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Nonlocal currents for supersymmetric nonlinear models

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Curvature-free covariant derivatives are given for the O(N) and CP(N) supersymmetric nonlinear models in two dimensions. The derivatives are used in a differential algorithm to explicitly construct sequences of nonlocal conserved currents. Properties of these currents and their charges are briefly discussed.

I. INTRODUCTION

At present there are few techniques for confidently extracting nonperturbative information from quantum chromodynamics (QCD). An interesting observation[1-5] which may lead to new techniques is that four-dimensional Yang-Mills theory is in many ways simulated by nonlinear σ models in two dimensions. For example, nonlinear models are self-coupled systems which are asymptotically free,[5] possess instanton solutions,[4,6] and exhibit dynamical mass generation.[7-9] All three of these properties are believed to be important characteristics of four-dimensional gauge theories. In addition, σ models have a renormalization structure similar to that of Yang-Mills theories.[10]

More recently, Polyakov[1] has emphasized an intriguing parallelism between the functional differential equations[11] satisfied by appropriately defined "string variables" for gauge theories and the equations governing the conventional field variables for two-dimensional σ models. This parallelism could well lead to the discovery of new and useful computational approaches to Yang-Mills theory by analogy with the σ models.

It has also been suggested that the coupling of fermions to gauge fields may be simulated by supersymmetric extensions of σ models.[12-15] These extensions retain many of the features of the original nonlinear theory, such as those examples cited above. In this paper we discuss some interesting nonlocal features[3] which are also retained by supersymmetric σ models (SSM's).

The dynamics of two-dimensional σ models are remarkable in that they yield a new class of unconventional conservation laws, both local[16] and nonlocal.[3] For quantized σ models these laws have been exploited to explicitly construct S matrices.[17,18] For higher-dimensional gauge theories it has been conjectured that analogous nonlocal conservation laws will provide crucial information for understanding large-distance effects.[1,2]

However, the dynamical basis of such nonlocal conservation laws has not been completely clarified, even within the context of two-dimensional systems. To ensure the presence of nonlocal charge sequences in these systems, several criteria have been proposed involving linear eigenvalue systems,[3] dual symmetries,[2,19] and zero-curvature conditions for the Noether currents.[20,21] Most of these criteria do not directly accommodate those nontrivial enlargements of the σ models which are needed to produce supersymmetric theories. Nonetheless, nonlocal charges for the O(N)-invariant SSM have been found.[22,23]

In this paper we clarify why these charges exist for nonlinear supersymmetric models by presenting an improved method for their construction. Our new construction illustrates that some of the previously offered criteria may be relaxed if only a suitably defined, curvature-free, fundamental vector functional can be found. Our discussion also shows the compatibility of the two previous methods used by us[22,23] to obtain the nonlocal charges of the SSM.

In Sec. II we briefly review the O(N) SSM and exhibit those algebraic properties of the model which yield a fundamental curvature-free family of currents, $C_\mu(\kappa)$.

Section III is devoted to a straightforward recursive derivation of nonlocal currents by an algorithm closely related to that used for ordinary σ models.[20,21] A generating functional for the nonlocal currents is obtained and then converted into a charge-generating functional $G(\kappa)$. A simple explanation for the conservation of the charges is given, and $G(\kappa)$ is related to the functional found previously[23] using elegant contour techniques.[2] Our results are shown to reduce in a simple way to describe either the plain σ model or the Gross-Neveu[7] sector of the theory.

We present in Sec. IV the details of another example, the supersymmetric CP($N-1$) model, and we comment on the general applicability of our approach. We conclude the paper in Sec. V with a brief discussion of additional features of the nonlocal currents: their supersymmetry transformations, their charge algebra, and the constraints they impose on the S matrix of the model. We close by commenting on the possibility of explicitly diagonalizing the SSM Hamiltonian.

II. PROPERTIES OF THE O(N) SUPERSYMMETRIC σ MODEL

The O(N)-invariant SSM in two dimensions is a hybridization of the plain nonlinear σ model,[24] formulated for real scalar fields n^a, and the Gross-Neveu model,[7] formulated for Majorana spinors ψ^a. Constraints are imposed on both n^a and ψ^a to ensure equal numbers of degrees of freedom in the Bose and Fermi sectors of the model. The properties of these two sectors are closely interrelated through supersymmetry transformations. Originally, the model was constructed using superfields.[12,13] Such superfields are not used in this paper for reasons of transparency and accessibility. However, in our discussion in Sec. V, we provide those ingredients needed to see how the nonlocal currents could be expressed in a superfield framework.

