -c^{\mu\nu}\,.$$

$\tilde{A}_{\mu\nu}$ denotes the Fourier transform of $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. To (5.12) we apply the projection operator $p$ on the subspace $\mathcal{H}^+$ of positive metric and take the norm

$$(\Phi_+, \Phi_+) = \int dk\, dl\, \bar{c}^{\mu\nu}(k) c^{\rho\lambda}(-l)(k^2)^{-1} \times \langle \tilde{A}_{\mu\nu}(k) p \tilde{A}_{\rho\lambda}(l)\rangle,\ \Phi_+ = p\Phi\,.$$

From (2.9) we find

$$\langle \tilde{A}_{\mu\nu}(k) p \tilde{A}_{\rho\lambda}(l)\rangle = 2(2\pi)^4 \delta(k+l)\,\theta(k_0) \times (k_\mu k_\rho g_{\nu\lambda} - k_\mu k_\lambda g_{\nu\rho} - k_\nu k_\rho g_{\mu\nu} + k_\nu k_\lambda g_{\mu\rho})\,\mathrm{Im}D^+\,.$$

Thus

$$-8(2\pi)^4 \int dk\, c_\mu(k)\bar{c}^\mu(k)\theta(k_\nu)(k^2)^{-1}\,\mathrm{Im}D^+(k^2) = (\Phi_+, \Phi_+) \gtrsim 0\,, \tag{5.12}$$

with

$$c_\mu = k^\rho c_{\rho\mu}\,.$$

For $k^2 \geq 0$, $k_0 \geq 0$ we have

$$c_\mu \bar{c}^\mu \leq 0 . \tag{5.13}$$

Equations (5.12) and (5.13) combined imply that

$$\rho^{+aa} = \frac{1}{\pi} \mathrm{Im} D^{+aa}(k^2) \geq 0 ,$$

in the sense of distributions.

The mass parameters in (5.11) either vanish:

$$m_{ab} = 0 ,$$

or have the $g$ dependence

$$m^2{}_{ab} = \kappa^2 c_{V0}^{ab} \exp\left[\int_{g^2}^{g_0{}^2} dx\, \beta^{-1}(x)\right], \; c_{V0}^{ab} > 0 . \tag{5.14}$$

If $m_{ab} \neq 0$ the projected transverse propagator approaches an entire function in the limit $g \to g_\infty$.

It remains to be shown that the Lehmann representations of the propagators as well as the projected propagators hold without subtractions. As a consequence of Lorentz invariance and spectrum conditions the structure functions of the propagators are regular analytic in the $k^2$ plane cut along $k^2 \geqslant 0$. From this, the Lehmann representations follow without subtractions since it will be seen in the next section that the structure functions of the propagator approach zero along every direction of the cut complex plane.

## VI. GENERAL ASYMPTOTIC PROPERTIES

### A. Complex plane

A convenient starting point for studying the asymptotic behavior of the diagonal gluon propagators $D = D^{aa}, D^{+aa}$ is given by the relations (3.17), which can be continued from the Euclidean region to any point $k^2$ of the cut complex plane. With the help of Eqs. (3.19)–(3.22), we may write Eq. (3.17) in the form

$$R\left(\frac{k^2}{\kappa^2}, g\right) = R\left(\frac{\kappa'^2}{\kappa^2}, g\right) R\left(\frac{k^2}{\kappa'^2}, g'\right) , \; g' = Q\left(\frac{\kappa'^2}{\kappa^2}, g\right) \tag{6.1}$$

for the components $R = R^{aa}, R^{+aa}$. Here and in the following we suppress indices $aa$ or $+aa$. In (6.1) the functions $R^{aa}$ and $R^{+aa}$ are normalized by Eq. (2.3) or Eq. (2.11), respectively.

