Some properties of an analog of the chiral model

Chiara R. Nappi*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 31 May 1979)

A two-dimensional, renormalizable scalar-field model, equivalent to the chiral model at the classical level, is shown not to be asymptotically free and to have particle production. Therefore, the classical analogy fails at the quantum level.

I. THE MODEL

We are going to investigate a two-dimensional renormalizable scalar-field theory which, classically, is equivalent to the O(N) chiral model. The Lagrangian of the chiral model in two-dimensional Minkowski space reads

$$\mathcal{L}=\mathrm{Tr}(\partial_\mu g\partial^\mu g^{-1}). \quad (1)$$

The equation of motion associated with it is simply

$$\partial_\mu(g^{-1}\partial^\mu g)=0. \quad (2)$$

The chiral model, recently studied by several authors,[1-3] is closely related to the nonlinear σ model. They are both known to be asymptotically free[4,5] and in both the interaction arises from the fact that the field takes values in a curved space. It is actually the chiral model that is related to Yang-Mills theories through Midgal's recursion relations.[6] Also, as will be explained later, the chiral model of O(3) or SU(2) symmetry is exactly equivalent to the nonlinear σ model of O(4) symmetry.

The chiral Lagrangian (1) is invariant under $G\times G$ and the invariance under this symmetry gives the conserved currents

$$J_\mu^{ij}=(g^{-1}\partial_\mu g)^{ij},\ \ K_\mu^{ij}=(g\partial_\mu g^{-1})^{ij}.$$

Here we choose G to be O(N). Then J_μ and k_μ are antisymmetric $N\times N$ matrices.

It is easy to check, by using the orthogonality condition $gg^T=1$, that

$$F_{\mu\nu}\equiv\partial_\mu J_\nu-\partial_\nu J_\mu+[J_\nu,J_\mu]=0. \quad (3)$$

Because J_μ is conserved, i.e., $\partial^\mu J_\mu=0$, there exists an $N\times N$ antisymmetric matrix ϕ, whose entries are scalar fields, such that $J_\mu=\epsilon_{\mu\nu}\partial_\nu\phi$. By virtue of (3), ϕ satisfied the equation

$$\partial_\mu\partial^\mu\phi-\tfrac{1}{2}\epsilon_{\mu\nu}[\partial_\mu\phi,\partial_\nu\phi]=0. \quad (4)$$

This is the equation of motion associated with the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}\mathrm{Tr}\{\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{3}\epsilon_{\mu\nu}\phi[\partial_\mu\phi,\partial_\nu\phi]\}. \quad (5)$$

Therefore, given a solution g of the equation of motion (2) we get, through the above procedure, a solution of the equation (4) as well.

Conversely, given an $N\times N$ antisymmetric matrix ϕ, a solution of the equation (4), we define $J_\mu=\epsilon_{\mu\nu}\partial_\nu\phi$. Then, in terms of J_μ, the equation (4) becomes (3)

$$\partial_\mu J_\nu-\partial_\nu J_\mu-[J_\mu,J_\nu]=0.$$

This implies that J_μ is a pure gauge, i.e., there exists an orthogonal matrix g such that

$$J_\mu=g^{-1}\partial_\mu g.$$

Moreover, from the definition of J_μ it follows that the equation (2) is satisfied. This means that to any solution ϕ of (4) corresponds a solution, g, of (2). Thus, the scalar field theory (5) is, at the classical level, equivalent to the O(N) chiral model.

The two theories, however, are not equivalent at the quantum level. We will in fact show that, in contrast to the O(N) chiral model, the theory (5) or, more precisely, the theory with Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi+\lambda\epsilon_{\mu\nu}\phi\partial_\mu\phi\partial_\nu\phi) \quad (6)$$

[where we have introduced in (5) a coupling constant λ by rescaling ϕ] is not asymptotically free.[7]

Moreover, we will show that this model has particle production. The question of whether the chiral model has particle production has not been studied in the general case, but in view of the results of Refs. 2 and 3, it quite likely does not. This is definitely true in the particular case of O(3) or SU(2), when the chiral model is equivalent to the O(4) σ model, which is known not to have particle production.[8]

The equivalence between the SU(2) chiral model and the O(4) σ model can be seen by expressing g in terms of 4 real parameters (n_0,n_1,n_2,n_3) as $g=n_0I+i\vec{n}\cdot\vec{\sigma}$, where σ are the Pauli matrices. For g to be in SU(2), n_i must satisfy $n_0^2+n_1^2+n_2^2+n_3^2=1$ which is the constraint of the σ model. Then a straightforward calculation shows that

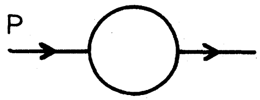

FIG. 1. Two-point function $\Gamma^{(2)}$.

$$\mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] = 2\partial_\mu n_i \partial^\mu n_i, \quad i=0,\ldots,3$$

where the left-hand side is the Lagrangian of the O(4) nonlinear σ model.

Therefore, at least for O(3) or SU(2) and probably in general particle production is absent in the O(N) chiral model; this is another notable difference between the quantum theory based on Lagrangian (6) and the O(N) chiral model.