The SSM is defined by the action

$$S=\int d^2x\left[\tfrac{1}{2}\partial_\mu n^a\partial^\mu n^a+\tfrac{1}{2}i\bar\psi^a\not\partial\psi^a+\tfrac{1}{8}(\bar\psi^a\psi^a)^2\right], \tag{2.1}$$

together with the constraints

$$n^an^a=1,\quad n^a\psi^a=0. \tag{2.2}$$

Our Lorentz index and Dirac matrix conventions are summarized in the Appendix. All fields in the action transform as vector representations of O(N) with $a=1,\ldots,N$. The equations of motion for the fields are

$$\begin{aligned}\Phi^a&\equiv\partial^2n^a+n^a(\partial_\mu n^b\partial^\mu n^b)+i\bar\psi^a(n^b\not\partial\psi^b)=0\,,\\ \Psi^a&\equiv i\not\partial\psi^a+\tfrac{1}{2}\psi^a(\bar\psi^b\psi^b)-in^a(n^b\not\partial\psi^b)=0\,.\end{aligned} \tag{2.3}$$

The model is invariant under the supersymmetry transformations

$$\delta n^a=\bar\epsilon\psi^a,\quad \delta\psi^a=\left[-i\not\partial n^a+\tfrac{1}{2}n^a(\bar\psi^b\psi^b)\right]\epsilon\,, \tag{2.4}$$

as well as under O(N) rotations.

For reasons to become apparent, we split the Noether currents generating the O(N) rotations into boson and fermion components,

$$\begin{aligned}J_\mu^{(0)ab}&=A_\mu^{ab}+2B_\mu^{ab}\,,\\ A_\mu^{ab}&=2n^a\overleftrightarrow{\partial}_\mu n^b\,,\\ B_\mu^{ab}&=-i\bar\psi^a\gamma_\mu\psi^b\,.\end{aligned} \tag{2.5}$$

Derivatives of these components may be evaluated with the help of the identities in the Appendix. Using (2.2) and (2.3), we find

$$\begin{aligned}\partial^\mu A_\mu^{ab}&=(A^\mu)^{ac}B_\mu^{cb}-(B^\mu)^{ac}A_\mu^{cb}+2(n^a\Phi^b-\Phi^an^b)\,,\\ \partial^\mu B_\mu^{ab}&=-\tfrac{1}{2}(A^\mu)^{ac}B_\mu^{cb}+\tfrac{1}{2}(B^\mu)^{ac}A_\mu^{cb}-(\bar\psi^a\Psi^b-\bar\Psi^a\psi^b)\,,\\ \epsilon^{\mu\nu}\partial_\mu A_\nu^{ab}&=-\epsilon^{\mu\nu}A_\mu^{ac}A_\nu^{cb}\,,\\ \epsilon^{\mu\nu}\partial_\mu B_\nu^{ab}&=-\epsilon^{\mu\nu}B_\mu^{ac}B_\nu^{cb}-\tfrac{1}{2}\epsilon^{\mu\nu}(A_\mu^{ac}B_\nu^{cb}+B_\mu^{ac}A_\nu^{cb})\\ &\quad-(\bar\psi^a\gamma_5\Psi^b-\bar\Psi^a\gamma_5\psi^b)\,.\end{aligned} \tag{2.6}$$

Note that on-shell, where Φ and Ψ vanish, each of these derivatives is a bilinear in A_μ and/or B_μ. Furthermore, observe that the local O(N) currents are conserved on-shell as a simple consequence of the first two relations in (2.6):

$$\partial^\mu J_\mu^{(0)ab}=0\,. \tag{2.7}$$

In the following we suppress all O(N) indices.

