PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Stability and bifurcation in Yang-Mills theory

R. Jackiw and P. Rossi*
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 9 October 1979)

Whenever gyroscopic forces are present, stable static solutions to dynamical equations of motion need not minimize the energy. We show that this happens in the classical Yang-Mills theory with sources, and we identify the stable fluctuations which lower the energy. The finite form of these infinitesmal, time-dependent deformations of the known static solutions is obtained for weak external sources, and a unified description of both the static and time-varying solutions is given. Also, we demonstrate that the previously found bifurcation in the presence of strong sources is characterized by a zero-eigenvalue mode which dominates the behavior of the solutions near the bifurcation point. The stability properties of the bifurcating solutions are assessed.

I. INTRODUCTION

Recent investigations of classical Yang-Mills theory have concentrated on solutions to the field equations in the presence of static, prescribed sources. A variety of results has been obtained, but no systematic discussion that organizes them is in hand. In this paper we offer an analysis, based on properties of fluctuations about steady motion, which helps towards a comprehensive description. Also we take the opportunity to review the general theory of stability—a topic widely studied in former times by physicists but now known mostly to mathematicians—and assess the stability of some of the available solutions. The general theory, which does not rely on minima of the energy, is found to be applicable to the Yang-Mills model, here shown to share the physical attributes of an upright top: Some static solutions—critical points of the (constrained) energy—are stable even though they are not (local) minima.

Section II is devoted to general remarks about stability, which are then exemplified in a toy model that provides a simple setting for some of the features of Yang-Mills theory. In Sec. III, we review the solutions which we then analyze in Sec. IV. Stability is established for weak external sources. When the source is sufficiently strong to produce a bifurcation, a zero-eigenvalue mode is present in the stability equations. The stability behavior near the bifurcation is analyzed in terms of this mode and is related to the stability properties at the bifurcation point. Various technical computations comprise the Appendices.

II. STABILITY IN DYNAMICAL SYSTEMS

A. Definition of stability

Consider a time-translation-invariant system whose equations of motion for the $2N$ dynamical variables P_n and Q_n, $n=1,\ldots,N$, can be obtained from a Hamiltonian $H(P,Q)$, which is also the conserved energy,

$$\dot{P}_n = -\frac{\partial H(P,Q)}{\partial Q_n}, \qquad \dot{Q}_n = \frac{\partial H(P,Q)}{\partial P_n}. \tag{2.1}$$

A static solution, one for which $\dot{P}$ and $\dot{Q}$ vanish, is a critical point of H, and *vice versa*, stationary points of the energy define static solutions. (An overdot means differentiation with respect to time.)

A natural question with which we shall concern ourselves here is whether or not a static solution $\{P^{(s)}, Q^{(s)}\}$ is stable. By "stable" one means that if initial data are given to be $\{P^{(s)}+\delta P, Q^{(s)}+\delta Q\}$, where δP and δQ are "small" quantities, then the subsequent time evolution keeps the solution "near" the static configurations. In order to convert the above qualitative notion to a well-posed definition, we shall take stability as the requirement that the equations of motion for δP and δQ, when linearized about the static solution, do not yield exponential growth in time. In other words, for stable motion the small quantities $\{\delta P, \delta Q\}$ fluctuate harmonically in time with real frequency, while complex frequencies signal instability.

The above criterion for stability is also in accord with quantum-mechanical ideas. The first quantum correction to the energy of a state involves the fluctuation frequencies. That quantity must be real for the state to be quantum-mechanically stable.

Note that growth with time of the fluctuations smaller than exponential, say polynomial, is not a sign of instability. In such a circumstance, the eigenfrequencies are degenerate, but still real, and the quantal energy remains real. We shall not

 © 1980 The American Physical Society

discuss the very difficult problem of the relationship between stability in the small fluctuations and true global stability. No general analysis exists, and the question continues to be the subject of mathematical research.[1]

B. Conditions for stability

An intuitively appealing idea is that stability should be connected with energy minima; and it is widely assumed by contemporary physicists that a static solution, which necessarily is a stationary point, also is a (local) minimum when the solution is stable. More precisely, the minimality condition is the requirement that only non-negative eigenvalues occur for the quadratic Hamiltonian matrix, $\mathcal{H}$, defined by expanding $H(P,Q)$ about $\{P^{(s)},Q^{(s)}\}$ and retaining quadratic terms in $\{\delta P,\delta Q\}$ (linear terms vanish since the expansion is about a critical point):

$$\begin{aligned} H(P,Q) &= H(P^{(s)},Q^{(s)})+\tfrac{1}{2}\delta P_n T_{nm}\delta P_m \\ &\quad +\delta P_n G_{nm}\delta Q_m+\tfrac{1}{2}\delta Q_n V_{nm}\delta Q_m+\cdots \\ &= H(P^{(s)},Q^{(s)})+\tfrac{1}{2}\tilde{X}\mathcal{H}X+\cdots , \end{aligned} \tag{2.2}$$

$$\mathcal{H}=\begin{bmatrix} T & G \\ \tilde{G} & V\end{bmatrix},\quad X=\begin{bmatrix}\delta P\\ \delta Q\end{bmatrix}, \tag{2.3}$$

$$\det(\mathcal{H}-\lambda I)=0\Rightarrow\lambda\geq 0 . \tag{2.4}$$

In (2.2) we make use of the summation convention and the tilde indicates transposition.

In fact minimality is a sufficient condition for stability—a result, known as Dirichlet's theorem,[1] which will become apparent below—but by no means is it a necessity. Indeed there are familiar physical systems (tops, gyroscopes, planetary configurations) which are stable, even though their energy is not locally minimal. To derive a more general condition, we expand (2.1) around its static solution and find

$$\mathcal{H}X=i\eta\dot{X}, \tag{2.5}$$

$$\eta=\begin{bmatrix} 0 & -iI\\ iI & 0\end{bmatrix}. \tag{2.6}$$

By making a monochromatic *Ansatz* for X,

$$X=e^{-i\omega t}x, \tag{2.7}$$

we recognize the (constant) x as simplectic eigenvectors of $\mathcal{H}$ with simplectic eigenvalue ω:

$$\mathcal{H}x=\omega\eta x . \tag{2.8}$$

It is clear that our definition of stability requires the ω's be real; this is known as Liapunov's theorem[1]:

$$\det(\mathcal{H}-\omega\eta)=0\Rightarrow\omega \text{ real} . \tag{2.9}$$

The point is that (2.9) is in general different from (2.4) and can be satisfied when (2.4) fails.

If (2.8) is premultiplied by $x^\dagger$, where the dagger indicates transposition and complex conjugation,

$$x^\dagger\mathcal{H}x=\omega x^\dagger\eta x , \tag{2.10}$$

we see that the left-hand side is real, $\mathcal{H}$ being real symmetric, hence Hermitian. Also $x^\dagger\eta x$ is real since η is Hermitian, and we conclude that ω can fail to be real only when $x^\dagger\mathcal{H}x$ and $x^\dagger\eta x$ vanish. So when $\mathcal{H}$ is positive definite, ω is real and Dirichlet's theorem is established: Minimality implies stability. The more general situation reflects the possibility that ω may be real without $\mathcal{H}$ being positive definite.

One may consider η as a metric on the vector space of the x's. Then (2.8) is the condition that $x^\dagger\mathcal{H}x$ be stationary against variations of x which preserve the simplectic length $x^\dagger\eta x$. Instability can occur only when there are zero-length simplectic eigenvectors of $\mathcal{H}$. The simplectic eigenvalue equation in (2.9) is relevant to the program of diagonalizing by simplectic matrices, just as the corresponding equation in (2.4) arises when diagonalizing with orthogonal matrices. (A matrix M is simplectic when $\tilde{M}\eta M=\eta$.)

Conditions (2.4) and (2.9) are clearly different, and no simple relationship exists between the two in the general case. In practice, we can specialize somewhat. Firstly, the kinetic-energy matrix T in (2.2) and (2.3) is taken to be positive definite; with an appropriate definition of coordinates, we may choose it to be the identity. Secondly, the off-diagonal matrix G arising from mixed p-q terms in the Hamiltonian, which are frequently called gyroscopic or Coriolis terms, is always antisymmetric, when the theory is derivable from a Lagrangian. The reason is that any symmetric piece in such velocity-dependent forces corresponds to a total time derivative in the Lagrangian and may be dropped. Thus we are led to a simpler form for $\mathcal{H}$,

$$\mathcal{H}=\begin{bmatrix} I & G\\ -G & V\end{bmatrix},\qquad \tilde{G}=-G ,\quad \tilde{V}=V . \tag{2.11}$$

With this $\mathcal{H}$, the simplectic eigenvalue problem (2.8) reduces to

$$[(i\omega+G)(i\omega+G)+V]\,\delta Q=0 , \tag{2.12}$$

and the stability condition (2.9) becomes

$$\det(2i\omega G+G^2+V-\omega^2 I)=0\Rightarrow\omega \text{ real} . \tag{2.13}$$

To bring out the difference from the minimality condition (2.4), we first recognize that positivity of $\mathcal{H}$ is entirely equivalent to positivity of $\tilde{M}\mathcal{H}M$, with

$$M = \begin{pmatrix} I & -G \\ 0 & I \end{pmatrix}, \quad \tilde{M}\mathcal{H}M = \begin{pmatrix} I & 0 \\ 0 & G^2+V \end{pmatrix}.$$

Then (2.4) becomes equivalent to

$$\det(G^2+V-\lambda I)=0 \Rightarrow \lambda \geq 0. \tag{2.14}$$

This is analogous to (2.13), but an obvious difference exists when G is present. In the absence of gyroscopic forces, the two conditions coincide: ω^2 may be identified with λ, and instability occurs only for imaginary ω. In the presence of gyroscopic terms, there may occur stable, static solutions which do not minimize the energy, while instability can exist with complex ω. When $\mathcal{H}$ is as in (2.11), the condition for instability, $x^\dagger \eta x=0$, is equivalent to

$$\mathrm{Re}\,\omega = i\,\frac{\delta Q_n^* G_{nm}\delta Q_m}{\delta Q_n^*\delta Q_n}. \tag{2.15}$$

We shall use the phrase "gyroscopic stability" when we make a distinction with the more familiar "energetic stability." A hint for gyroscopic stability occurs when we can find arbitrarily close to a static solution harmonic fluctuations that lower the energy. As we shall show, such solutions exist in the Yang-Mills theory. Instability would be indicated when there are, arbitrarily close to the static solution, time-dependent solutions which decrease the energy and grow exponentially in time.

To conclude our review of stability theory, let us remark that although we discussed Dirichlet's sufficient condition in terms of the energy constant of motion, a similar criterion can be formulated by reference to other constants of motion. This generalization is useful when analyzing solutions invariant with respect to the symmetry transformation which is associated with the constant in question.[1]

C. Example

The above remarks are well illustrated by the following Hamiltonian[1]:

$$H=\tfrac{1}{2}(P_1^2+P_2^2)+g(P_1Q_2-P_2Q_1)+\tfrac{1}{2}\Omega^2(Q_1^2+Q_2^2), \quad g>0, \quad \Omega \geq 0. \tag{2.16}$$

(A physical realization is the motion of a symmetric top, with Q_1, Q_2 being the direction cosines for the axis of symmetry relative to two orthogonal horizontal axes. The angular momentum along the symmetry axis is $2g$, and $g^2-\Omega^2$ is the potential energy of the center of mass. The symmetric moments of inertia are set equal to unity.) Vanishing P_n and Q_n, the origin in phase space, provide a static solution, while (2.16) is the quadratic Hamiltonian which determines stability. Applying the minimality test (2.14), we find that the origin is a minimum of the energy only when $\Omega \geq g$. However, the stability criterion (2.13) is always fulfilled since the four frequencies are $\omega = \pm g \pm \Omega$. (In this example the motion is stable in the absence of gyroscopic terms. This is not a general condition for gyroscopic stability; gyroscopic terms can stabilize an otherwise unstable configuration.) For $\Omega \neq 0$, the solution to the fluctuation equations is

$$\begin{aligned} Q_1 &= a_1\cos(\Omega+g)t + a_2\sin(\Omega+g)t \\ &\quad + a_3\cos(\Omega-g)t + a_4\sin(\Omega-g)t, \\ Q_2 &= -a_1\sin(\Omega+g)t + a_2\cos(\Omega+g)t \\ &\quad + a_3\sin(\Omega-g)t - a_4\cos(\Omega-g)t, \end{aligned} \tag{2.17}$$

with energy

$$\mathcal{E} = \Omega(a_1^2+a_2^2)(\Omega+g) + \Omega(a_3^2+a_4^2)(\Omega-g). \tag{2.18}$$

For $\Omega > g$ this is positive, confirming that the vanishing energy of the static solution is minimal. However, when $g>\Omega$, the energy can be lowered below zero without loss of stability.

When $\Omega=0$, the frequencies are pairwise degenerate and the solution becomes

$$\begin{aligned} Q_1 &= a_1\cos gt + a_2\sin gt - b_1 gt\sin gt + b_2 gt\cos gt, \\ Q_2 &= -a_1\sin gt + a_2\cos gt - b_1 gt\cos gt - b_2 gt\sin gt. \end{aligned} \tag{2.19}$$

The energy can again be lowered below zero,

$$\mathcal{E} = \tfrac{1}{2}g^2[(a_1+b_1)^2+(a_2+b_2)^2-a_1^2-a_2^2]. \tag{2.20}$$

Note that the degeneracy of the frequencies produces in the fluctuations a linear rise with t. Nevertheless, we consider them to be stable.