The discussion of the asymptotic behavior for $|k^2| \to \infty$ is particularly simple along the ray

$$k^2 = -|k^2| e^{i\varphi}, \quad |\varphi| < \pi . \tag{6.2}$$

Setting $\kappa'^2 = -|k^2|$, the relation (6.1) yields

$$R\left(\frac{k^2}{\kappa^2}, g\right) = R\left(\left|\frac{k^2}{\kappa^2}\right|, g\right) R(e^{i\varphi}, Q), \quad Q = Q\left(\left|\frac{k^2}{\kappa^2}\right|, g\right) . \tag{6.3}$$

Using Eqs. (4.12) or (4.13) and their continuations, we obtain

$$\lim_{|k^2| \to \infty} R(e^{i\varphi}, Q) = \lim_{Q \to +0} R(e^{i\varphi}, Q) = 1 .$$

Together with the asymptotic formulas (4.5) and (4.7), we find then

$$D(k^2, g, \kappa^2) \simeq D_{\mathrm{as}}(k^2, g, \kappa^2) \tag{6.4}$$

for $|k^2| \to \infty$ along the ray (6.2), with the asymptotic form $D_{\mathrm{as}}$ given by

$$D_{\mathrm{as}} = -C_V k^{-2} \left(\ln\left|\frac{k^2}{\kappa^2}\right|\right)^{-\alpha_0/\beta_0} ,$$
$$C_V = (g^2 |\beta_0|)^{-\alpha_0/\beta_0} \exp\left[\int_{g^2}^{0} dx\, \tau(x)\right] > 0 . \tag{6.5}$$

By a slightly different method, this result can be generalized to any direction in the cut plane including lines parallel to the real axis. In general, we consider the propagator functions along any straight line

$$\frac{k^2}{|\kappa^2|} = c_1 + c_2 s, \quad s \geqslant 0 \tag{6.6}$$

which lies entirely within the cut $k^2$ plane. Since (3.17) is valid for complex $k^2$, the functions $D$ satisfy the Callan-Symanzik equation (3.23), (3.24)

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \beta \frac{\partial}{\partial g^2} + \gamma_V\right) D = 0 , \tag{6.7}$$

with the same coefficients at every point $k^2$ of the cut $k^2$ plane. Thus the differential equation

$$s \frac{\partial R}{\partial s} = \beta \frac{\partial R}{\partial g^2} + \gamma_V R \tag{6.8}$$

follows for

$$R = R(c_1 + c_2 s, g), \quad s \geqslant 0$$

considered as a function of $s$ and $g^2$. The solution of (6.8) is

$$R(c_1 + c_2 s, g) = R(c_1 + c_2, Q) \exp\left[\int_{g^2}^{Q^2} dx\; \gamma_V(x) \beta^{-1}(x)\right] ,$$
$$Q = Q(s, g), s \geqslant 0 \tag{6.9}$$

from which follows the asymptotic formula

$$D \simeq -C_V (c_2 s)^{-1} (\ln s)^{-\alpha_{V0}/\beta_0}, \text{ for } s \to \infty \tag{6.10}$$

with the same coefficient $C_V$ as in Eq. (6.5). In particular, for lines $k^2 \pm i\epsilon$ ($k^2 \geqslant 0$) parallel to the real axis, we have

$$D(k^2 \pm i\epsilon, g, \kappa^2) \simeq -C_V k^{-2} \times \left(\ln \frac{k^2}{|\kappa^2|}\right)^{-\alpha_{V0}/\beta_0} \quad \text{for } k^2 \to \infty . \tag{6.11}$$

In a similar way, the asymptotic behavior of the other propagators is obtained. We only quote the final results. The generalization of (6.1) is

$$R^{ab}\left(\frac{k^2}{\kappa^2},g\right)=\left[R^{aa}\left(\frac{\kappa'^2}{\kappa^2},g\right)R^{bb}\left(\frac{\kappa'^2}{\kappa^2},g\right)\right]^{1/2}\times R^{ab}\left(\frac{k^2}{\kappa'^2},g'\right),$$

$$S^{jl}\left(\frac{k^2}{\kappa^2},g\right)=\left[T^{jj}\left(\frac{\kappa'^2}{\kappa^2},g\right)T^{ll}\left(\frac{\kappa'^2}{\kappa^2},g\right)\right]^{1/2}\times S^{jl}\left(\frac{k^2}{\kappa'^2},g'\right),\tag{6.12}$$