In general, the construction of nonlocal currents for a model requires examining integrability conditions for certain vector functionals. To this end we define "covariant" derivatives involving arbitrary local vector fields:

$$D_\mu^V=\partial_\mu+V_\mu\,. \tag{2.8}$$

All cases of interest for the O(N) SSM concern vector fields which are linear combinations of A_μ, B_μ, and their duals $\epsilon^{\mu\nu}A_\nu$, $\epsilon^{\mu\nu}B_\nu$. The vector field is a "pure gauge," i.e.,

$$V_\mu=U^{-1}\partial_\mu U\,, \tag{2.9}$$

if and only if the covariant derivative satisfies

$$[D_\mu^V,D_\nu^V]=0\,. \tag{2.10}$$

This is referred to as the "zero-curvature" condition. The commutator in (2.10) may be written as

$$\begin{aligned}[D_\mu^V,D_\nu^V]&=\partial_\mu V_\nu-\partial_\nu V_\mu+[V_\mu,V_\nu]\\ &=-\epsilon_{\mu\nu}(\epsilon^{\kappa\lambda}D_\kappa^VV_\lambda)\,.\end{aligned} \tag{2.11}$$

The second of these identities is peculiar to two dimensions.

If V_μ is a conserved pure gauge, a sequence of nonlocal conserved currents can be simply constructed using the algorithm of Brézin *et al.*[20] For the plain σ model, where $B_\mu=0$, or for the Gross-Neveu model, where $A_\mu=0$, this algorithm can be applied [cf. (2.6)]. For the SSM, however, A_μ satisfies (2.10) but is not conserved, B_μ is neither conserved nor curvature-free, and the conserved Noether currents $J^{(0)}$ do not yield zero curvature. Instead we have[23]

$$\epsilon^{\mu\nu}(\partial_\mu+J_\mu^{(0)})J_\nu^{(0)}=-2\epsilon^{\mu\nu}\partial_\mu B_\nu\,, \tag{2.12}$$

and so the algorithm of Brézin *et al.* is not applicable to the SSM. Nevertheless, because the right-hand side (RHS) of (2.12) is a curl of a local field, it is still relatively straightforward[25] to construct the first nonlocal current.[22] Higher nonlocal charges could possibly be generated through the charge algebra (cf. Sec. V), but as we show in the next section, the full nonlocal charge sequence emerges from a much more elegant algorithm.

III. NONLOCAL CURRENTS AND CHARGES

We previously noted that there is a curvature-free covariant derivative in the SSM, D_μ^A. Unfortunately, the currents involved, A_μ, are not conserved, so the methods of Brézin *et al.*[20] cannot be used to construct nonlocal currents. However, there is an alternate method for constructing nonlocal charges, which was found by Polyakov for the plain σ model.[2] In this other method a one-parameter family of curvature-free derivatives is the sole ingredient. The currents involved in this family need not be conserved. Motivated by Polyakov's method, a curvature-free family of derivatives was discovered for the SSM by investigating arbitrary linear combinations of A_μ, B_μ, and their duals.[23] This new family of currents allowed the construction of the nonlocal charges for the SSM using techniques similar to Polyakov's.

In this section we use this same family of zero-curvature covariant derivatives in a differential algorithm for directly constructing nonlocal currents. The method is reminiscent of that used by Brézin *et al.*, but more general. The basic technique was introduced by Lüscher and Pohlmeyer[3] and further discussed by others[21] in analyses of the plain σ model.

A convenient parametrization of the curvature-free current family for the SSM is given by

$$C_\mu(\kappa)=\frac{-\kappa}{1-\kappa^2}\left[\kappa A_\mu+\epsilon_{\mu\nu}A^\nu+\frac{4\kappa}{1-\kappa^2}B_\mu+\frac{2(1+\kappa^2)}{1-\kappa^2}\epsilon_{\mu\nu}B^\nu\right]. \tag{3.1}$$

Using (2.6), we find these currents to be nonconserved on-shell for all nontrivial κ. More importantly, defining

$$D_\mu=\partial_\mu+C_\mu(\kappa;x,t) \tag{3.2}$$

and using (2.6) again, we find these covariant derivatives all give zero curvature, (2.10). Thus the integrability condition for the equation

$$D_\mu\chi(\kappa;x,t)=0 \tag{3.3}$$

is satisfied, and the solution $\chi(\kappa)$ can be obtained.