It is also interesting to examine the quantum-mechanical version of (2.16). (We do not here take the variables $\{P_n, Q_n\}$ to be angular, but rather Cartesian.) The Hamiltonian is recognized to be a two-dimensional harmonic oscillator augmented by gM, where M is the angular momentum with eigenvalues $\hbar m$, $m=0,\pm1,\ldots$. The harmonic-oscillator ground state has energy $\hbar\Omega$, and since $m=0$ this is also an energy eigenvalue of H. The first excited state of the oscillator has energy $2\hbar\Omega$ and is doubly degenerate; $m=1,-1$. Hence the energy eigenvalues of H are $\hbar(2\Omega \pm g)$. When $g>\Omega$, the state with $m=-1$ has lower energy than the harmonic-oscillator ground state. The quantal ground state is not the one with maximal symmetry (corresponding to the static classical solution) but rather one with nonvanishing angular momentum (corresponding to a time-dependent classical solution).

When $\Omega=0$, the motion becomes essentially free; it is unbounded, but entirely stable.

III. YANG-MILLS FIELDS

We list here some finite-energy solutions to the Yang-Mills equations with sources:

$$\begin{aligned} &\mathfrak{D}_\mu F^{\mu\nu}=\delta^{\nu 0}\rho\,, \\ &F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+[A^\mu,A^\nu]\,, \\ &E^i=F^{i0}\,,\quad B^i=-\tfrac{1}{2}\epsilon^{ijk}F_{jk}\,, \\ &\mathfrak{D}_\mu=\partial_\mu+[A_\mu,\ ]\,. \end{aligned} \tag{3.1}$$

[We study the SU(2) theory with coupling strength scaled to unity and use interchangeably component notation and anti-Hermitian matrix notation, e.g., ρ_a, $a=1,2,3$; $\rho=\rho_a\sigma^a/2i$, $\sigma^a=$ Pauli matrices.] The source, assumed extended, is a given time-independent function, $\partial_t\rho=0$. The covariant conservation law for the source is reduced to

$$[A^0,\rho]=0\,, \tag{3.2a}$$

or in component notation

$$\epsilon_{abc}A_b^0\rho_c=0\,, \tag{3.2b}$$

which means that in a solution the time component of the vector potential always is parallel to the source. The energy constant of motion is

$$\mathcal{E}=\tfrac{1}{2}\int d\vec{r}(\vec{E}_a{}^2+\vec{B}_a{}^2)\,. \tag{3.3}$$

For all solutions of concern to us this is finite; consequently we do not discuss point sources.

Presentation of solutions is complicated by the gauge covariance of (3.1): If A^μ solves the equations with source ρ, then the equations with a gauge-rotated source ρ'

$$\rho'=U^{-1}\rho U \tag{3.4a}$$

are solved by the gauge transformation of the previous solution

$$A'^\mu=U^{-1}A^\mu U+U^{-1}\partial^\mu U\,. \tag{3.4b}$$

[Here U is an SU(2) matrix, i.e., a 2×2 unitary matrix with unit determinant.] We shall take the point of view that two solutions related by a gauge transformation, as described above, represent the same solution in different "gauge frames." Frequently we shall speak of an "Abelian gauge frame"—one in which the source points in the third direction:

$$\rho_a=\delta_{a3}q\,. \tag{3.5}$$

Of course, results for gauge-invariant quantities, like the energy, are frame independent.

In addition to the above gauge covariance, there exists also a gauge invariance with respect to gauge transformations which leave the source unchanged. From (3.2), we see that gauge transformations involving A^0 in the gauge function do not affect the source. Thus it is always possible to pass to the temporal gauge where A^0 vanishes, without changing the gauge frame.

Solutions naturally fall into two classes: Those that are present for arbitrary sources and those that require a critical, nonvanishing source strength. We list these in turn.

A. Sources with arbitrary strength

When a solution exists for arbitrary source strength, it will in particular be present for weak sources, where an expression for it can be given perturbatively in the source strength. So that we can speak of order of perturbation, we shall take the source to be $O(Q)$, where Q is a convenient scale of magnitude for the source.

Of course the most obvious solution is the static Coulomb one, which is easily presented in the Abelian gauge frame by an exact formula[2]:

$$A_a^0=\delta_{a3}\varphi\,, \tag{3.6a}$$

$$\vec{A}_a=0\,, \tag{3.6b}$$

$$\varphi=\frac{-1}{\nabla^2}q\,. \tag{3.6c}$$

An alternative description, still in the same gauge frame, is given by passing to the temporal gauge:

$$A_a^0=0\,, \tag{3.7a}$$

$$\vec{A}_a=\delta_{a3}\vec{\nabla}\varphi t\,. \tag{3.7b}$$

The energy of this, according to (3.3), is the familiar Coulomb expression

$$\begin{aligned}\mathcal{E}_c&=\tfrac{1}{2}\int q\,\frac{-1}{\nabla^2}q \\ &=\frac{1}{8\pi}\int d\vec{r}d\vec{r}'q(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}q(\vec{r}')\,.\end{aligned} \tag{3.8}$$

Note that in the Abelian frame the solution vanishes with the source.

Another static solution is also known. It differs from the Coulomb one by the property that in the Abelian frame it does not vanish with the source; rather it becomes a pure gauge,[3]

$$A^\mu \underset{\rho\to 0}{\longrightarrow} U^{-1}\partial^\mu U\,. \tag{3.9}$$

A closed expression cannot be given; only a perturbative formula is available. Its most economical description is in the frame obtained by transforming from the Abelian frame with the gauge function U, occurring in (3.9):

$$\rho' = U\rho U^{-1} = U\frac{\sigma^3}{2i}U^{-1}q\,, \tag{3.10a}$$

$$A'^0 = \Phi + O(Q^3)\,, \tag{3.10b}$$

$$\vec{A}' = \frac{1}{\nabla^2}[\Phi, \vec{\nabla}\Phi] + O(Q^4)\,, \tag{3.10c}$$

$$\Phi = \frac{-1}{\nabla^2}\rho'\,. \tag{3.10d}$$

The primes remind that the solution is being presented in a gauge frame other than the Abelian one. The gauge function U is not arbitrary but must be chosen so that (3.2) is satisfied. It is a consequence of that condition and of (3.10b) that we must have

$$[\Phi, \nabla^2\Phi] = 0\,. \tag{3.11}$$

The following is the temporal gauge equivalent to (3.10):

$$A'^0 = 0\,, \tag{3.12a}$$

$$\vec{A}' = \vec{\nabla}\Phi t + \left(\frac{1}{2}t^2 + \frac{1}{\nabla^2}\right)[\Phi, \vec{\nabla}\Phi] + O(Q^3)\,, \tag{3.12b}$$

with an $O(Q)$ electric field

$$\vec{E}' = -\vec{\nabla}\Phi - t[\Phi, \vec{\nabla}\Phi] + O(Q^3)\,, \tag{3.12c}$$

and an $O(Q^2)$ magnetic field

$$\vec{B}' = \vec{\nabla}^{-1}\times[\Phi, \vec{\nabla}\Phi] + O(Q^3)\,, \quad \vec{\nabla}^{-1} = \vec{\nabla}/\nabla^2\,. \tag{3.12d}$$

[The time dependence in (3.12) is, of course, a consequence of the gauge choice, as comparison with (3.10) shows.] In the primed frame, the solution appears similar to the Coulomb one, (3.6) or (3.7), except that the nonvanishing commutator $[\Phi, \vec{\nabla}\Phi]$ prevents the expressions from closing. Hence we call the above a "non-Abelian Coulomb" solution to contrast it with the "Abelian Coulomb" solution of Eqs. (3.6) and (3.7). The energy of the non-Abelian Coulomb solution also exhibits similarities with the Abelian Coulomb case. The formula to lowest order in Q follows from (3.3) and (3.12c),

$$\mathcal{E} = \tfrac{1}{2}\int \rho'_a \frac{-1}{\nabla^2}\rho'_a + O(Q^4) = \frac{1}{8\pi}\int d\vec{r}\,d\vec{r}\,'\rho'_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}\,'|}\rho'_a(\vec{r}\,') + O(Q^4)\,. \tag{3.13}$$

A specific example of a non-Abelian Coulomb solution can be given when the source in the Abelian frame is spherically symmetric,

$$\rho_a = \delta_{a3}q(r)\,. \tag{3.14a}$$

One then verifies that (3.11) is satisfied with the charge density in the radial frame;

$$\rho'_a = \hat{r}^a q(r)\,, \tag{3.14b}$$

i.e., U is the gauge transformation which rotates the third axis into the radial axis. A further interesting feature is that the present solution carries less energy than the corresponding Coulomb one,[3]

$$\mathcal{E} = \frac{1}{8\pi}\int d\vec{r}\,d\vec{r}\,' q(r)\frac{\hat{r}\cdot\hat{r}'}{|\vec{r}-\vec{r}\,'|}q(\vec{r}\,') + O(Q^4) \leq \mathcal{E}_c\,. \tag{3.14c}$$

In addition to the above static solutions, there are also time-dependent ones that we shall want to study. One family comprises a generalization of the Abelian Coulomb solutions (3.6) and (3.7) in that it possesses the same magnetic field, i.e., a vanishing amount (this is a gauge-invariant property):

$$\vec{B}'_a = \vec{\nabla}\times\vec{A}'_a - \tfrac{1}{2}\epsilon_{abc}\vec{A}'_b\times\vec{A}'_c = 0\,. \tag{3.15}$$

Primes again are used as an indication that the solution is being presented in a frame other than the Abelian Coulomb one; see (3.19) below. It is easy to see that all potentials satisfying the Yang-Mills equations and (3.15) are characterized in the gauge $A'^0 = 0$ by conditions on the electric field, which has to be static:

$$\vec{A}' = -\vec{E}'t\,, \tag{3.16a}$$

$$\vec{E}' = -\vec{\nabla}\Phi\,, \tag{3.16b}$$

$$[\vec{\nabla}\Phi, \vec{\nabla}\Phi] = 0\,. \tag{3.16c}$$

According to Gauss's law, the time-component of Yang-Mills equations, the source which gives rise to these solutions is related to Φ by Poisson's equation

$$\nabla^2\Phi = -\rho'\,. \tag{3.17}$$

The energy expression following from (3.3), (3.16b), and (3.17) is of the Coulomb form:

$$\mathcal{E} = \tfrac{1}{2}\int \rho'_a \frac{-1}{\nabla^2}\rho'_a = \frac{1}{8\pi}\int d\vec{r}\,d\vec{r}\,'\rho'_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}\,'|}\rho'_a(\vec{r}\,')\,. \tag{3.18}$$

For the problem at hand, we are not interested in arbitrary configurations satisfying (3.16), but only in those for which the source in (3.17) is gauge equivalent to the source given in the Abelian frame:

$$\rho' = U\rho U^{-1} = U\frac{\sigma^3}{2i}U^{-1}q\,. \tag{3.19}$$

In that case the solution can be presented in the Abelian frame by gauge transforming with U,

$$A^0 = 0\,, \tag{3.20a}$$

$$\vec{A} = -\vec{E}t - U^{-1}\vec{\nabla}U\,, \tag{3.20b}$$

$$\vec{E} = -U^{-1}\vec{\nabla}\Phi U\,. \tag{3.20c}$$

To recapitulate, the solution for a given source (3.5) is constructed by choosing a gauge function U; computing ρ' from (3.19); Φ from (3.17) and finally the potentials from (3.16) or (3.20). When (3.16c) is met, one has a time-dependent solution, which can be viewed as a continuous deformation of the static Abelian Coulomb solution, obtained by a continuous development of I to U. We shall demonstrate in Sec. IV that the energy (3.18) of this time-dependent generalization can be lowered by an arbitrary amount below the Coulomb value (3.8).

One can show that in general these solutions are essentially time dependent—a time translation cannot be compensated by a gauge transformation. The only member of the family for which the time dependence can be absorbed in a gauge function is the Coulomb one where $U = I$. (This and the previous assertions are proven in Appendix A.)