$$T^{jl}\left(\frac{k^2}{\kappa^2},g\right)=\left[T^{jj}\left(\frac{\kappa'^2}{\kappa^2},g\right)T^{ll}\left(\frac{\kappa'^2}{\kappa^2},g\right)\right]^{1/2}\times T^{jl}\left(\frac{k^2}{\kappa'^2},g'\right),$$

$$g'=Q\left(\frac{\kappa'^2}{\kappa^2},g\right),$$

with analogous expressions for the projected functions. Along the ray (6.2), the asymptotic formulas for the diagonal elements of the quark propagator are

$$A^{jj}(k^2,g,\kappa^2)\simeq\frac{S_j}{|\beta_0|}C_F^{jj}\,|k|^{-1}e^{-(i/2)\varphi}\ln^{-1}\left|\frac{k^2}{\kappa^2}\right|,\tag{6.13}$$

$$B^{jj}(k^2,g,\kappa^2)\simeq C_F^{jj}\,|k|^{-2}e^{-i\varphi},\ \text{for}\ |k|\to\infty\tag{6.14}$$

with the coefficient given by Eq. (4.10). The asymptotic expression $B^{+jj}$ has the same form as Eq. (6.14). Using Eq. (4.13), we find for $A^{+jj}$ the weaker statement

$$A^{+jj}(k^2,g,\kappa^2)\lesssim|k|^{-1}e^{-(i/2)\varphi}\ln^{-\epsilon}\left|\frac{k^2}{\kappa^2}\right|.\tag{6.15}$$

The symbol $\lesssim$ means that the ratio of $A^{+jj}$ to the expression on the right-hand side approaches a finite, possibly vanishing limit for $|k^2|\to\infty$. For the color off-diagonal elements, we obtain

$$\lim_{|k^2|\to\infty}\ln^n\left|\frac{k^2}{\kappa^2}\right|G(k^2,g,\kappa^2)=0,$$

where (6.16)

$$G=|k^2|D^{ab},\ |k|A^{jl},\ |k^2|B^{jl},\ a\neq b, j\neq l$$

along the ray (6.2) for any $n>0$. If the expansion coefficients of the projected propagators with respect to powers of $g^2$ (and possibly $\ln g^2$) are color symmetric, one finds a decrease of the color off-diagonal elements similar to Eq. (6.16), Otherwise, only

$$D^{+ab}\lesssim|k|^{-2}\left(\ln\left|\frac{k^2}{\kappa^2}\right|\right)^{-\gamma_{V0}/\beta_0-\epsilon},$$

$$S^{+jl}\lesssim|k|^{-1}\ln^{-\epsilon}\left|\frac{k^2}{\kappa^2}\right|,\tag{6.17}$$

$$T^{+jl}\lesssim|k|^{-2}\ln^{-\epsilon}\left|\frac{k^2}{\kappa^2}\right|,\ a\neq b,\ j\neq l,\ 0<\epsilon\leq1$$

follows from (3.39) and (6.12).

Corresponding formulas are obtained along any line (6.6). Hence all structure functions vanish along any direction of the cut complex plane so that the unsubtracted Lehmann representations (5.1)–(5.3), (5.8) and (5.9), and (5.11) follow.

### B. Minkowski region

Generally, the propagators are tempered distributions in the Minkowski region $k^2\geq0$. However, if distribution type singularities are present only below a finite momentum value $K$, the propagators may be treated as ordinary functions for $k^2>K^2$. In this case the derivation of the asymptotic behavior given above also applies to the real positive axis, and the relations (6.4) and (6.5) and (6.13)–(6.17) hold for large Minkowski momenta with $\varphi=\pm\pi$. If distribution-type singularities occur for arbitrarily large momenta, the asymptotic behavior of the propagators should be studied in terms of the average values which are obtained by smearing with suitable test functions. After substituting $\lambda^2k^2$ for $k^2$, we may interpret the propagators as distributions in the dimensionless variable $\lambda^2$ at given positive $k^2$. We then form the average values