It is convenient to write Eq. (3.3) in the equivalent form

$$\epsilon_{\mu\nu}\partial^\nu\chi(\kappa)=\kappa\left[(\partial_\mu+A_\mu)+\frac{2}{1-\kappa^2}(B_\mu+\kappa\epsilon_{\mu\nu}B^\nu)\right]\chi(\kappa). \tag{3.4}$$

This equation has a series solution obtained by substituting

$$\chi(\kappa;x,t)=\sum_{n=0}^{\infty}\kappa^n\chi^{(n-1)}(x,t). \tag{3.5}$$

To each order in κ, the potentials $\chi^{(n)}$ on the left-hand side (LHS) of Eq. (3.4) are then of higher degree, n, than those on the RHS. Consequently, the $\chi^{(n)}$'s may be constructed iteratively through a recursion relation. Defining the current $J_\mu^{(n)}$ as the space-time curl of $\chi^{(n)}$, this recursion relation is

$$\begin{aligned}J_\mu^{(n+1)}&\equiv\epsilon_{\mu\nu}\partial^\nu\chi^{(n+1)}\\&=(\partial_\mu+J_\mu^{(0)})\chi^{(n)}+2B_\mu[\chi^{(n-2)}+\chi^{(n-4)}+\cdots]\\&\quad+2\epsilon_{\mu\nu}B^\nu[\chi^{(n-1)}+\chi^{(n-3)}+\cdots],\end{aligned} \tag{3.6}$$

where

$$\chi^{(n)}(x,t)=\begin{cases}0, & n<-1\\ 1, & n=-1\\ \int_{-\infty}^{x}dy\,J_0^{(n)}(y,t), & n\geq 0.\end{cases} \tag{3.7}$$

The two terms in Eq. (3.6) involving explicit series of lower-degree potentials correspond to the $\zeta_\mu^{(n)}$'s employed in the recursion relation of Ref. 22.

The currents $J_\mu^{(n+1)}$ are manifestly conserved by (3.6) and are nonlocal functions of currents of a lower degree:

$$J_\mu^{(n+1)}=J_\mu^{(0)}\chi^{(n)}+\epsilon_{\mu\nu}(2J^{(n)\nu}-A^\nu\chi^{(n-1)})-J_\mu^{(n-1)}. \tag{3.8}$$

For example, the first two nonlocal currents are

$$\begin{aligned}J_\mu^{(1)}(x,t)&=J_\mu^{(0)}(x,t)\int_{-\infty}^{x}dy\,J_0^{(0)}(y,t)+\epsilon_{\mu\nu}[J^{(0)\nu}(x,t)+2B^\nu(x,t)],\\J_\mu^{(2)}(x,t)&=J_\mu^{(0)}(x,t)\int_{-\infty}^{x}dy\,J_0^{(1)}(y,t)+2\epsilon_{\mu\nu}B^\nu(x,t)\int_{-\infty}^{x}dy\,J_0^{(0)}(y,t)\\&\quad+\epsilon_{\mu\nu}J^{(1)\nu}(x,t)+2B_\mu(x,t).\end{aligned} \tag{3.9}$$

The reader may find it instructive to show directly these are conserved using Eq. (2.6).[26]

Formally we may also solve Eq. (3.3) without recourse to a κ expansion by using ordered exponentials:

$$\chi(\kappa;x,t)=P\exp\left(-\int_{-\infty}^{x}dy\,C_1(\kappa;y,t)\right) \tag{3.10}$$

is a solution satisfying the boundary condition $\chi(\kappa;-\infty,t)=1$. The exponential is path ordered with y decreasing to the right. This solution permits us to obtain a generating functional for the nonlocal currents.

$$\epsilon_{\mu\nu}\partial^\nu\chi(\kappa;x,t)=-\epsilon_{\mu\nu}C^\nu(\kappa;x,t)\times P\exp\left(-\int_{-\infty}^{x}dy\,C_1(\kappa;y,t)\right)=\sum_{n=1}^{\infty}\kappa^n J_\mu^{(n-1)}(x,t)\,. \quad (3.11)$$