For a given source, there are as many members of this family of solutions as there are gauge functions U. A specific example can be given for the radial source (3.14a),[4]

$$\vec{A}_a = -\vec{E}_a t + \delta_{a1}\vec{\nabla}\left(\alpha r^2 \frac{d\varphi}{dr}\right), \tag{3.21a}$$

$$\vec{E}_a = -\frac{\hat{r}}{r^2}\frac{1}{\alpha}\left\{\delta_{a3}\sin\left(\alpha r^2\frac{d\varphi}{dr}\right) + \delta_{a2}\left[1-\cos\left(\alpha r^2 \frac{d\varphi}{dr}\right)\right]\right\}, \tag{3.21b}$$

$$\varphi = -\frac{1}{\nabla^2}q\,. \tag{3.21c}$$

This is of the form (3.20) with

$$U = \exp\left(\frac{i\sigma^1}{2}\alpha r^2 \frac{d\varphi}{dr}\right). \tag{3.21d}$$

Here q and ϕ depend only on r. The solution is characterized by an arbitrary parameter α, which when set to zero gives $U = I$ and the Abelian Coulomb solution is regained. The energy is

$$\mathcal{E} = 8\pi \int_0^\infty dr\, \frac{1}{\alpha^2 r^2}\sin^2\left(\frac{\alpha}{2} r^2 \frac{d\varphi}{dr}\right). \tag{3.21e}$$

Just as the above generalizes in a well-defined way the Abelian Coulomb solution into time-dependent generalizations of the non-Abelian Coulomb solution discussed in Eqs. (3.10)–(3.12). The feature which characterizes the present family is that there exists a gauge in which the magnetic field coincides to $O(Q^2)$ with that of the non-Abelian Coulomb solution, (3.12d). The time-dependent solution is constructed perturbatively as is its time-independent antecedent. Define the $O(Q)$ matrix Φ by

$$\Phi = \frac{-1}{\nabla^2}\rho''\,. \tag{3.22}$$

We use double primes to distinguish this source from ρ—the source in Abelian frame—and from ρ'—the source in the gauge transformed frame where the non-Abelian Coulomb solution has a simple perturbative expansion; see (3.10). We are interested only in those ρ'' that are gauge transforms of ρ',

$$\rho'' = U\rho' U^{-1}\,. \tag{3.23}$$

In the temporal gauge, our time-dependent solution is represented by

$$A''^0 = 0\,, \tag{3.24a}$$

$$\vec{A}'' = \vec{\nabla}\Phi t + \left(\frac{1}{2}t^2 + \frac{1}{\nabla^2}\right)([\Phi, \vec{\nabla}\Phi] - \vec{\nabla}^{-1}[\Phi, \nabla^2\Phi]) + O(Q^3)\,. \tag{3.24b}$$

One readily computes the electric field, which is $O(Q)$,

$$\vec{E}'' = -\vec{\nabla}\Phi - t([\Phi, \vec{\nabla}\Phi] - \vec{\nabla}^{-1}[\Phi, \nabla^2\Phi]) + O(Q^3)\,. \tag{3.24c}$$

The magnetic field is indeed $O(Q^2)$ and has the same static form as in (3.12d),

$$\vec{B}'' = \vec{\nabla}^{-1}\times[\Phi, \vec{\nabla}\Phi] + O(Q^3)\,. \tag{3.24d}$$

The energy, to lowest order in Q, is

$$\mathcal{E} = \tfrac{1}{2}\int \rho''_a \frac{-1}{\nabla^2}\rho''_a + O(Q^4)$$
$$= \frac{1}{8\pi}\int d\vec{r}\,d\vec{r}'\,\rho''_a(\vec{r})\,\frac{1}{|\vec{r}-\vec{r}'|}\rho''_a(\vec{r}') + O(Q^4)\,. \tag{3.25}$$

In Sec. IV, we show that also the above energy can lie below the corresponding energy of the static solution (3.13) by an arbitrary amount.

In comparing this time-dependent solution to the static non-Abelian Coulomb solution, we see that the generalization does not require (3.11). Similarly upon comparing with the time-dependent generalization of the Abelian Coulomb solution (3.16), we note that (3.24b) differs from (3.16a) and (3.16b) in the term involving $[\Phi, \vec{\nabla}\Phi] - \vec{\nabla}^{-1}[\Phi, \nabla^2\Phi]$, which may also be written as $-\vec{\nabla}^{-1}\times\vec{\chi}$, $\chi^i = \epsilon^{ijk}[\partial_j\Phi, \partial_k\Phi]$. But according to (3.16c) the vanishing of $\vec{\chi}$ is precisely one of the characteristics of that solution. So we may summarize the four solutions by the following scheme.

Given a static source, construct Φ by solving Poisson's equation. Consider next the vector $\vec{C}=[\Phi, \vec{\nabla}\Phi]$. If this is zero, one is dealing with the static Abelian Coulomb case; otherwise construct $\vec{\nabla}\times\vec{C}$ and $\vec{\nabla}\cdot\vec{C}$. When the former vanishes, (3.16c) is satisfied; one has the time-dependent generalization of the Abelian Coulomb. When the latter vanishes (3.11) is true; one is dealing with the static non-Abelian Coulomb case. Finally when none of the above happens, one can construct the time-dependent generalization of the non-Abelian Coulomb. The different possibilities arise for different sources. We are interested only in those differences in the sources which can be achieved by gauge transformations. In each case, in the temporal gauge

$$A^0=0, \quad \vec{A}=\frac{-1}{\nabla^2}\left[\vec{\nabla}\rho t+\left(\frac{1}{2}t^2+\frac{1}{\nabla^2}\right)\vec{\nabla}\times\vec{\nabla}\times\vec{C}\right], \tag{3.26}$$

and the energy has the form

$$\mathcal{E}=\tfrac{1}{2}\int\rho\,\frac{-1}{\nabla^2}\rho = \frac{1}{8\pi}\int d\vec{r}\,d\vec{r}\,'\rho_a(\vec{r})\,\frac{1}{|\vec{r}-\vec{r}\,'|}\rho_a(\vec{r}\,'), \tag{3.27}$$

with ρ being the appropriate charge density in the different solutions. This is exact for the Abelian Coulomb and its time-dependent generalization, while in the non-Abelian Coulomb case and time-dependent generalization, (3.26) and (3.27) comprise the $O(Q^2)$ contribution.

B. Sources with critical strength

When the source strength Q increases, the previous solutions continue to be present. For the Abelian Coulomb, and its time-dependent generalization, there are closed expressions which hold for arbitrary Q. For the non-Abelian Coulomb with its time-dependent generalization, one must calculate terms higher order in Q; a tedious procedure with unknown convergence properties. Alternatively one can do numerical computations.

Furthermore as Q increases, solutions appear which require a critical, minimal source strength to support them. Very little is known about these, and the numerical method is presently the only effective means of investigation. We review one such example.[3,5]

When the source is radially symmetric, as in (3.14a), we have the spherically symmetric Abelian Coulomb solution. Also by passing to the radial frame (3.14b) we exhibit the perturbative non-Abelian Coulomb solution. By iterating Eqs. (3.10) a few orders in Q, it is found that the form of the potentials remains within the following *Ansatz*:

$$A^0=\frac{\hat{r}\cdot\vec{\sigma}}{2i}\,\frac{1}{r}f(r/r_0), \tag{3.28a}$$

$$\vec{A}=\frac{\hat{r}\times\vec{\sigma}}{2i}\,\frac{1}{r}[a(r/r_0)-1]. \tag{3.28b}$$

Here r_0 is a length scale. In this subsection we shall always remain in the radial frame,

$$\rho_a=\frac{\hat{r}^a}{r_0^{\,3}}q(r/r_0); \tag{3.29}$$

hence primes on the potentials are dropped. We postulate the above *Ansatz* for the complete static solution and derive the following nonlinear differential equations:

$$-f''+\frac{2a^2}{x^2}f=xq, \tag{3.30a}$$

$$-a''+\frac{a^2-1-f^2}{x^2}a=0. \tag{3.30b}$$

All functions depend only on $x=r/r_0$, and the prime indicates differentiation with respect to that variable. Equation (3.30a) is what remains of Gauss's law, and (3.30b) is Ampère's law, the spatial components of the Yang-Mills equations (3.1). More general radially symmetric *Ansätze* can be given, but it has been proven that static, radial solutions necessarily fall into the above restrictions.[3] [We emphasize that the Abelian Coulomb solution does not lie within the above *ansatz*, and cannot be found in the solutions to (3.30); in the radial frame, the Abelian Coulomb solution is not radially symmetric.] Requiring that the energy be finite

$$\mathcal{E}=\frac{4\pi}{r_0}\int_0^\infty dx\left[(a')^2+\frac{1}{2x^2}(a^2-1)^2+\tfrac{1}{2}(f')^2+\frac{1}{x^2}f^2a^2\right] \tag{3.31}$$

—the above is the form that (3.3) takes within the *Ansatz* (3.28)—imposes boundary conditions at the origin and at infinity. At the origin the potentials must vanish rapidly: $f(0)=0$, $a(0)=1$, $A^0(0)=0$, $\vec{A}(0)=0$. At infinity two types of behavior are allowed: Type I, where the potentials vanish as in the origin; type II, where the vector potential tends to a pure gauge $a(\infty)=-1$,

$$\vec{A}\underset{r\to\infty}{\sim} i\,\frac{\hat{r}\times\vec{\sigma}}{r}=-(i\vec{\sigma}\cdot\hat{r})\vec{\nabla}(-i\sigma\cdot\hat{r}).$$

The type-I solution is the previously perturbatively encountered non-Abelian Coulomb. The type II is a new, nonperturbative solution.

Numerical computation confirms the above, with the further surprise that type II comes in two branches, once Q exceeds a critical magnitude.[3] Figure 1 shows a plot of the energy versus source strength for solutions with a δ-shell charge den-

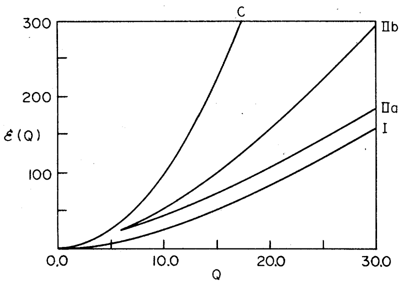

FIG. 1. Energy, in units of $2\pi/r_0$, as a function of Q for a δ-shell source of strength Q. The curve C is the Abelian Coulomb parabola. The curve I is the non-Abelian Coulomb solution. Curves IIa and IIb are the two branches of the bifurcating solution. The bifurcation point occurs at $Q=5.835$. The plot is taken from Ref. 3.

sity

$$\rho_a = \frac{\hat{r}^a}{r_0^{\,2}} Q\delta(r - r_0). \tag{3.32}$$

The Coulomb parabola (which does not lie within the *Ansatz*) is also exhibited for comparison. The bifurcation point where the two type-II solutions first occur is found numerically to be at $Q = 5.835$. We shall call this type-II solution the "bifurcating solution."

IV. STABILITY ANALYSIS FOR YANG-MILLS THEORY

We turn now to the stability analysis of static solutions for the Yang-Mills equations, and we use the ideas sketched in Sec. II. There are two ways in which the Yang-Mills field theory differs from the simple Hamiltonian previously analyzed. Firstly, rather than $2N$ degrees of freedom, there is now an infinite number. This causes matrices to be replaced by differential operators, summations by integrations, etc., thus raising questions of convergence and uniformity. We shall not concern ourselves with this complication, even though there will be occasion to refer to it in the course of our development. Secondly, because one is dealing with a gauge theory, the Hamiltonian formulation of the dynamics is not straightforward; rather Gauss's law must be viewed as a constraint. For our purposes this complication may be handled in the following manner.

We retain (3.3) as the Hamiltonian, identifying $-\vec{E}_a$ with the canonical momentum, conjugate to the canonical coordinate $\vec{A}_a$, $\vec{B}_a$ being constructed from the latter in the standard way. However the variations of $H=\mathcal{E}$, with respect to $\vec{A}_a$ and $\vec{E}_a$, which are needed for obtaining the dynamical equations are not unrestricted; they are constrained by the three relations that comprise Gauss's law. These constraints may be implemented with three Lagrange multipliers, which we call A_a^0, and unrestricted variations are performed on

$$\overline{\mathcal{E}} = \mathcal{E} - \int d\vec{r}\, A_a^0(\vec{\nabla}\cdot\vec{E}_a - \epsilon_{abc}\vec{A}_b\cdot\vec{E}_c - \rho_a). \tag{4.1}$$

In this way the full Yang-Mills equations are regained,

$$0 = -\frac{\delta\overline{\mathcal{E}}}{\delta A_a^0} = \vec{\nabla}\cdot\vec{E}_a - \epsilon_{abc}\vec{A}_b\cdot\vec{E}_c - \rho_a \tag{4.2a}$$

(Gauss's law constraint—time component of Yang-Mills equations),

$$\partial_t \vec{E}_a = \frac{\delta\overline{\mathcal{E}}}{\delta\vec{A}_a} = \vec{\nabla}\times\vec{B}_a - \epsilon_{abc}\vec{A}_b\times\vec{B}_c - \epsilon_{abc}A_b^0\vec{E}_c \tag{4.2b}$$

(Ampère's law—spatial components of Yang-Mills equations),

$$\partial_t \vec{A}_a = -\frac{\delta\overline{\mathcal{E}}}{\delta\vec{E}_a} = -\vec{E}_a - \vec{\nabla}A_a^0 + \epsilon_{abc}\vec{A}_b A_c^0$$

$$\text{(definition of } \vec{E}_a\text{)}. \tag{4.2c}$$

No equation is needed to determine the magnetic field since $\vec{B}_a$ is always $\vec{\nabla}\times\vec{A}_a - \frac{1}{2}\epsilon_{abc}\vec{A}_b\times\vec{A}_c$. The static equations emerge as conditions that $\mathcal{E}$ be stationary when $\vec{E}_a$ and $\vec{A}_a$ are subject to variations restricted by Gauss's law (4.2a).[3,6] Provided the constraint on the variations is taken into account, we can apply the general theory of Sec. II.