$$D^t(k^2,g,\kappa^2)=N^{-1}\int d\lambda^2 t(\lambda^2)\lambda^2 D(\lambda^2k^2,g,\kappa^2),$$

$$B^t(k^2,g,\kappa^2)=N^{-1}\int d\lambda^2 t(\lambda^2)\lambda^2 B(\lambda^2k^2,g,\kappa^2),\tag{6.18}$$

$$A^t(k^2,g,\kappa^2)=N^{-1}\int d\lambda^2 t(\lambda^2)\lambda A(\lambda^2k^2,g,\kappa^2),$$

$$N=\int d\lambda^2 t(\lambda^2),\ k^2>0$$

and study their asymptotic behavior for $k^2\to\infty$. Here $t(\lambda^2)$ is an arbitrary test function of positive support.

Let us first derive the asymptotic behavior for the averaged diagonal elements of the gluon propagator. If, in Eq. (6.3), the variable $k^2$ is replaced by $\lambda^2k^2$ with $k^2>0$, we have a relation between distributions in $\lambda^2$. Setting $\kappa'^2=-k^2$ and averaging with a test function $t(\lambda^2)$, we get

$$\int d\lambda^2 t(\lambda^2)\lambda^2 D(\lambda^2k^2,g,\kappa^2)=-k^{-2}R\left(\frac{k^2}{|\kappa^2|},g\right)\int d\lambda^2 t(\lambda^2)R(-\lambda^2,Q),$$

$$Q=Q\left(\frac{k^2}{|\kappa^2|},g\right),\ k^2>0.\tag{6.19}$$

With

$$\lim_{Q\to0}N^{-1}\int d\lambda^2 t(\lambda^2)R(-\lambda^2,Q)=1,$$

and Eqs. (4.5) and (4.7), we find

$$D^t(k^2,g,\kappa^2) \simeq D_{as}(k^2,g,\kappa^2) \text{ for } k^2 \to \infty\,. \tag{6.20}$$

We see that the averaged gluon propagator in the Minkowski region has again the asymptotic form (6.4), independent of the chosen test function.

In a similar way, the asymptotic behavior of the other structure functions is obtained. As examples, we quote the asymptotic forms for the average values of $A^{jj}$, $B^{jj}$, and $B^{+jj}$:

$$\begin{aligned} &A^{tjj} \simeq i\,\frac{S_j}{|\beta_0|} C_F'^j |k|^{-1} \ln^{-1}\frac{k^2}{|\kappa^2|}, \\ &B^{tjj} \simeq -C_F'^j |k|^{-2}, \quad B^{t+jj} \simeq -C_F''^j |k|^{-2}, \quad k^2 \to \infty. \end{aligned} \tag{6.21}$$

In the remainder of this section we discuss the asymptotic behavior for the weight functions of the Lehmann representations. We begin with the case of the diagonal elements of the gluon propagator

$$\pi\rho = \operatorname{Im} D = -k^{-2} \operatorname{Im} R$$

(with indices $aa, +aa$ suppressed). If $\rho$ is an ordinary function for large $k^2$, we set $\kappa'^2 = -k^2$ in (6.3) and take the imaginary part

$$\pi\rho(k^2, g, \kappa^2) = -k^{-2} R\left(\frac{k^2}{|\kappa^2|}, g\right) \operatorname{Im} R(-1, Q)\,,$$

$$Q = Q\left(\frac{k^2}{|\kappa^2|}, g\right).$$

From (4.12) and (4.13), the Taylor formula

$$\operatorname{Im} R(-1, Q) = -\pi\gamma_0 Q^2 + Q^2 h$$

follows with the remainder $h$ vanishing for $Q \to 0$. Hence

$$\rho(k^2, g, \kappa^2) \simeq \rho_{as}(k^2, g, \kappa^2) \text{ for } k^2 \to \infty \tag{6.22}$$

or

$$\lim_{k^2 \to \infty} \frac{\rho}{\rho_{as}} = 1\,,$$

with the asymptotic form

$$\rho_{as} = -\frac{\gamma_{V0}}{\beta_0} C_V |k|^{-2} \left(\ln\frac{k^2}{|\kappa^2|}\right)^{-\gamma_{V0}/\beta_0 - 1}, \quad C_V > 0\,. \tag{6.23}$$