Integrating the timelike component of Eq. (3.11) produces a generating functional for the nonlocal charges:

$$G(\kappa;t)=\sum_{n=1}^{\infty}\kappa^n Q^{(n-1)}(t)=-\int_{-\infty}^{\infty}dx\,C_1(\kappa;x,t)\times P\exp\left(-\int_{-\infty}^{x}dy\,C_1(\kappa;y,t)\right). \quad (3.12)$$

Because of the zero-curvature condition, we may simplify this expression by writing C_μ in pure gauge form,

$$C_\mu(\kappa;x,t)=U^{-1}(\kappa;x,t)\partial_\mu U(\kappa;x,t)\,, \quad (3.13)$$

from which we obtain the relation

$$P\exp\left(-\int_a^b C_1 dy\right)=U^{-1}(b)U(a)\,. \quad (3.14)$$

Thus (3.12) may be rewritten as

$$G(\kappa;t)=U^{-1}(\kappa;\infty,t)U(\kappa;-\infty,t)-1=P\exp\left(-\int_{-\infty}^{\infty}dy\,C_1(\kappa;y,t)\right)-1\,. \quad (3.15)$$

Note that $G(\kappa;t)+1$ is the inverse of the contour integral $I(\kappa;t)$ defined in Ref. 23. Also, $G(\kappa;t)+1$ could be obtained directly from $\chi(\kappa;x=+\infty,t)$, as is clear from (3.5) and (3.10).

Since the argument of every exponential vanishes at $\kappa=0$, the direct expansion of $G(\kappa;t)$ immediately gives

$$G(\kappa;t)=\kappa\int_{-\infty}^{\infty}dx J_0^{(0)}(x,t)+\kappa^2\int_{-\infty}^{\infty}dx\left(J_1^{(0)}(x,t)+2B_1(x,t)+J_0^{(0)}(x,t)\int_{-\infty}^{x}dy\,J_0^{(0)}(y,t)\right)+\cdots \quad (3.16)$$

as expected. We note in passing that $G(\kappa;t)$ is obviously t independent when A_μ, B_μ, and hence C_μ vanish at $x=\pm\infty$, since $U(\kappa;\pm\infty,t)$ is then a constant.

The fundamental functional $C_\mu(\kappa)$ in Eq. (3.1) may be reparametrized in a variety of ways, or expanded around different values of κ. For instance, $\lambda=1/\kappa$ leaves invariant the form of the coefficients of B_μ and $\epsilon_{\mu\nu}B^\nu$. Alternatively, $\rho=2\kappa/(1+\kappa^2)$ gives

$$\frac{4\kappa^2}{(1-\kappa^2)^2}=\frac{\rho^2}{1-\rho^2}\,,\quad \frac{2\kappa(1+\kappa^2)}{(1-\kappa^2)^2}=\frac{\rho}{1-\rho^2}\,,$$

and produces ρ-dependent coefficients of B_μ and its dual of the same form as the original κ-dependent coefficients of A_μ and its dual. This enables us to recover the structure of the Gross-Neveu theory through the restriction $C_\mu(\rho;A_\mu=0,B_\mu)$, in complete analogy to the recovery of the plain σ model through $C_\mu(\kappa;A_\mu,B_\mu=0)$. The singularities in parameter space ($\kappa=\pm1$) may also be moved to infinity by defining

$$\sinh\theta=\frac{2\kappa}{1-\kappa^2}\,,\quad \cosh\theta=\frac{1+\kappa^2}{1-\kappa^2}\,, \quad (3.17)$$

so that

$$C_\mu(\theta)=-\tfrac{1}{2}[(\cosh\theta-1)A_\mu+\sinh\theta\epsilon_{\mu\nu}A^\nu+(\cosh2\theta-1)B_\mu+\sinh2\theta\epsilon_{\mu\nu}B^\nu]\,. \quad (3.18)$$

With this parametrization, the recovery of the Gross-Neveu model is more transparent ($2\theta\to\theta$).

IV. ANOTHER EXAMPLE: THE CP($N-1$) MODEL

In this section we discuss a U(N)-invariant supersymmetric nonlinear model, the CP($N-1$) theory.[15] We emphasize those system properties which are relevant to the construction of nonlocal currents. Pleasingly enough, the structure of this theory is in one-to-one correspondence with the O(N) model of Sec. II. The algorithm in Sec. III may thus be applied without modification to obtain the nonlocal SU(N) currents.