To determine the quadratic Hamiltonian, we set $\vec{A}_a \to \vec{A}_a^{(s)} + \delta\vec{A}_a$, $\vec{E}_a \to \vec{E}_a^{(s)} + \delta\vec{E}_a$ and expand $\mathcal{E}$. With the help of the static equations satisfied by $\{\delta\vec{E}_a^{(s)}, \delta\vec{A}_a^{(s)}\}$, we find up to quadratic terms

$$\mathcal{E} = \mathcal{E}^{(s)} + \tfrac{1}{2}\int d\vec{r}\,[(\delta\vec{E}_a)^2 + (\delta\vec{B}_a)^2 - \delta A_a^i(\epsilon^{ikj}\epsilon_{acb}B_c^k)\,\delta A_b^j + 2\vec{E}_a\cdot\delta\vec{E}_a + 2\epsilon_{abc}A_a^0\vec{E}_b\cdot\delta\vec{A}_c]. \tag{4.3a}$$

Here $\delta\vec{B}_a = \vec{\nabla}\times\delta\vec{A}_a - \epsilon_{abc}\vec{A}_b\times\delta\vec{A}_c$ and $\mathcal{E}^{(s)}$ is the energy of the static solution. In the above we have suppressed the label (s) on the static background field; this practice is followed henceforth. Note that linear terms do not disappear, because as yet the constraint (4.2a) has not been taken into account. Expanding that equation gives

$$\vec{\nabla}\cdot\delta\vec{E}_a - \epsilon_{abc}\delta\vec{A}_b\cdot\vec{E}_c - \epsilon_{abc}\vec{A}_b\cdot\delta\vec{E}_c - \epsilon_{abc}\delta\vec{A}_b\cdot\delta\vec{E}_c = 0, \tag{4.3b}$$

and use of this to reexpress the last term in (4.3a) does eliminate linear terms, leaving

$$\mathcal{E} = \mathcal{E}^{(s)} + \tfrac{1}{2}\int d\vec{r}\,[(\delta\vec{E}_a)^2 - 2\delta E_a^i\epsilon_{abc}A_c^0\,\delta A_b^i + (\delta\vec{B}_a)^2 - \delta A_a^i(\epsilon^{ikj}\epsilon_{acb}B_c^k)\,\delta A_b^j]. \tag{4.3c}$$

Thus we see that the quadratic Hamiltonian matrix is exactly of the form (2.11) with

$$G=\delta^{ij}\epsilon_{abc}A_c^0\,\delta(\vec{r}-\vec{r}\,')\,, \tag{4.4a}$$

$$V=(\epsilon^{ikm}\mathfrak{D}_{ac}^m\mathfrak{D}_{cb}^n\,\epsilon^{nkj}-\epsilon^{ikj}\epsilon_{acb}B_c^k)\delta(\vec{r}-\vec{r}\,')\,. \tag{4.4b}$$

In particular we observe the presence of gyroscopic terms (4.4a).

Derivation of small-fluctuation equations proceeds from the quadratic Hamiltonian as explained in Sec. II—the vector X is now

$$\begin{pmatrix} -\delta\vec{E}_a \\ \delta\vec{A}_a \end{pmatrix}$$

—except that the constraint of Gauss's law for the fluctuations must be satisfied. The latter is merely the condition (4.3b), taken without its last term which is not needed in this order. The subsidiary condition is implemented with a Lagrange multiplier, which we here call δA_a^0. Thus the effective quadratic Hamiltonian is

$$H^{(2)}=\int d\vec{r}\,[\tfrac{1}{2}(\delta\vec{E}_a)^2-\delta E_a^i\,\epsilon_{abc}A_c^0\,\delta A_b^i+\tfrac{1}{2}(\delta\vec{B}_a)^2-\tfrac{1}{2}\delta A_a^i(\epsilon^{ikj}\epsilon_{acb}B_c^k)\,\delta A_b^j$$
$$-\delta A_a^0(\vec{\nabla}\cdot\delta\vec{E}_a-\epsilon_{acb}\vec{A}_c\cdot\delta\vec{E}_b+\epsilon_{acb}\vec{E}_c\cdot\delta\vec{A}_b)]\,. \tag{4.5}$$

The fluctuation equations are

$$0=-\frac{\delta H^{(2)}}{\delta(\delta A_a^0)}=\vec{\nabla}\cdot\delta\vec{E}_a-\epsilon_{acb}\vec{A}_c\cdot\delta\vec{E}_b+\epsilon_{acb}\vec{E}_c\cdot\delta\vec{A}_b\,, \tag{4.6a}$$

$$\partial_t\,\delta\vec{E}_a=\frac{\delta H^{(2)}}{\delta(\delta\vec{A}_a)}=\vec{\nabla}\times\delta\vec{B}_a-\epsilon_{acb}\vec{A}_c\times\delta\vec{B}_b-\epsilon_{acb}\vec{B}_c\times\delta\vec{A}_b+\epsilon_{acb}\vec{E}_c\,\delta A_b^0-\epsilon_{acb}A_c^0\,\delta\vec{E}_b\,, \tag{4.6b}$$

$$\partial_t\,\delta\vec{A}_a=-\frac{\delta H^{(2)}}{\delta(\delta\vec{E}_a)}=-\delta\vec{E}_a-\vec{\nabla}\delta A_a^0+\epsilon_{abc}\vec{A}_b\,\delta A_c^0-\epsilon_{abc}A_b^0\,\delta\vec{A}_c\,. \tag{4.6c}$$

One may check directly that these are the linearization of (4.2) around a static background field. When a monochromatic *Ansatz* is made for the time dependence, the above take the form of a simplectic eigenvalue problem.

Before making use of Eqs. (4.6) to analyze the stability of the two static solutions described in the previous section, we comment on their properties. An integrability condition follows from (4.6b). By taking the covariant divergence, one finds that the infinitesimal version of (3.2b) must be satisfied,

$$\epsilon_{abc}\,\delta A_b^0\rho_c=0\,. \tag{4.7}$$

Also *vice versa*: (4.7) and the integrability condition on (4.6b) imply, together with (4.6c), that the covariant time derivative of the right-hand side in (4.6a) vanishes.

Equations (4.6) possess a local gauge invariance,

$$\delta\vec{E}_a\to\delta\vec{E}_a-\epsilon_{abc}\vec{E}_b\theta_c\,, \tag{4.8a}$$

$$\delta A_a^0\to\delta A_a^0-\partial_t\,\theta_a-\epsilon_{abc}A_b^0\,\theta_c\,, \tag{4.8b}$$

$$\delta\vec{A}_a\to\delta\vec{A}_a+\vec{\nabla}\theta_a-\epsilon_{abc}\vec{A}_b\,\theta_c\,. \tag{4.8c}$$

Here θ_a is a local function which must be parallel to the source,

$$\epsilon_{abc}\,\theta_b\,\rho_c=0\,. \tag{4.8d}$$

Owing to the linearity of the equations, the finite transformation and the infinitesimal are of the same form (4.8). The energy (4.5) of all configurations that satisfy Gauss's law (4.6a) is gauge invariant. [There is also a gauge covariance to the Eqs. (4.6): A gauge transformation on the background fields is compensated by a homogeneous gauge transformation on the fluctuating quantities. We shall not make use of this property.]

It is clear from (4.7) and (4.8d) that the external charge density defines a direction in group space which we can call the "electromagnetic" direction, while the orthogonal directions can be termed "charged." Thus A_a^0, δA_a^0, θ_a, and ρ_a all lie in the electromagnetic direction, and vanish in the charged direction. This reduces the allowed gauge transformations, in that the last term in (4.8b) must vanish. Observe also that the gyroscopic term (4.4a) affects only the charged direction; the electromagnetic fluctuations are free of gyroscopic terms.

It is possible to derive a gauge-invariant fluctuation equation in the following way. The quantity

$$\begin{aligned}\vec{e}_a&=\delta\vec{E}_a+\epsilon_{abc}A_b^0\,\delta\vec{A}_c\\&=-\partial_t\,\delta\vec{A}_a-\vec{\mathfrak{D}}_{ab}\,\delta A_b^0\end{aligned} \tag{4.9}$$

is gauge invariant with respect to the gauge transformations (4.8). Then by taking a covariant time derivative of (4.6b), we arrive after a set of steps at

$$\mathfrak{D}_{ab}^0\,\mathfrak{D}_{bc}^0\,\vec{e}_c+\vec{\mathfrak{D}}_{ab}\times\vec{\mathfrak{D}}_{bc}\times\vec{e}_c-\epsilon_{abc}\,\vec{B}_b\times\vec{e}_c=0\,. \tag{4.10}$$

[This is readily derived in the $\delta A^0=0$ gauge, which can always be achieved with the transformation (4.8b).]

Equation (4.10) is gauge invariant and involves the unconstrained fluctuation variable $\vec{e}_a$. It is remarkable that such an equation can be derived; the possibility to do so is intimately linked with the existence of an external charge density that defines a direction with respect to which the small fluctuations are constrained by (4.7).

With the monochromatic *Ansatz*

$$\vec{e}_a = e^{-i\omega t}\,\delta\vec{Q}_a\,, \tag{4.11}$$

(4.10) may be written as

$$[(i\omega+G)(i\omega+G)+V]\,\delta Q=0\,, \tag{4.12}$$

with G and V given by (4.4). We see that (4.12) is precisely as in (2.12). This puts the Yang-Mills model into the formalism described in Sec. II, which we now apply to assess the stability of the static solutions discussed in Sec. III.

A. Abelian Coulomb solution

For the Abelian Coulomb solution (3.5), the small-oscillation equations are best presented by introducing complex quantities in the charged directions, 1 and 2,

$$\begin{aligned}\delta\vec{E} &= \frac{1}{\sqrt{2}}(\vec{E}_1+i\vec{E}_2)\,,\\ \delta\vec{A} &= \frac{1}{\sqrt{2}}(\vec{A}_1+i\vec{A}_2)\,,\\ \vec{e} &= \delta\vec{E}+i\varphi\delta\vec{A}\,.\end{aligned} \tag{4.13}$$

Equation (4.10) in the electromagnetic direction, that is direction 3, decouples completely,

$$\partial_t{}^2\vec{e}_3+\vec{\nabla}\times\vec{\nabla}\times\vec{e}_3=0\,, \tag{4.14}$$

while the equation in the charged directions simply becomes

$$\begin{aligned}&(\partial_t+i\varphi)^2\vec{e}+\vec{\nabla}\times\vec{\nabla}\times\vec{e}=0\,,\\ &\varphi=-\frac{1}{\nabla^2}q\,.\end{aligned} \tag{4.15}$$

The electromagnetic fluctuations are free; the charged ones describe the motion of charged vector mesons in an external potential ϕ.[7]

Detailed analysis of the equations can be performed in frequency space. Note that the electromagnetic equation involves ω^2 as an eigenvalue of a Hermitian operator, hence it is real. Only the issue remains whether ω^2 is positive or negative. In the charged equation there appears $(\omega-\phi)^2$ and ω can be complex; it is not related to the eigenvalue of a Hermitian operator. This difference merely reflects the fact previously remarked upon: In the charge-neutral, electromagnetic direction there are no gyroscopic terms, hence stability is equivalent to minimality. In the charged direction, gyroscopic terms are present; they are responsible for the more complicated equation.

The electromagnetic fluctuations are obviously stable. Those in the charged directions are stable in the absence of the external potential and by continuity they remain stable for a sufficiently small external potential.[2] As the external charge density increases in strength, an instability is expected to appear. This is not the instability of the Klein-Gordon equation to an $1/r$ (Coulomb) potential, which has previously been remarked upon,[8] and which is a consequence of the (presumably unphysical) singularity at the origin which is absent in our examples. Instead it is the instability of the Klein-Gordon equation in a strong external potential.[9]

For the δ-shell source (3.32), the instability sets in at $Q=1.5$. This number can be extracted from Mandula's calculations concerning a point source.[8] Indeed in order to regulate the singularity in the Coulomb potential, he replaced that expression by a δ-shell potential, with radius r_0. The finite radius is a regulator, set to zero by Mandula at the end, while we are concerned with results for nonzero radius. The coincidence of the δ-shell value for the magnitude of Q at the onset of instability with that of the (limiting) point-source value is a consequence of special scaling properties of the δ-shell source, and is not expected for arbitrary sources. Nor does there seem to be any transparent relation to the value of the critical charge at the bifurcation.

In spite of stability for weak sources, we expect as a consequence of the gyroscopic terms to find modes which, though harmonic, lower the energy. These can be readily exhibited, without passing to frequency space. We remain with the first-order equations (4.6), and seek a solution with $\delta\vec{B}_a=0$. In that case the charged portions of (4.6), with an Abelian Coulomb solution as the background field, reduce to

$$0=\vec{\nabla}\cdot\delta\vec{E}-i\vec{\nabla}\varphi\cdot\delta\vec{A}\,, \tag{4.16a}$$

$$0=(\partial_t+i\varphi)\,\delta\vec{E}\,, \tag{4.16b}$$

$$\delta\vec{E}=-(\partial_t+i\varphi)\,\delta\vec{A}\,, \tag{4.16c}$$

$$\delta\vec{B}=\vec{\nabla}\times\delta\vec{A}=0\,. \tag{4.16d}$$

The solution of (4.16b) and (4.16c) is

$$\delta\vec{A}=[\vec{a}_0(\vec{r})+t\,a_1(\vec{r})]\exp[-it\varphi(\vec{r})]\,, \tag{4.17a}$$

$$\delta\vec{E}=-a_1(\vec{r})\exp[-it\varphi(\vec{r})]\,, \tag{4.17b}$$

while (4.16a) and (4.16d) demand

$$\vec{\nabla}\times\vec{a}_0=0\,, \tag{4.18a}$$

$$\vec{a}_1\times\vec{\nabla}\varphi=0\,, \tag{4.18b}$$

$$\vec{a}_0\times\vec{\nabla}\varphi=i\vec{\nabla}\times\vec{a}_1\,, \tag{4.18c}$$

$$\vec{a}_0\times\vec{\nabla}\varphi=i\vec{\nabla}\times\vec{a}_1\,. \tag{4.18d}$$

Owing to (4.18a) $\vec{a}_0$ is written as a gradient,

$$\vec{a}_0=\vec{\nabla}\theta\,. \tag{4.19a}$$

Equations (4.18a) and (4.18c) show that the vector quantities $\vec{a}_1$ and $i\phi\vec{a}_0$ differ by a gradient of a scalar, which together with (4.19a) means we should set $\vec{a}_1=-i\theta\vec{\nabla}\phi+\vec{\nabla}X$, where X is undetermined. But (4.18d) requires that $i\nabla^2X=\theta q$; hence we take

$$\vec{a}_1=-i\theta\vec{\nabla}\varphi-i\vec{\nabla}^{-1}(\theta q)\,. \tag{4.19b}$$

Finally, to fulfill (4.18b), $\vec{\nabla}^{-1}(\theta q)$ must be parallel to $\vec{\nabla}\phi$, which can be easily achieved, for example, by setting $\theta q=\nabla^2F(\phi)$, where F is an arbitrary function. Thus Eqs. (4.16) can be satisfied in terms of one arbitrary function.