II. ASYMPTOTIC FREEDOM

The renormalization-group equations for the one-particle irreducible Green's functions are

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right]\Gamma^{(n)} = 0 .$$

In order to determine whether the theory (6) is asymptotically free, we need to know the sign of the coefficient of the first nonzero term in the expansion series of $\beta(\lambda)$ in terms of the physical coupling constant λ. In this case the lowest term turns out to be cubic in λ. To this purpose, it is sufficient to know the expansions, up to one loop order, of the two- and three-point functions $\Gamma^{(2)}$ and $\Gamma^{(3)}$, which are determined by the diagrams of Figs. 1 and 2, respectively.

Apart from irrelevant constants, the normalized inverse propagator $\Gamma^{(2)}$ turns out to be

$$\Gamma^{(2)} = 1 - \lambda^2 \frac{(N-2)}{4\pi} \ln \frac{-p^2}{\mu^2} .$$

The diagram of Fig. 2, however, turns out to be finite. Therefore $\Gamma^{(3)} = \lambda + \lambda^3 B$, where B is a number, independent of p and μ.

Based on these results, one finds that, to the lowest order in λ, $\gamma(\lambda) = [(N-2)/4\pi]\lambda^2$ and $\beta(\lambda) = [3(N-2)/4\pi]\lambda^3$. Therefore, the first nonzero coefficient in β is positive, and the theory is not asymptotically free.

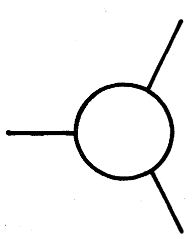

FIG. 2. Three-point function $\Gamma^{(3)}$.

III. PARTICLE PRODUCTION

Here we will show that the scalar-field theory with derivative coupling (6) has particle production.

For the sake of simplicity, let us consider our model in the case when the symmetry group is O(3). For general N, we would consider a process involving particles in an O(3) subgroup of O(N), and reach the same answer. So, in the case of O(3), we have only three independent fields ϕ^{ij}, which we will call A, B, and C. In terms of these fields the Lagrangian (6) becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + \partial_\mu C \partial^\mu C + \lambda \epsilon_{\mu\nu} A \partial_\mu B \partial_\nu C) . \tag{7}$$

The simplest process with particle production compatible with the symmetries of (6) is a process with two particles going to three such as $B + C \to A + A + A$, which corresponds to the Feynman diagram shown in Fig. 3. The amplitude for this process, apart from symmetrization between identical particles, is

$$\lambda^3 (2\pi)^2 \epsilon_{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} p_\mu q_\delta \frac{(p-p_1)_\nu (p-p_1)_\beta}{(p-p_1)^2} \frac{(q-p_3)_\alpha (q-p_3)_\gamma}{(q-p_3)^2} . \tag{8}$$

Equation (8), by simple algebraic manipulations, becomes

$$\lambda^3 \pi^2 \frac{\epsilon_{\gamma\delta} q_\delta p_{3\gamma}}{q \cdot p_3} (q - p_3) \cdot (p + p_1) .$$

Let us consider now the process where p_3 is parallel to p, and p_1 and p_2 parallel to q. By conservation laws then

$$p_3 = p \quad \text{and} \quad p_1 + p_2 = q$$

and we will have to symmetrize between p_1 and p_2.

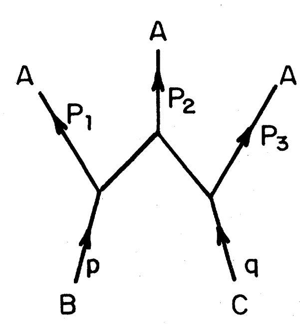

FIG. 3. A process with two particles going to three whose amplitude is given by Eq. (8).

Therefore, the amplitude of the process in Fig. 3 is

$$\frac{\pi^2}{2}\lambda^3[(p-q)\cdot p_2+(p-q)\cdot p_1]=\frac{\pi^2}{2}\lambda^3 p\cdot q\neq 0\,,$$

which shows that our model has particle production.

ACKNOWLEDGMENTS

The author thanks E. Witten for discussions. This research was partially supported by the National Science Foundation under Grant No. PHY77-18762.

*On leave from Istituto di Fisica Teorica, Napoli, Italy.

[1]A. M. Polyakov, Phys. Lett. 82B, 247 (1979).
[2]E. Brezin, C. Itzykson, J. Zinn-Justin, and J.-B. Zuber, Phys. Lett. 82B, 442 (1979).
[3]H. J. de Vega, Saclay report, 1979 (unpublished).
[4]A. M. Polyakov, Phys. Lett. 59B, 79 (1975).
[5]A. M. Polyakov, lecture at Harvard University, 1978 (unpublished).
[6]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys. JETP 42, 743 (1975)].

[7]Incidentally, this Lagrangian has an unusual symmetry $\phi\rightarrow\phi+C$, where C is a constant. It is for this reason that (6) is renormalizable without the need for additional counterterms, such as mass renormalization, and that ϕ particles remain massless in perturbation theory.

[8]A. Zamolodchikov and Al. Zamolodchikov, Pis'ma Zh. Eksp. Teor. Fiz. 26, 608 (1977) [JETP Lett. 26, 457 (1977)].