At this point, it is important to note that the sign of the asymptotic discontinuity $\rho_{as}$ is determined by the ratio $\gamma_{V0}/\beta_0$. It is *negative* for $\gamma_{V0}/\beta_0 > 0$. Furthermore, we see from Eqs. (6.4), (6.5), and (6.10) that the transverse gluon propagator is *superconvergent* for $\gamma_{V0}/\beta_0 > 0$. We discuss the implications of these results in the last section.

For the diagonal weight functions of the quark propagator, we find

$$\begin{aligned} &\rho_1^{jj} = \pi^{-1} \operatorname{Im} B^{jj} \simeq \frac{\gamma'_{F1}}{\beta_0{}^2} C_F'^j |k|^{-2} \ln^{-2}\frac{k^2}{|\kappa^2|}, \\ &|k|\rho_1^{jj} - \rho_2^{jj} = \pi^{-1} \operatorname{Im} A^{jj} \\ &\qquad \simeq \frac{S_j}{|\beta_0|} C_F'^j |k|^{-1} \ln^{-1}\frac{k^2}{|\kappa^2|}, \\ &\rho_1^{+jj} \lesssim |k|^{-2} \ln^{-1-\epsilon}\frac{k^2}{|\kappa^2|}, \\ &|k|\rho_1^{+jj} - \rho_2^{+jj} \lesssim |k|^{-1} \ln^{-\epsilon}\frac{k^2}{|\kappa^2|}, \quad 0 < \epsilon \leq 1\,, \\ &\quad \text{for } k^2 \to \infty. \end{aligned} \tag{6.24}$$

The asymptotic behavior of the off-diagonal elements of the weight functions is analogous to Eqs. (6.16) and (6.17).

If the weight functions have distribution-type singularities at arbitrarily large momenta, we consider instead the asymptotic behavior of the average values:

$$\rho^t(k^2, g, \kappa^2) = N^{-1} \int d\lambda^2 t(\lambda^2) \lambda^2 \rho(\lambda^2 k^2, g, \kappa^2), \quad k^2 > 0 \tag{6.25}$$

with similar expressions for $\rho_1$ and $\rho_2$. As an example, we state the result for the diagonal weight functions of the gluon propagator. We obtain

$$\rho^t(k^2, g, \kappa^2) \simeq \rho_{as}(k^2, g, \kappa^2) \text{ for } k^2 \to \infty \tag{6.26}$$

with the same asymptotic form (6.23).

From the boundedness properties we have derived in this section, it follows immediately that the structure functions of the propagators cannot be entire functions for values of the coupling constant in the interval $0 < g < g_\infty$. In the presence of mass gaps, all branch points move to infinity at the infrared fixed point $g_\infty$. Although the color-off-diagonal propagators may well vanish, the color-diagonal structure function cannot be identically zero for $0 < g < g_\infty$, as may be seen from Eqs. (6.4) and (6.13). Hence the latter must have singularities on the positive real $k^2$ axis above any given point; in particular, there must be a singularity at infinity. It is sufficient to have one or more branch lines drawn to infinity. From a mathematical point of view, there is also the possibility of an infinite sequence of poles with an accumulation point at $k^2 = +\infty$.[24] However, even for ghost states, one would expect that poles are accompanied by branch points associated with the appropriate multiparticle states, so that meromorphy at very large values of $k^2$ is excluded. It is possible to give more general arguments for this exclusion of pole accumulations, but we do not want to pursue this point in the present paper.[25]

## VII. CONCLUSIONS

It may be helpful to have a summary of the assumptions made for obtaining the various results of the paper. We first list the general postulates. The existence of solutions to massless quantum chromodynamics is assumed, uniquely parametrized by a coupling constant $g$ and a normalization mass $\kappa$. For small $g$ these solutions should be represented by the formal perturbation expansion of Lagrangian field theory. Essential is the restriction to models with asymptotic freedom, i.e., $\beta_0 < 0$. The usual postulates of abstract quantum field theory are assumed to hold, but we allow for a state space with indefinite metric.[27] The spectral conditions are only used in the general sense that negative eigenvalues of $P_\mu P^\mu$ and $P_0$ are excluded. In particular, the existence of nontrivial discrete eigenvalues of $P_\mu P^\mu$ corresponding to particle states is not required. Consequently the postulate of asymptotic completeness need not hold.