The quadratic energy (4.5) is seen to be negative (at least for sources which do not change sign—actually this restriction can be removed; see below):

$$\mathcal{E}^{(2)}=\int\left[(q\theta)\frac{-1}{\nabla^2}(q\theta^*)-(|\theta|^2q)\frac{-1}{\nabla^2}(q)\right]$$
$$=-\frac{1}{8\pi}\int d\vec{r}\,d\vec{r}\,'\,\frac{q(\vec{r})\,q(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}|\theta(\vec{r})-\theta(\vec{r}\,')|^2\leq 0\,. \tag{4.20}$$

This rather peculiar fluctuation gives evidence that the Coulomb solution is gyroscopically stable since the energy decreases below its Coulomb value. Note, in particular, the linear growth with time, which, however, does not produce instability. Because the frequency is position dependent, it is not clear how to locate this mode among superpositions of functions with definite frequency. Nevertheless we have encountered it already. It is merely the time-dependent generalization of the Abelian Coulomb solution, discussed in Sec. III, when the latter is brought arbitrarily close to the static solution. To recognize this, let us chose ρ' in (3.19) to differ from the Abelian frame charge density by an infinitesimal gauge transformation,

$$\rho_a'=\delta_{a3}q-\epsilon_{ab3}\theta_b q\,. \tag{4.21a}$$

Here θ_a is the gauge function, taken without an $a=3$ component. It follows from (3.17) that

$$\Phi_a=\delta_{a3}\varphi+\epsilon_{ab3}\frac{1}{\nabla^2}(\theta_b q)\,, \tag{4.21b}$$

and from (3.16)

$$\vec{A}_a'=\delta_{a3}\vec{\nabla}\varphi t+\epsilon_{ab3}\vec{\nabla}^{-1}(\theta_b q)t\,, \tag{4.21c}$$

$$\vec{E}_a'=-\delta_{a3}\vec{\nabla}\varphi-\epsilon_{ab3}\vec{\nabla}^{-1}(\theta_b q)\,, \tag{4.21d}$$

$$\vec{\nabla}\varphi\times\vec{\nabla}^{-1}(\theta_a q)=0\,. \tag{4.21e}$$

Next we make a gauge transformation, so as to return the charge density (4.21a) into the Abelian frame. Accordingly (3.20) takes the form

$$A_a^0=0\,, \tag{4.22a}$$

$$\vec{A}_a=\delta_{a3}\vec{\nabla}\varphi t+\epsilon_{ab3}\vec{\nabla}^{-1}(\theta_b q)t+\epsilon_{ab3}\theta_b\vec{\nabla}\varphi t+\vec{\nabla}\theta_a\,, \tag{4.22b}$$

$$\vec{E}_a=-\delta_{a3}\vec{\nabla}\varphi-\epsilon_{ab3}\vec{\nabla}^{-1}(\theta_b q)-\epsilon_{ab3}\theta_b\vec{\nabla}\varphi\,. \tag{4.22c}$$

To effect a comparison with the expressions for the small fluctuations, we pass out of the temporal gauge; a formula gauge-equivalent to (4.22) is

$$A_a^0=\delta_{a3}\varphi\,, \tag{4.23a}$$

$$\vec{A}_a=\epsilon_{ab3}[\vec{\nabla}^{-1}(\theta_b q)+\theta_b\vec{\nabla}\varphi]\,t\cos\varphi t$$
$$-[\vec{\nabla}^{-1}(\theta_a q)+\theta_a\vec{\nabla}\varphi]\,t\sin\varphi t$$
$$+\vec{\nabla}\theta_a\cos\varphi t+\epsilon_{ab3}\vec{\nabla}\theta_b\sin\varphi t\,, \tag{4.23b}$$

$$\vec{E}_a=-\vec{\nabla}\varphi\delta_{a3}-\epsilon_{ab3}[\theta_b\vec{\nabla}\varphi+\vec{\nabla}^{-1}(\theta_b q)]\cos\varphi t$$
$$+[\theta_a\vec{\nabla}\varphi+\vec{\nabla}^{-1}(\theta_a q)]\sin\varphi t\,. \tag{4.23c}$$

Equivalently, in the complex notation for the charged direction, the above become identical to (4.17) and (4.19),

$$\delta\vec{A}=\{\vec{\nabla}\theta-i[\theta\vec{\nabla}\varphi+\vec{\nabla}^{-1}(\theta q)]\,t\}e^{-i\varphi t}\,, \tag{4.23d}$$

$$\delta\vec{E}=i[\theta\vec{\nabla}\varphi+\vec{\nabla}^{-1}(\theta q)]e^{-i\varphi t}\,, \tag{4.23e}$$

while (4.21e) reproduces the requirement that $\vec{\nabla}^{-1}(\theta q)$ be parallel to $\vec{\nabla}\phi$.

Finally we may compare with the explicit time-dependent solution presented in (3.21). Expanding those formulas to first-order in α and equating with the expressions in (4.22) gives agreement with $\theta_1=\alpha r^2 d\phi/dr$, $\theta_2=0$. Thus $\theta=(\alpha/\sqrt{2}\,)r^2d\phi/dr$, and the energy $\mathcal{E}^{(2)}$ in (4.20) becomes evaluated to

$$-\frac{\pi}{6}\alpha^2\int_0^\infty dr'(r')^6\left(\frac{d\phi}{dr'}\right)^4,$$

which is always negative, regardless of the sign of the source. This demonstrates explicitly that the static Abelian Coulomb solution, when stable, is only gyroscopically stable.

B. Non-Abelian Coulomb solution

The non-Abelian Coulomb solution, (3.10) and (3.12), follows in many respects, at least for weak sources, the behavior of the Abelian Coulomb solution. The stability equations are now highly coupled, and we have not solved them; see however Sec. IV D. Nevertheless, by continuity with the sourceless problem one expects stability for weak sources.[2] Moreover, one can show that this again must be an instance of gyroscopic stability, since the energy is lowered by the time-dependent generalization presented in (3.24), which can be taken arbitrarily close to the non-Abelian Coulomb solution. This is achieved by making ρ'', the source for the time-dependent solution, to be an infinitesimal gauge transformation of ρ', the source in the non-Abelian Coulomb solution. The energy can then be computed from (3.25). The details are the following. Set

$$\rho''_a=(1-\tfrac{1}{2}\theta_b\theta_b)\rho'_a-\epsilon_{abc}\theta_b\rho_c\,. \tag{4.24}$$

This ensures that ρ''_a is a gauge transformation of ρ'_a taken through second order in θ_a which lies only in the charged direction,

$$\theta_a(\vec{r})\rho'_a(\vec{r})=0\,. \tag{4.25}$$

It now follows from (3.25) that the $O(Q^2)$ energy is

$$\mathcal{E}=\frac{1}{8\pi}\int d\vec{r}d\vec{r}'\rho'_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\rho'_a(\vec{r}')-\frac{1}{4\pi}\int d\vec{r}d\vec{r}'\rho'_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\epsilon_{abc}\rho'_b(\vec{r}')\theta_c(\vec{r}')$$
$$-\frac{1}{16\pi}\int d\vec{r}d\vec{r}'\rho'_a(r)\frac{1}{|\vec{r}-\vec{r}'|}\rho'_a(\vec{r}')[\theta_b(\vec{r})-\theta_b(\vec{r}')]^2-\frac{1}{8\pi}\int d\vec{r}d\vec{r}'\rho'_a(\vec{r})\theta_b(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\rho'_b(\vec{r}')\theta_a(\vec{r}')+O(Q^4)\,. \tag{4.26}$$

The first term is the $O(Q^2)$ non-Abelian Coulomb energy. The second may also be written as

$$\int d\vec{r}\Phi_a(\vec{r})\epsilon_{abc}\nabla^2\Phi_b(\vec{r})\theta_c(\vec{r})\,,$$

whence it is seen to vanish owing to (3.11). The remaining two terms give the energy of the fluctuation. Unlike (4.20), one cannot determine the sign by inspection, but in Appendix B we show that in the generic, spherically symmetric case the terms are negative.

C. Bifurcating solution

The bifurcating solution, described in Sec. III B where it is also called the type-II radial solution, exists only for sufficiently strong sources. Consequently, we have no closed-form expressions to describe it; yet precisely because there is a bifurcation, we can say a considerable amount without explicit computations. Consider first a solution to the static Yang-Mills equations—Eqs. (4.2) with the left-hand side of (4.2b) and (4.2c) set to zero—for a definite source ρ. Next imagine changing the source slightly, $\rho\to\rho+\delta\rho$, and looking for a new static solution. If the new solution is regularly related to the old one, the increments in the Yang-Mills fields will satisfy linear equations which are of the same form as the fluctuation Eqs. (4.6) except that $\delta\rho$ occurs in the left-hand side of (4.6a) and the left-hand sides of (4.6b) and (4.6c) are zero. However, if we are at the bifurcation point, it must be impossible to solve these equations, and this happens if the homogeneous system [left-hand side zero in all Eqs. (4.6)] has a nontrivial solution. In this way we arrive at the important observation that at the bifurcation point the stability equations have a zero-eigenvalue mode, and *vice versa*: A zero-eigenvalue mode is a hint for a bifurcation (or generalizations thereof).

The zero-eigenvalue mode dominates the behavior of the solutions near the bifurcation, and general properties of bifurcation phenomena allow for the development of a rather detailed theory, without using the explicit form of the solutions. We present this here, but the discussion is complicated by a proliferation of equations and indices. We therefore begin with an analysis of a simple, one-component model, in order to exemplify the theory. Then we state the results for the Yang-Mills model, which follow in every respect the features of the example.

Consider a nonlinear field equation for a field $\Phi(t,\vec{r})$ in the presence of an external, static source $\rho(\vec{r})$,

$$\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\Phi+U'(\Phi)=-\rho\,. \tag{4.27}$$

Here U is a potential for the field, and the prime indicates differentiation with respect to argument. We suppose that a bifurcation occurs at $\rho=\rho_c$. By hypothesis ρ_c supports a unique static solution $\phi_c(\vec{r})$. Correspondingly, there is a real, normalized zero-eigenvalue mode $\psi(\vec{r})$ in the small-fluctuation equations,

$$-\nabla^2\varphi_c+U'(\varphi_c)=-\rho_c\,, \tag{4.28a}$$

$$[-\nabla^2+U''(\varphi_c)]\psi=0\,. \tag{4.28b}$$

Let us now replace ρ by $\rho_c+\epsilon\delta\rho$, where ϵ is a small parameter, chosen to be positive, which systematizes the study of the theory around the bifurcation point. It is then appropriate to expand the static field ϕ according to

$$\varphi=\varphi_c+\epsilon^{1/2}c\psi+\epsilon\delta\varphi+\cdots\,, \tag{4.29}$$

with c being a numerical factor which together with $\delta\phi$ is to be determined. Expansion of (4.27) shows that terms independent of ϵ as well as those of order $\epsilon^{1/2}$ vanish by virtue of (4.28). The order-ϵ equation leaves

$$[-\nabla^2 + U''(\varphi_c)]\,\delta\varphi + \tfrac{1}{2}c^2 U'''(\varphi_c)\,\psi^2 = -\delta\rho\,. \qquad (4.30)$$

Equation (4.28b) implies a consistency condition on (4.30),

$$\tfrac{1}{2}c^2 \int d\vec{r}\, U'''(\varphi_c)\,\psi^3 = \int d\vec{r}\,\psi\,\delta\rho\,. \qquad (4.31)$$

For subsequent use let us observe that $U'''(\phi_c)\psi^3$ is also proportional to the ψ^3 term in the expansion of the Lagrange density, evaluated at $\phi_c+\psi$, in powers of ψ. Calling that contribution $\mathcal{L}^{(3)}(\phi_c+\psi)$, we may rewrite (4.31) as

$$3c^2 \int d\vec{r}\, \mathcal{L}^{(3)}(\varphi_c+\psi) = \int d\vec{r}\,\psi\,\delta\rho\,. \qquad (4.32)$$

With generic $\delta\rho$, the right-hand side of (4.32) is nonvanishing. We shall further assume that the integral in the left-hand side also is nonvanishing. (This assumption is a prerequisite for the subsequent development. Although we have not checked its validity in our Yang-Mills application, the fact that our analysis of the bifurcation is verified in the explicit numerical results, see below, provides an *a posteriori* justification.) Thus (4.31) determines both the magnitude of c and the direction of the bifurcation,

$$c^2 = 2\,\frac{-\int d\vec{r}\psi\delta\rho}{\int d\vec{r}\,U'''(\varphi_c)\,\psi^3}\,. \qquad (4.33a)$$

Obviously the functional sign of $\delta\rho$ must be such that the right-hand is positive. [We seek real solutions, hence c in (4.29) must be real.] Two solutions for c are obtained,

$$c = \pm\left|\frac{2\int d\vec{r}\,\psi\,\delta\rho}{\int d\vec{r}\,U'''(\varphi_c)\,\psi^3}\right|^{1/2}, \qquad (4.33b)$$

and consequently (4.29) shows that ϕ bifurcates around ϕ_c. Once the consistency condition (4.33) is satisfied, (4.29) may be solved for $\delta\phi$.