The supersymmetric CP($N-1$) theory is a complex extension of the O(N) SSM, defined by the action

$$S=\int d^2x\{(\mathfrak{D}_\mu z^a)^*\mathfrak{D}_\mu z^a+i\bar\psi^a\not{\mathfrak{D}}\psi^a+\tfrac{1}{4}[(\bar\psi^a\psi^a)^2-(\bar\psi^a\gamma_5\psi^a)^2-(\bar\psi^a\gamma_\mu\psi^a)(\bar\psi^b\gamma^\mu\psi^b)]\}\,, \quad (4.1)$$

with the constraints

$$z^{a*}z^a=1\,,\quad z^{a*}\psi^a=0=\bar\psi^a z^a\,. \quad (4.2)$$

Here z^a is a complex scalar field transforming as the fundamental representation ($a=1,\ldots,N$) of SU(N). z^* is the complex conjugate of z. The field ψ^a is a Dirac (complex) spinor also transforming as the fundamental representation of SU(N). The covariant derivative appearing in (4.1) involves an auxiliary vector field bilinear in z:

$$\mathfrak{D}_\mu = \partial_\mu - z^{a*}\partial_\mu z^a . \tag{4.3}$$

This model is locally invariant under phase changes of z and ψ, globally invariant under complex supersymmetry,[15] and also chirally symmetric. The field equations for z and ψ are

$$\Phi^a \equiv \mathfrak{D}^\mu(\mathfrak{D}_\mu z^a) + [(\mathfrak{D}^\mu z^b)^*\mathfrak{D}_\mu z^b]z^a + i(\bar\psi^b \not{\mathfrak{D}} z^b)\psi^a = 0 ,$$
$$\Psi^a \equiv i\not{\mathfrak{D}}\psi^a + \tfrac{1}{2}[(\bar\psi^b\psi^b) - (\bar\psi^b\gamma_5\psi^b)\gamma_5 - (\bar\psi^b\gamma_\mu\psi^b)\gamma^\mu]\psi^a + iz^a\gamma_\mu(\mathfrak{D}^\mu z^b)^*\psi^b = 0 . \tag{4.4}$$

These may be obtained using standard Lagrange multiplier techniques to maintain (4.2).

As done for the O(N) SSM, we split the conserved SU(N) Noether currents into boson and fermion contributions:

$$J_\mu^{(0)ab} = A_\mu^{ab} + 2B_\mu^{ab} ,$$
$$A_\mu^{ab} = 2[z^{a*}(\mathfrak{D}_\mu z^b) - (\mathfrak{D}_\mu z^a)^* z^b] , \tag{4.5}$$
$$B_\mu^{ab} = -i\bar\psi^a\gamma_\mu\psi^b .$$

These currents are all anti-Hermitian, $A_\mu^{ab*} = -A_\mu^{ba}$, etc., as compared to the real, antisymmetric currents of the O(N) model.

With the definitions in Eq. (4.5), one can use the field equations (4.4) and the constraints (4.2) to obtain exactly the same results (2.6) for the divergences and curls of A_μ and B_μ, with obvious substitutions of the above SU(N) expressions for the previous O(N) quantities. Thus the construction of a full sequence of nonlocal currents for the CP($N-1$) model follows directly from the analysis of Sec. III using the same definition for the curvature-free currents (3.1). This sequence of nonlocal currents has precisely the same form for either the O(N) or CP($N-1$) models, when written in terms of the appropriate A_μ's and B_μ's.

We conjecture that this correspondence between the O(N) and CP($N-1$) models can be generalized to include other models, and that Eqs. (2.6) and (3.1) are valid for essentially all two-dimensional supersymmetric nonlinear theories.[27]

V. DISCUSSION

We now discuss some properties and implications of the nonlocal currents and their charges. First, consider the effects of supersymmetry transformations (2.4) on the sequences of potentials, currents, and charges for the O(N) SSM. The on-shell transform of Eq. (3.3) is

$$D_\mu \delta\chi = -\delta C_\mu \chi . \tag{5.1}$$

This equation is easily solved for $\delta\chi$ upon observing that

$$\delta C_\mu = \frac{-\kappa}{(1-\kappa^2)}\bar\epsilon(1-\kappa\gamma_5)(\partial_\mu E + [C_\mu , E]) , \tag{5.2}$$

where $E^{ab} = -2\gamma_5(n^a\psi^b - \psi^a n^b)$ and $\delta\chi^{(0)} = \bar\epsilon E$. We immediately obtain the solution

$$\delta\chi = \frac{\kappa}{(1-\kappa^2)}\bar\epsilon(1-\kappa\gamma_5)E\chi . \tag{5.3}$$