For some results additional information is used concerning the properties of a state space $\mathcal{H}^+$ with positive-definite metric. $\mathcal{H}^+$ should be a Lorentz- and translational-invariant, linear subspace of the full state space[28] which does not involve new dimensional parameters in its construction. The propagators obtained by projecting the field operators onto $\mathcal{H}^+$ are assumed to approach their free-field values in the weak-coupling limit $g \to 0$.[18]

There are several interesting conclusions which can be drawn from the analyticity and boundedness properties of the propagators as obtained in the previous sections. Let us first consider the asymptotic behavior of the transverse gluon propagator $D^{aa}(k^2, g)$. We have shown that it has an unsubtracted Lehmann representation Eq. (5.1). From Eqs. (6.4), (6.5), and (6.11), it is seen that the propagator vanishes faster than $(k^2)^{-1}$ in all directions of the complex $k^2$ plane provided $\gamma_{V0}/\beta_0 > 0$. Taking the limit of $k^2 D^{aa}(k^2, g)$ in Eq. (5.1), for example in the direction $k^2 \to -\infty$, we obtain the superconvergence relation

$$\int_0^\infty dk^2 \operatorname{Im} D^{aa}(k^2 + i0, g) = 0 . \tag{7.1}$$

Furthermore the asymptotic form of the discontinuity of $D$ along the positive real $k^2$ axis is proportional to $-\gamma_{V0}/\beta_0$, and it is negative for $\gamma_{V0}/\beta_0 > 0$. Hence there must be a real point $k^2 = t(g) > 0$ above which $\operatorname{Im} D^{aa}(k^2, g) < 0$.

For the unprojected propagator, these properties imply that the ghosts determine the absorptive part for $k^2 > t(g)$ if $\gamma_{V0}/\beta_0 > 0$. However, within the framework of our assumptions, the same results should be valid for the projected propagator, and here they lead to contradictions since $\operatorname{Im} D^{+aa} \geq 0$. Given the reasonable assumption[18] that the projected propagator approaches the free-field expression for $g^2 \to +0$ as described in Eqs. (3.37) and (3.38), we find the consistency condition[26]

$$\gamma_{V0}/\beta_0 \leq 0 . \tag{7.2}$$

For the color gauge group SU(3), with all quarks in the fundamental (triplet) representation, this condition would correspond to the requirement

$$10 \leq N_f \leq 16 \tag{7.3}$$

for the number of flavors (triplets).[29] However, it is important to remember that $\gamma_{V0}$ is gauge dependent, and that all our results in this paper have been obtained in the Landau gauge. We have analyzed in detail the more complicated situation for general covariant gauges, and we will report on our results in a sequel to this paper, where we will also discuss the flavor condition in greater detail. Generally, we find that the results described above are not restricted to the Landau gauge.

In the case $\gamma_{V0}/\beta_0 < 0$, the gluon propagator vanishes less fast than $(k^2)^{-1}$ at infinity, and hence there are no difficulties. From the asymptotic formulas for the gluon propagator, it is seen that the structure of the theory becomes particularly simple for the special value $\gamma_{V0}/\beta_0 = -1$, which corresponds to $N_f = 12$ in the SU(3) model or $N_f = 4N$ for SU($N$), $N \geq 2$. The special role of $\gamma_{V0}/\beta_0 = -1$ is also apparent in our more detailed studies of the properties of projected propagators, which will be reported elsewhere.[18]