The energy near the bifurcation point also is readily determined. The energy of a static solution to (4.27) is

$$\mathcal{E} = \int d\vec{r}\,[\tfrac{1}{2}(\vec{\nabla}\varphi)^2 + U(\varphi) + \rho\varphi]\,. \qquad (4.34a)$$

When this is differentiated with respect to ϵ we find

$$\frac{\partial\mathcal{E}}{\partial\epsilon} = \int d\vec{r}\left(\frac{\delta\mathcal{E}}{\delta\varphi(\vec{r})}\,\frac{\partial\varphi(\vec{r})}{\partial\epsilon} + \varphi(\vec{r})\,\frac{\partial\rho(\vec{r})}{\partial\epsilon}\right). \qquad (4.34b)$$

The first term on the right-hand side vanishes, since a static solution stationarizes the energy, while in the second term we use our assumed form for ϕ and ρ. Thus we find

$$\frac{\partial\mathcal{E}}{\partial\epsilon} = \int d\vec{r}(\varphi_c + \epsilon^{1/2}c\psi + \cdots)\,\delta\rho\,, \qquad (4.34c)$$

$$\mathcal{E} = \mathcal{E}_c + \epsilon\int d\vec{r}\,\varphi_c\,\delta\rho + \tfrac{2}{3}c\epsilon^{3/2}\int d\vec{r}\,\psi\,\delta\rho + \cdots\,. \qquad (4.34d)$$

Since c can take on two different signs according to (4.33), the above shows that the energy bifurcates, with an energy difference rising as $\epsilon^{3/2}$.

Finally we examine stability of the bifurcating solutions. The oscillatory modes associated with (4.27) satisfy

$$\Phi = \varphi + e^{-i\omega_n t}\Psi_n\,, \qquad (4.35a)$$

$$[-\nabla^2 + U''(\varphi)]\Psi_n = \omega_n^2\Psi_n\,. \qquad (4.35b)$$

We concentrate on the mode Ψ_0, which at the bifurcation point is the zero-eigenvalue mode, $\varphi=\varphi_c$, $\Psi_0=\psi$, and examine what happens immediately above the bifurcation, where we may set

$$\varphi = \varphi_c + \epsilon^{1/2}c\psi + \cdots\,, \qquad (4.36a)$$

$$\Psi_0 = \psi + \delta\Psi_0\,, \qquad (4.36b)$$

and take ω_0^2 to be a small quantity, since in lowest order it vanishes. Inserting (4.36) into (4.35b) gives, to first order in small quantities,

$$\omega_0^2\psi = [-\nabla^2 + U''(\varphi_c)]\,\delta\Psi_0 + c\epsilon^{1/2}U'''(\varphi_c)\,\psi^2\,. \qquad (4.37a)$$

Equation (4.28b) implies a consistency condition on the above; this evaluates ω_0^2,

$$\omega_0^2 = c\epsilon^{1/2}\int d\vec{r}\,U'''(\varphi_c)\,\psi^3\,. \qquad (4.37b)$$

Again, since c can have either sign, we see that for one of the branches ω_0^2 is negative, hence there is an instability. Comparison with (4.33a) and (4.34d) shows that ω_0^2 has the opposite sign from the energy difference. This means that it is the upper branch which is unstable, while the lower branch shares the stability properties of the unique solution at the bifurcation point: If the latter is stable (the zero-eigenvalue mode is the lowest mode) so is the lower-branch solution; if there is instability at the bifurcation point, (there exist complex eigenfrequencies) it will persist in the lower mode even beyond bifurcation point.

Before proceeding to the Yang-Mills problem, let us remark that when gyroscopic terms are present in the small-oscillation equation (they are absent in the above example), the formula for ω_0^2 is modified. It is sufficient to assume that the gyroscopic terms enter as in (2.11) and that the eigenvalue problem which replaces (4.35b) has the form (2.12). It is then a straightforward application of the expansion techniques exemplified above to derive the following result:

$$\omega_0^2 = \frac{c\epsilon^{1/2}\langle\psi|\Delta W|\psi\rangle}{1 + 4\langle G\psi|W^{-1}|G\psi\rangle}\,. \qquad (4.38)$$

The notation here is the following: W stands for the combination $V-\tilde{G}G=V+G^2$, where V is the quadratic potential and G the gyroscopic term, both evaluated with the solution at the bifurcation point. [In the above scalar example V is $U''(\phi_c)$.] $c\epsilon^{1/2}\Delta W$ is the first-order change in W as the background solution at the bifurcation point is increased by $c\epsilon^{1/2}$ times the zero-eigenvalue mode. [In the above this is $U''(\phi_c+c\epsilon^{1/2}\psi)-U''(\phi_c) \simeq c\epsilon^{1/2}U'''(\phi_c)\psi$.] The bracket notation involves taking matrix elements over whatever degrees of freedom are appropriate. $|\psi\rangle$ is the zero-eigenvalue mode of W and $|G\psi\rangle$ is that mode transformed by the gyroscopic term. W^{-1} is the inverse of W, projected on the nonzero eigenmodes of W. When operating on $|G\psi\rangle$, W^{-1} is well-defined since that vector is orthogonal to $|\psi\rangle$, owing to the antisymmetry of G. Thus (4.38) differs from (4.37b) by the presence of the denominator; it is of course assumed that the denominator does not vanish.

For the Yang-Mills theory, after repeating a calculation which differs from the above merely in that many degrees of freedom are involved, one too finds that when the source ρ is replaced by $\rho+\epsilon\delta\rho$, then the static solution above the bifurcation point has an expansion as in (4.29),

$$A^\mu=a_c^\mu+\epsilon^{1/2}c\,\mathfrak{A}^\mu+\cdots\,, \qquad (4.39)$$

where a_c^μ is the static solution at the bifurcation point, and $\mathfrak{A}^\mu$ is the normalized, zero-eigenvalue small-oscillation mode. The constant c is evaluated up to sign by a formula like (4.32),

$$3c^2\int d\vec{r}\,\mathcal{L}^{(3)}(a_c^\mu+\mathfrak{A}^\mu)=-2\,\mathrm{tr}\int d\vec{r}\,\mathfrak{A}^0\,\delta\rho\,, \qquad (4.40)$$

with $\mathcal{L}^{(3)}$ now signifying the expansion coefficient in the Yang-Mills Lagrangian. Since $\overline{\mathcal{E}}$ of Eq. (4.1) provides for a variational derivation of the static equations, the derivation of the bifurcation in the energy goes through as in (4.34), and we find

$$\mathcal{E}=\mathcal{E}_c-2\,\mathrm{tr}\left(\epsilon\int d\vec{r}\,a_c^0\,\delta\rho+\tfrac{2}{3}c\epsilon^{3/2}\int d\vec{r}\,\mathfrak{A}^0\delta\rho\right)+\cdots\,. \qquad (4.41)$$

Finally, the small-oscillation frequency above the bifurcation point is evaluated from (4.38) with G and V defined in (4.4).

The general theory here presented is verified by the numerical results obtained by solving the differential equations with the δ-shell source (3.32).[3] Thus, as can already be seen from Fig. 1, the mean energy rises linearly with $(Q-Q_c)$ above the bifurcation point occurring at $Q=Q_c$, and a detailed analysis shows that the energy difference rises as $(Q-Q_c)^{3/2}$ as predicted by (4.41). Similarly, the functions a and f of Eqs. (3.28) and (3.30), given in Ref. 3, bifurcate with $(Q-Q_c)^{1/2}$, validating (4.39). Accordingly the upper mode is expected to be unstable, and the lower one follows the stability properties of the bifurcation point. We have not solved the small-oscillation equations there, so we cannot report on the result. In the next subsection, a general discussion of the small-oscillation equations is given.

D. Analysis of stability equations

While general considerations in subsections B and C above provide much understanding of the structure of the radially symmetric solutions, detailed information about stability requires solving the small-fluctuation Eq. (4.10). This formidable task is simplified somewhat by using the known *Ansatz* for the background field, Eqs. (3.28); but still the problem has remained intractable, especially since we do not have closed-form analytic expressions for the potentials.

The radial symmetry of the background field allows for separation of the equations according to the total angular momentum

$$\vec{J}=\vec{L}+\vec{S}+\vec{T}\,,$$

where $\vec{L}$ is the orbital angular momentum; $\vec{S}$, the internal spin; and $\vec{T}$, the SU(2) rotation generator. Entirely conventional, but extraordinarily tedious expansion in terms of spherical harmonics results in nine coupled ordinary differential equations. For the $J=0$ fluctuations they are markedly different from those with $J\neq 0$; the former leading to a Hermitian problem, involving ω^2; the latter remain with the simplectic problem which reflects the full gyroscopic structure, involving both ω^2 and $i\omega$. Thus radial instability modes, if any, have purely imaginary ω; unstable modes with higher angular momentum, if any, can have complex ω. This difference is understood by realizing that the radial modes are necessarily charge-neutral; hence according to our general discussion they are not affected by gyroscopic terms.

So as not to lose a set of formulas that might eventually prove useful, we reproduce here the radial fluctuation equations, for which we make the following *Ansatz*:

$$e_a^i=\epsilon^{iaj}\hat{r}^j\,\frac{\varphi_1}{r}+(\delta^{ia}-\hat{r}^i\hat{r}^a)\,\frac{\varphi_2}{r}+\hat{r}^i\,\hat{r}^a\,\frac{e}{r^2}\,. \qquad (4.42)$$

By standard manipulation the following set of equations is obtained:

$$-\varphi_1''+\frac{3a^2-1-f^2}{r^2}\,\varphi_1=\omega^2\varphi_1-2i\omega\,\frac{1}{r}\,\varphi_2\,, \qquad (4.43a)$$

$$-\varphi_2'' + \frac{a^2-1-f^2}{r^2}\varphi_2 + \frac{a}{r^2}e' + \frac{2a'}{r^2}e - \frac{2a}{r^3}e = \omega^2\varphi_2 + 2i\omega\frac{1}{r}\varphi_1\,, \quad (4.43\text{b})$$

$$\frac{2a^2}{r^2}e - 2a\varphi_2' + 2a'\varphi_2 = \omega^2 e\,. \quad (4.43\text{c})$$

All quantities depend solely on r, and the primes signify differentiation with respect to that variable. The three equations may be combined into an alternative form as follows. Subtract half the derivative of (4.43c) from a times (4.43b) to get

$$-\omega^2 e' + 2\omega^2 a\varphi_2 + 4i\omega\frac{f}{r}a\varphi_1 = 0\,, \quad (4.44)$$

which allows for the elimination of ϕ_2, when $\omega \neq 0$

$$-e'' + \frac{2a^2}{r^2}e + \frac{2a'}{a}e' + \frac{4a^2 i}{\omega}\left(\frac{f}{ra}\varphi_1\right)' = \omega^2 e\,, \quad (4.45\text{a})$$

$$-\varphi_1'' + \frac{3a^2-1+3f^2}{r^2}\varphi_1 + \frac{\omega f}{ra}ie' = \omega^2\varphi_1\,. \quad (4.45\text{b})$$

This is equivalent to a Hermitian problem

$$L\psi = \omega^2\psi\,, \quad (4.46)$$

with

$$L = L^\dagger = \begin{pmatrix} p^2 + \dfrac{3a^2-1+3f^2}{r^2} & \dfrac{2f}{ra}pa \\ 2ap\dfrac{f}{ra} & p^2 + \dfrac{a^2+1+f^2}{r^2} + 2\left(\dfrac{a'}{a}\right)^2 \end{pmatrix}, \quad (4.47)$$

$$\psi = \begin{pmatrix} -\dfrac{1}{\omega}\varphi_1 \\ \dfrac{e}{2a} \end{pmatrix}, \quad p = \frac{1}{i}\frac{d}{dr}\,.$$

It is interesting that for the type-II, bifurcating solutions the operator L is singular owing to the change in sign of a as r ranges from 0 to ∞.

The equation for the zero-eigenvalue modes is most easily extracted from (4.43a), (4.43c), and (4.44). Upon defining $\phi_1 = i\omega\delta a$, $\phi_2(-a/r)\delta f$, we find from (4.43c), when $\omega^2 = 0$

$$e = -r^2\frac{\partial}{\partial r}\frac{\delta f}{r}\,, \quad (4.48)$$

while (4.43a) and (4.44) become, respectively,

$$-(\delta a)'' + \frac{3a^2-1-f^2}{r^2}\delta a - \frac{2af}{r^2}\delta f = 0\,, \quad (4.49\text{a})$$

$$-(\delta f)'' + \frac{2a^2}{r^2}\delta f + \frac{4fa}{r^2}\delta a = 0\,. \quad (4.49\text{b})$$

Equations (4.49) are just the deformations of (3.30). This is as it should be: The zero-eigenvalue mode is a deformation of the static equations.

The zero-eigenvalue mode arising from the occurrence of the bifurcation is necessarily radially symmetric, since it is proportional to the differenece of the two bifurcating solutions immediately above the critical value of the source. Hence, that mode will satisfy (4.49). An interesting application of the formalism, as well as a check on our theory, is to determine theoretically the bifurcation point by inquiring what source strength supports a zero-eigenvalue solution to (4.49). This problem can be solved approximately and the result gives the value of 5.892 to Q_{critical}, which is in excellent agreement with the numerically determined 5.835. Details of this computation are presented in Appendix C.

V. CONCLUSION

Various static solutions to Yang-Mills theory with static, extended but weak sources, do not minimize the energy. This, however, is not a sign of instability, rather gyroscopic terms are present which stabilize the configuration. In this way the Yang-Mills problem shares the physics of a top.[10] The time-dependent fluctuations which are stable, yet lower the energy, can be explicitly constructed.

The analogy with a top may be extended. In the mechanical system, the gyroscopic forces stabilizing steady motion, which does not minimize the energy, arise from the constraint of conservation of angular momentum. In the field theory, the gyroscopic terms also arise from the imposition of constraints by Gauss's law,

$$-\epsilon_{abc}\vec{A}_b\cdot\vec{E}_c + \vec{\nabla}\cdot\vec{E}_a = \rho_a\,.$$

The left-hand side is the generator of local rotations in group space; it is like a group-space angular momentum. ($-\epsilon_{abc}\vec{A}_b\cdot\vec{E}_c$ is analogous to $\vec{q}\times\vec{p}$.) In other words, a nonvanishing source establishes at each point in ordinary space a nonvanishing angular momentum in group space, which then stabilizes configurations, which otherwise would be unstable.

As the source strength increases, the Abelian Coulomb solution destabilizes. It is not known what happens to the non-Abelian Coulomb solution. Also the bifurcating solution, present for sources above a critical strength, gives rise to a zero eigenvalue mode at the critical source strength. Above that point, the upper branch becomes unstable, the other retains the stability properties of the bifurcation point. It is gratifying that such detailed information can be obtained from general considerations about bifurcations; the rather intractable stability equations need not be solved.

A succinct formula for all the weak-source solutions, both static and time dependent, has been given in (3.26).

Some further computations obviously suggest themselves, especially for strong sources. But the most pressing open question at the present time concerns the relevance of these mathematical investigations to the quantum physics of Yang-Mills theory.

ACKNOWLEDGMENTS

Some of the material in this paper was presented at various meetings in the Summer, 1979. The text of these lectures will be published by Springer, in their *Lecture Notes in Physics* series, as a part of the proceedings of the Canadian Mathematical Society Summer Research Institute in Montreal, Canada. We benefited from conversations with S. Coleman, R. Giles, J. Goldstone, K. Johnson, and C. Rebbi, which we gratefully acknowledge. This research was supported by funds provided by the U. S. Department of Energy (DOE) under Contract No. EY-76-C-02-3069.

APPENDIX A: ELECTRIC YANG-MILLS FIELDS

Properties of solutions to Yang-Mills equations with $\vec{B}=0$ are derived. Since vanishing $\vec{B}$ is a gauge-invariant characteristic, we may work in a special gauge, we chose $A^0=0$. From Ampère's law, Eq. (4.2b), it follows that $\vec{E}$ is static, and from the definition of $\vec{E}$, Eq. (4.2c), we learn that $\vec{A}$ is at most linear in the time t,

$$\vec{A}_a = -\vec{E}_a t + \vec{\mathfrak{A}}_a . \tag{A1}$$

The $\vec{B}$ implied by (A1) is

$$\begin{aligned}\vec{B}_a &= \vec{\nabla}\times\vec{\mathfrak{A}}_a - \tfrac{1}{2}\epsilon_{abc}\vec{\mathfrak{A}}_b\times\vec{\mathfrak{A}}_c \\ &\quad - t(\vec{\nabla}\times\vec{E}_a - \epsilon_{abc}\vec{\mathfrak{A}}_b\vec{E}_c) \\ &\quad - \tfrac{1}{2}t^2\epsilon_{abc}\vec{E}_b\times\vec{E}_c .\end{aligned} \tag{A2}$$

If this is to vanish, individual terms in t must vanish, which means that $\vec{\mathfrak{A}}$ is a pure gauge, and can be removed by performing a static gauge transformation which preserves the temporal gauge. We describe the transformed quantities with a primed notation, and thus find

$$\vec{A}'_a = -\vec{E}'_a t , \tag{A3}$$

$$\vec{B}'_a = -t\vec{\nabla}\times\vec{E}'_a - \tfrac{1}{2}t^2\epsilon_{abc}\vec{E}'_b\times\vec{E}'_c . \tag{A4}$$

The vanishing of (A4) in turn implies that $\vec{E}'$ is a gradient of a scalar and that its different components commute,

$$\vec{E}' = -\vec{\nabla}\Phi , \tag{A5}$$

$$[\partial_i\Phi, \partial_j\Phi] = 0 . \tag{A6}$$

Equations (A3), (A5), and (A6) reproduce (3.16), while (3.17) is just Gauss's law, Eq. (4.2a).

Next we show that in general, the time dependence in (A3) is not a gauge artifact. To prove our assertion, let us assume the contrary; that a time translation can be compensated by a gauge transformation,

$$A'^\mu(t+\tau) = U^{-1}A'^\mu(t)U + U^{-1}\partial^\mu U . \tag{A7}$$

The temporal component of this equation requires U to be independent of t, but depending on τ,

$$U = U(\tau) , \quad U(0) = I . \tag{A8}$$

In view of (A3), the spatial components of (A7) state

$$(t+\tau)\vec{E}' = tU^{-1}\vec{E}'U + U^{-1}\vec{\nabla}U , \tag{A9a}$$

or equating powers of t,

$$\vec{E}' = U^{-1}\vec{E}'U , \tag{A9b}$$

$$\vec{E}'\tau = U^{-1}\vec{\nabla}U . \tag{A9c}$$

Equation (A9c) shows that $\vec{A}'$ is a pure gauge, as it must be since $\vec{B}=0$. Equations (A9a) and (A9c) combine into a condition on U,

$$\begin{aligned}U^{-1}(t+\tau)\vec{\nabla}U(t+\tau) &= U^{-1}(\tau)U^{-1}(t)\vec{\nabla}U(t)U(\tau) \\ &\quad + U^{-1}(\tau)\vec{\nabla}U(\tau) ,\end{aligned} \tag{A10a}$$

which implies that

$$\begin{aligned}&U(t+\tau) = U(t)U(\tau) , \\ &U(t) = e^{-tA^0} .\end{aligned} \tag{A10b}$$

This defines a static A^0. Thus from (A9) and (A10) we conclude that if the time dependence is gauge artifactual, as is assumed in (A7), then there exists a gauge such that the spatial components of the potential vanish and the temporal component is time independent. In this gauge, the electric field is the negative gradient of A^0 and continues to be static. Ampère's law, with static quantities and no magnetic field reduces to the condition that

$$\epsilon_{abc}A_b^0\vec{\nabla}A_c^0 = 0 , \tag{A11}$$

which means that the direction in which A_a^0 points in SU(2) space is constant in position space. This can only happen for the Abelian Coulomb solution.

To summarize, the time dependence in the solution under discussion can be removed by a gauge transformation only when the solution is the Abelian Coulomb one. Otherwise the time dependence is truly present. This is somewhat surprising since the electric and magnetic fields are static (the latter vanishes) and so is the energy density.

One may wonder how to construct a gauge-invariant but time-dependent quantity. Of course, the general answer involves the path-order nonintegrable phase factor, $\mathrm{tr}P\exp[\oint_c A^\mu(z)dz_\mu]$.[11] However, a simpler object is available. Consider the second covariant derivative of the electric field,

$$E_a^{ijk} = \mathfrak{D}_{ab}^i \mathfrak{D}_{bc}^j E_c'^k . \tag{A12a}$$

This transforms homogeneously under gauge transformations and $E_a^{ijk}E_a^{i'j'k'}$ is gauge invariant. To recognize the time dependence, observe that the first covariant derivative of $\vec{E}'$ is static since components of $\vec{E}'$ commute with those of $\vec{A}' = -\vec{E}'t$,

$$E_a^{ijk} = -\mathfrak{D}_{ab}^i \partial_j E_b'^k . \tag{A12b}$$

But, $\vec{A}'$ does not in general commute with $\vec{\nabla}\vec{E}'$, and we encounter a linear time dependence,

$$E_a^{ijk} = \partial_i \partial_j E_a'^k - \epsilon_{abc} E_b'^i \partial_j E_c'^k t . \tag{A12c}$$

So this is yet another example of the phenomenon characteristic of non-Abelian gauge theories: Electric and magnetic fields do not carry all the gauge-invariant content of the model.

APPENDIX B: SIGN OF AN ENERGY CONTRIBUTION

We prove that for generic, radially symmetric charge distributions the last two terms in (4.26) can become negative, so that $\mathcal{E}$ can be made smaller than the non-Abelian Coulomb energy, the first term on the right-hand side of (4.26). Consider

$$\Delta = \frac{1}{16\pi}\int d\vec{r}d\vec{r}'\rho_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\rho_a(\vec{r}')[\theta_b(\vec{r})-\theta_b(\vec{r}')]^2 + \frac{1}{8\pi}\int d\vec{r}d\vec{r}'\rho_a(\vec{r})\theta_b(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\rho_b(\vec{r}')\theta_a(\vec{r}') . \tag{B1}$$

θ_a is constrained to be orthogonal to the charge density,

$$\theta_a(\vec{r})\rho_a(\vec{r}) = 0 . \tag{B2}$$

For notational clarity, we have dropped the prime on ρ. We wish to show that $\Delta \geq 0$. The first term on the right-hand side of (B1) can be rewritten as

$$\frac{1}{8\pi}\int d\vec{r}d\vec{r}'\rho_a(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}\rho_a(\vec{r}')[\theta_b(\vec{r})\theta_b(\vec{r})-\theta_b(\vec{r})\theta_b(\vec{r}')] .$$

Thus

$$\begin{aligned}\Delta &= \frac{1}{8\pi}\int \frac{d\vec{r}d\vec{r}'}{|\vec{r}-\vec{r}'|}[\rho_a(\vec{r})\rho_a(\vec{r}')][\theta_b(\vec{r})\theta_b(\vec{r})] + \frac{1}{8\pi}\int\frac{d\vec{r}d\vec{r}'}{|\vec{r}-\vec{r}'|}[\rho_a(\vec{r})\theta_b(\vec{r})][\rho_b(\vec{r}')\theta_a(\vec{r}')-\rho_a(\vec{r}')\theta_b(\vec{r}')] \\ &= \frac{1}{8\pi}\int \frac{d\vec{r}d\vec{r}'}{|\vec{r}-\vec{r}'|}[\rho_a(\vec{r})\rho_a(\vec{r}')][\theta_b(\vec{r})\theta_b(\vec{r})] - \frac{1}{16\pi}\int\frac{d\vec{r}d\vec{r}'}{|\vec{r}-\vec{r}'|}[\rho_a(\vec{r})\theta_b(\vec{r})-\rho_b(\vec{r})\theta_a(\vec{r})] \\ &\qquad \times[\rho_a(\vec{r}')\theta_b(\vec{r}')-\rho_b(\vec{r}')\theta_a(\vec{r}')] .\end{aligned} \tag{B3}$$

We define a unit vector in the direction of the charge density, which is assumed not to vanish at finite r,

$$\rho_a(\vec{r}) = \hat{\rho}_a(\vec{r})\,q(\vec{r}) . \tag{B4}$$

A third vector completes the orthogonal set of $\hat{\rho}_a$ and θ_a; it has the same length as θ_a,

$$\eta_a(\vec{r}) = \epsilon_{abc}\hat{\rho}_b(\vec{r})\theta_c(\vec{r}) , \tag{B5a}$$

$$\eta^2(\vec{r}) = \theta^2(\vec{r}) . \tag{B5b}$$

In terms of η, (B3) becomes

$$\Delta = \frac{1}{8\pi}\int d\vec{r}d\vec{r}'q(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}q(\vec{r}')[\hat{\rho}_a(\vec{r})\hat{\rho}_a(\vec{r}')\eta_b(\vec{r})\eta_b(\vec{r})-\eta_a(\vec{r})\eta_a(\vec{r}')] . \tag{B6}$$

Thus far we have not used the spherical symmetry of the charge density. But now let us assert that

$$\hat{\rho}_a(\vec{r}) = \hat{r}^a , \quad q(\vec{r}) = q(r) , \tag{B7}$$

and expand the inverse Laplacian in terms of spherical harmonics,

$$\begin{aligned}\Delta &= \tfrac{1}{2}\int_0^\infty dr\,dr'\,r^2 r'^2 q(r)\,q(r')\sum_{lm}\left(\int d\Omega Y_{lm}(\Omega)\hat{r}^a\eta^2(\vec{r})\int d\Omega' Y^*_{lm}(\Omega')\hat{r}'^a - \int d\Omega Y_{lm}(\Omega)\eta_a(\vec{r})\int d\Omega' Y^*_{lm}(\Omega')\eta_a(\vec{r}')\right) \\ &= \tfrac{1}{2}\int_0^\infty dr\,dr'\,r^2 r'^2 q(r)\,q(r')\left(\frac{1}{3}\frac{r_<}{r_>^2}\int d\Omega\,\eta^2(\vec{r}) - \sum_{lm}\frac{r_<^l}{r_>^{l+1}}\frac{1}{2l+1}\eta_a^{*lm}(r)\eta_a^{lm}(r')\right) .\end{aligned} \tag{B8}$$

In passing from the first to the second equality, we have made use of the formula

$$\sum_m Y_{lm}(\Omega)\int d\Omega' Y^*_{lm}(\Omega')\hat{r}'^a=\delta_{l1}\hat{r}^a .$$