There are no consistency problems arising from our results for the quark propagator (at least in the Landau gauge used here). From Eqs. (5.3), (6.14), and (6.24), it is seen that we have a relation of the form

$$\int_0^\infty dk^2 \operatorname{Im} B^{jj}(k^2 + i0, g) = b^2 , \tag{7.4}$$

with

$$b^2 = \lim_{k^2 \to \infty} C_F^j\left(\frac{k^2}{|\kappa^2|}, g\right) > 0 . \tag{7.5}$$

The anomalous dimension of the Fermi fields is given by $\gamma_F(g^2) = \gamma_{F1} g^4 + \cdots$, with the $g^2$ term vanishing in the Landau gauge. The coefficient $\gamma_{F1}$ depends upon the normalization conditions, in contrast to the corresponding coefficient $\gamma_{V0}$ for the gluon field.

We note that our assumptions, which lead to the inequality (7.2), are connected with the short-distance behavior of the theory. Because of asymptotic freedom, the short-distance limit is of direct physical relevance, and it should be adequately described by the topologically trivial sector of the theory. We do not expect, therefore, that the existence of instantons, and associated features of the theory, are important for our conclusions.

From a more physical point of view, the apparent preference of the massless non-Abelian gauge theory for models with a sufficient, but limited number of flavors could be of importance for the understanding of the particle spectrum.[30] If verified, it may explain why nature presents us with a certain number of quarks which are rather similar to each other.

We have demonstrated in this paper the asymptotic vanishing of the structure functions for quark and gluon propagators in all directions of the complex $k^2$ plane.[31] These bounds imply that the structure functions cannot be *entire* unless they are identically zero. The color off-diagonal propagators may, of course, vanish, but the color diagonal functions cannot be identically zero for values of the coupling constant in the interval

$$0<g<g_\infty ,$$

because the coefficients of the asymptotic expressions (6.5) and (6.14) approach nonvanishing limits. Here $g_\infty>0$ is the first nonintegrable singularity of $\beta^{-1}(g^2)$.

For values of the coupling parameter $g^2$ in the domain $0<g^2<g_\infty{}^2$, the discontinuities $\rho(k^2,g)$ of color diagonal propagator functions do not vanish identically for $k^2$ larger than any given value. This fact follows from relations like (6.23). Hence the structure functions must have branch lines drawn to infinity along the positive real $k^2$ axis, even in the case of complete confinement. As we have pointed out in Sec. VI, the meromorphic alternative can be excluded.

Spontaneous symmetry breaking, for example, via a dynamical Higgs mechanism, can lead to mass gaps for quarks and possibly also for gluons. These mass gaps result in $g$-dependent branch points on the positive real $k^2$ axis of the structure functions. For the quark propagator, we get a lowest branch point $M^2(g)>0$. Generally, the character of this branch point is gauge dependent, but for the projected propagator we know that, in the case of complete confinement, it should be an integrable singularity as discussed above. Also the projected gluon propagator then has a positive, lowest, integrable branch point $m^2(g)>0$, provided there is also a mass gap for the gauge field.

The $g$ dependence of all these branch points is given by the renormalization group. For $g\to+0$, we have an exponential approach like $M^2(g) \propto \exp(1/\beta_0 g^2)\sim 0$, since $\beta_0<0$, which shows that the mass gaps are not visible in the formal expansion for $g\to+0$ (perturbation expansion). If, with increasing $g$, we reach the infrared limit $g_\infty=Q(0,g)$ (independent of $g$), then the branch points move to infinity, i.e., $M^2(g_\infty)=\infty$. $g_\infty$ is the first positive zero of $\beta$ where $\beta^{-1}$ is nonintegrable, or $g_\infty=\infty$ if there is no such zero. It is only in the limit $g\to g_\infty$ that the propagators approach entire functions, provided there is a mass gap for $g<g_\infty$.

In a later paper, we will use the projected propagators, together with the quark-gluon vertex function, in order to give a definition of the effective coupling $Q(k^2,g)$ for nonasymptotic values of $k^2$. This definition leads to a function $Q^2(k^2,g)$ which satisfies an unsubtracted dispersion relation in $k^2$.

## ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant No. PHY78-01224.

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