The quantity $\eta_a(\vec{r})$ has been expanded according to

$$\eta_a(\vec{r})=\sum_{lm}\eta_a^{lm}(r)Y_{lm}(\Omega), \tag{B9}$$

and $r_<$ ($r_>$) is the lesser (greater) of r and r'. By virtue of (B9) the remaining angular integral in (B8) may be performed,

$$\int d\Omega\eta^2(\vec{r})=\sum_{lm}\eta_a^{*lm}(r)\eta_a^{lm}(r).$$

Thus the final result is

$$\begin{aligned}\Delta&=\tfrac{1}{6}\int_0^\infty dr\,dr'r^2r'^2q(r)\,q(r')\frac{r_<}{r_>^2}\sum_{lm}\eta_a^{*lm}(r)\left[\eta_a^{lm}(r)-\left(\frac{r_<}{r_>}\right)^{l-1}\frac{3}{2l+1}\eta_a^{lm}(r')\right]\\&=\tfrac{1}{6}\int_0^\infty dr\,dr'r^2r'^2q(r)\,q(r')\frac{r_<}{r_>^2}\sum_{lm}\left\{\tfrac{1}{2}[\eta_a^{*lm}(r)-\eta_a^{*lm}(r')][\eta_a^{lm}(r)-\eta_a^{lm}(r')]\right.\\&\qquad\left.+\eta_a^{*lm}(r)\eta_a^{lm}(r')\left[1-\frac{3}{2l+1}\left(\frac{r_<}{r_>}\right)^{l-1}\right]\right\}.\end{aligned}\tag{B10}$$

The first term in the curly brackets is manifestly non-negative. The second does not have a definite sign. But because the η_a^{lm} derive from largely arbitrary gauge functions θ_a, they are at our disposal and we can obviously arrange them so that the indefinite term is positive. We require $\eta_a^{00}=0$, then

$$1-\frac{3}{2l+1}\left(\frac{r_<}{r_>}\right)^{l-1}\geq 0 .$$

When we do not have control over $\eta_a^{*lm}(r)\eta_a^{lm}(r')$, we may arrange things so that only $l=1$ contributes, in which case that quantity vanishes. For the δ-shell potential, all the radial integrals are evaluated at the common shell radius r_0, and we get

$$\Delta=\frac{Q^2}{6a}\sum_{lm}\eta_a^{*lm}(r_0)\eta_a^{lm}(r_0)\left(1-\frac{3}{2l+1}\right),$$

which is obviously positive when $\eta_a^{00}=0$.

APPENDIX C: DETERMINATION OF THE BIFURCATION POINT

In this appendix we develop a crude computational scheme for an *a priori* determination of the bifurcation point. The final answer agrees well with the value found in the numerical solution. This then is a useful check on our understanding of the theory. Let us record the relevant Eqs. (3.30),

$$-f''+\frac{2a^2}{x^2}f=xQq, \tag{C1a}$$

$$-a''+\frac{a^2-1-f^2}{x^2}a=0. \tag{C1b}$$

A strength parameter Q for the source is explicitly exhibited. The type-II (bifurcating) solutions satisfy (C1) with boundary conditions $a^{\mathrm{II}}(0)=1$, $a^{\mathrm{II}}(\infty)=-1$; and two distinct solutions are found once Q is sufficiently large. The type-I (non-Abelian Coulomb) solution satisfies (C1) with boundary conditions $a^{\mathrm{I}}(0)=1$, $a^{\mathrm{I}}(\infty)=1$. One solution exists for arbitrary Q. For both types, $f(0)=0$, $f(\infty)=0$. We set ourselves the task of determining the minimum Q which supports the type-II solutions.

As a first step, we show how the type-II solutions can be approximately constructed from a knowledge of the type-I pair $\{f^{\mathrm{I}},a^{\mathrm{I}}\}$ for a definite Q. A crude approximation to the type-II solution, with the same Q, is

$$f_0^{\mathrm{II}}=f^{\mathrm{I}}, \tag{C2a}$$

$$a_0^{\mathrm{II}}=\begin{cases}a^{\mathrm{I}}, & x<x_0\\ -a^{\mathrm{I}}, & x>x_0 .\end{cases} \tag{C2b}$$

The pair $\{f_0^{\mathrm{II}},a_0^{\mathrm{II}}\}$ clearly has the correct boundary conditions. It satisfies the differential equation (C1) everywhere except at $x=x_0$. That point, defined to be the zero of a^{II}, remains undetermined. Another defect of (C2) is that one set of functions is obtained, while the type-II solution has two branches.

In order to improve on the crudest, zeroth-order approximation (C2), we write as a first approximation

$$f_1^{\rm II}=f^{\rm I}-\Delta f\,, \tag{C3a}$$

$$a_1^{\rm II}=\begin{cases} a^{\rm I}-\Delta a\,, & x<x_0 \\ -a^{\rm I}+\Delta a\,, & x>x_0\,, \end{cases} \tag{C3b}$$

where Δf and Δa are taken to be small. Substituting (C3) into (C1) and linearizing, we obtain first-order equations for the small quantities,

$$-(\Delta f)''+\frac{2(a^{\rm I})^2}{x^2}\Delta f+\frac{4f^{\rm I}a^{\rm I}}{x^2}\Delta a=0\,, \tag{C4a}$$

$$-(\Delta a)''+\frac{3(a^{\rm I})^2-1-(f^{\rm I})^2}{x^2}\Delta a-\frac{2a^{\rm I}f^{\rm I}}{x^2}\Delta f=0\,,$$

$$x\neq x_0\,. \tag{C4b}$$

The equations are identical to those governing the zero-eigenvalue mode (4.49). But the criteria for accepting a solution to (C4) differ from those applicable to (4.49). Clearly the boundary conditions are the same: Both sets of functions vanish at $x=0$ and ∞. The difference emerges when we impose the requirement that $\{f_1^{\rm II},a_1^{\rm II}\}$ be everywhere continuous, with a continuous first derivative. At $x=x^0$, $a_1^{\rm II}$ vanishes, by definition of x^0. Thus it follows that

$$a^{\rm I}(x^0)=\Delta a(x_+^0)=\Delta a(x_-^0)\,. \tag{C5a}$$

The continuity of derivatives requires

$$\frac{d}{dx^0}a^{\rm I}(x^0)-\Delta a'(x_-^0)=-\frac{d}{dx^0}a^{\rm I}(x^0)+\Delta a'(x_+^0)\,,$$

$$\Delta a'(x_-^0)=\frac{d}{dx^0}a^{\rm I}(x^0)+h(x^0)\,, \tag{C5b}$$

$$\Delta a'(x_+^0)=\frac{d}{dx^0}a^{\rm I}(x^0)-h(x^0)\,.$$

Here x_+^0 (x_-^0) is a point slightly above (below) x^0 and $h(x^0)$ is arbitrary. Thus the solutions to (C4) are seen to be everywhere continuous, but their first derivative possesses a discontinuity. [The zero-eigenvalue mode which solves Eqs. (4.49) is continuous both in its value and in its first derivative.]

The procedure for solving (C4) therefore is the following. We take one solution which has the correct behavior at the origin, but not at infinity; and another with correct behavior at infinity, but not at the origin. The norms of these solutions are obviously individually arbitrary and we choose them so that (C5a) is satisfied. While this condition can therefore always be met, the other condition, (C5b), is nontrivial. It may be combined with (C5a) into an equation free of irrelevant normalization constants,

$$\frac{d}{dx}\ln\frac{\Delta a(x)}{a^{\rm I}(x)}\bigg|_{x=x_-^0}=-\frac{d}{dx}\ln\frac{\Delta a(x)}{a^{\rm I}(x)}\bigg|_{x=x_+^0}. \tag{C6}$$

In general, Eq. (C6) cannot be satisfied for arbitrary x^0. Thus it may be viewed as a determination of x^0, and when Q is varied this defines x^0 as a function of Q. For sufficiently large Q, we expect to find two values of x^0 for which (C6) can be met. As Q is lowered, the two values of x^0 should approach each other, until at a critical value they coalesce and for still lower Q there is no solution. In this way two approximate type-II solutions are constructed, they are of the form (C3) with determined x^0, and $Q_{\rm critical}$ is obtained as the minimum in a curve of Q versus x^0.

In order to carry out this calculation explicitly, we need to have the type-I solution as a function of Q. Unfortunately this is lacking; we have only perturbative formulas in Q. However, since *a posteriori* we expect $Q_{\rm critical}$ not to be too large, we may approximate these formulas with the contribution of lowest order in Q. Within this approximation, we find another pleasant feature: For the δ-shell source the equations decouple and the solutions are expressed in terms of known functions. We now show this, but first we need to transform (C4).

Define [compare (4.48)]

$$e=-x^2\left(\frac{\Delta f}{x}\right)'\,, \tag{C7a}$$

$$e'=-x(\Delta f)''$$

$$=-\frac{2(a^{\rm I})^2}{x}\Delta f-\frac{4f^{\rm I}a^{\rm I}}{x}\Delta a\,. \tag{C7b}$$

Equation (C4a) has been used to obtain the second equality in (C7b). It follows from (C4b) and (C7b) that

$$-(\Delta a)''+\frac{3(a^{\rm I})^2-1+3(f^{\rm I})^2}{x^2}\Delta a+\frac{f^{\rm I}}{xa^{\rm I}}e'=0\,, \tag{C8a}$$

while differentiation of (C7b) also gives

$$-\left(\frac{e'}{(a^{\rm I})^2}\right)'+\frac{2}{x^2}e-\left(\frac{4f^{\rm I}}{xa^{\rm I}}\Delta a\right)'=0\,. \tag{C8b}$$

It is the set (C8), which is entirely equivalent to (C4) for $x\neq x_0$, that we shall approximate for small Q. The approximation consists of retaining the background potential in (C8) only through $O(Q)$. Thus we set $a^{\rm I}=1$, and drop $(f^{\rm I})^2$. Equations (C8) then reduce to [compare (4.46)]

$$-(\Delta a)''+\frac{2}{x^2}\Delta a+\frac{f^{\rm I}}{x}e'=0\,, \tag{C9a}$$

$$-e''+\frac{2}{x^2}e-\left(\frac{4f^{\rm I}}{x}\Delta a\right)'=0\,, \tag{C9b}$$

and the $O(Q)$ background field for the δ-shell source is

$$\frac{f^{\mathrm{I}}(x)}{x}=\frac{Q}{3}\alpha(x)\,, \tag{C10a}$$

$$\alpha(x)=\begin{cases} x, & x<1 \\ \dfrac{1}{x^2}\,, & x>1\,. \end{cases} \tag{C10b}$$

The fortunate circumstance is that α is an integrating factor. Equation (C9a) may be written as

$$-\alpha\frac{d}{dx}\frac{1}{\alpha^2}\frac{d}{dx}(\alpha\Delta a)+\alpha\frac{Q}{3}\frac{de}{dx}=0\,. \tag{C11}$$

Therefore, it follows that

$$\frac{1}{\alpha^2}\frac{d}{dx}(\alpha\Delta a)=\frac{Q}{3}e+c\,, \tag{C12}$$

where c is a constant. Use of this result casts (C9b) into a decoupled equation,

$$-e''+\frac{2}{x^2}e-\tfrac{4}{9}Q^2\alpha^2e=\tfrac{4}{3}Qc\alpha^2\,. \tag{C13}$$

Analysis of this equation is then straightforward and the minimum Q, found by a numerical computation is $Q_{\text{critical}}=5.892$, in excellent agreement with 5.835, the value from a direct solution of the differential equations.

---

*On leave of absence from the Scuola Normale Superiore, Pisa, Italy.

[1]For an introduction to current mathematical research about stability, see C. Siegel and J. Moser, *Lectures on Celestial Mechanics* (Springer, Berlin, 1971). A simple discussion, referring to older physics research, is found in H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics* (Cambridge Univ. Press, Cambridge, England, 1972).

[2]J. Mandula, Phys. Rev. D 14, 3497 (1976).

[3]R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D 20, 474 (1979).

[4]This formula is a generalization of the "total screening solution" found by P. Sikivie and N. Weiss, Phys. Rev. Lett. 40, 1411 (1978); Phys. Rev. D 18, 3809 (1978). The generalization consists of replacing a discrete parameter by the continuously varying α. The possibility of generalizing was also discussed by P. Pirilä and P. Prešnajder, Nucl. Phys. B142, 229 (1978), and by J. Kiskis, preceding paper, Phys. Rev. D 21, 421 (1980).

[5]Another example of a solution which exists when the source is of sufficient magnitude is Sikivie and Weiss's "magnetic dipole solution" (see Ref. 4). The source which supports this solution was studied by Y. Leroyer and A. Raychaudhuri, Phys. Rev. D 20, 2086 (1979).

[6]P. Sikivie and N. Weiss, Phys. Rev. D 20, 487 (1979).

[7]This form for the fluctuation equations was also derived by M. Magg, Phys. Lett. 74B, 246 (1978).

[8]J. Mandula, Phys. Lett. 67B, 175 (1977); see also M. Magg, Ref. 7.

[9]L. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57, 315 (1940); K. Johnson, Ph.D. thesis, Harvard, 1954 (unpublished); A. Migdal, Zh. Eksp. Teor. Fiz. 61, 2209 (1972) [Sov. Phys. JETP 34, 1184 (1972)]; A. Klein and J. Rafelski, Phys. Rev. D 11, 300 (1975).

[10]That the physics of a top is encountered in the Yang-Mills theory was previously remarked by J. Goldstone and R. Jackiw, Phys. Lett. 74B, 81 (1978).

[11]T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975).