We may use this result to obtain the supersymmetry transformations of the nonlocal currents through (3.11). This then gives the transform of the charge-generating functional upon integration:

$$\delta G = \int_{-\infty}^{\infty} dx\, \partial_1(\delta\chi) = \delta\chi(\kappa; +\infty, t) - \delta\chi(\kappa; -\infty, t) . \tag{5.4}$$

Consequently, if ψ and hence E vanish at spatial infinity, we have

$$\delta G = 0 , \tag{5.5}$$

so all nonlocal charges commute with supersymmetry.

We also observe that the potentials $\chi^{(n)}$ and their currents are components of a system closed under supersymmetry (i.e., a superfield). This follows from (5.2) and (5.3) upon noting that δE is linear in $J_\mu^{(0)}$ and $\bar\psi^a\gamma_5\psi^b$, while the transform of the latter is linear in E.

Next we note the consistency of the invariance (5.5) with the nonlocal charge algebra. The algebra may be obtained using the Dirac brackets given in Ref. 22. The Dirac bracket of $Q^{(n)}$ with $Q^{(m)}$ yields $Q^{(n+m)}$ plus a polynomial of lower-degree charges, generalizing previously noted results for the plain σ model.[3] Thus the algebra bears an intriguing resemblance to the internal symmetry sector of generalized Virasoro algebras.[28] [Similarly, the generating functional of Eq. (3.15) is reminiscent of vertex functions in dual theory, but a precise connection has not been established by us.]

An interesting question for the supersymmetric nonlinear models is whether the nonlocal currents and their charges survive quantization, as do their counterparts in the plain σ models.[29] We have not completed an investigation of this question. Nevertheless, we expect that the charges do survive, since only rather general assumptions about short-distance expansions and the renormalization properties of the theory are needed to show this. In this regard, we note that the supersymmetric models are asymptotically free, like the plain σ model, and can be renormalized in a way compatible with the supersymmetry.[13,15]

Given their existence in the quantized theories, the nonlocal charges would prohibit particle production and would provide the "factorization" properties[29] needed to construct the SSM's S matrix (e.g., as proposed in Ref. 18).

We believe it is also interesting to ask whether, because of the nonlocal currents or equivalently the zero-curvature derivatives in (3.2), the SSM Hamiltonian might be exactly diagonalizable using quantum inverse scattering techniques.[30] In particular, we conjecture that a supersymmetric version of the Bethe ansatz[31] exists and that it will play a role in this diagonalization. This subject is under further study.

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APPENDIX

We record here our conventions for two-dimensional theories and give some useful Lorentz index and Dirac matrix identities.

The timelike metric is $g^{00}=-g^{11}=1$, and the antisymmetric symbol is $\epsilon^{01}=\epsilon_{10}=-\epsilon^{10}=1$. From these we obtain the bilinear identities

$$\epsilon^{\kappa\lambda}\epsilon^{\mu\nu}=g^{\kappa\nu}g^{\lambda\mu}-g^{\kappa\mu}g^{\lambda\nu}\,, \tag{A1}$$

$$g^{\kappa\lambda}\epsilon^{\mu\nu}+g^{\kappa\mu}\epsilon^{\nu\lambda}+g^{\kappa\nu}\epsilon^{\lambda\mu}=0\,, \tag{A2}$$

which are easily checked by choosing specific components.

Dirac matrices are 2×2 in two dimensions and satisfy the relations

$$\gamma^\mu\gamma^\nu=g^{\mu\nu}1+\epsilon^{\mu\nu}\gamma_5\,, \tag{A3}$$

$$\gamma^\mu=\gamma_5\epsilon^{\mu\nu}\gamma_\nu\,, \tag{A4}$$

where the pseudoscalar matrix is

$$\gamma_5=\gamma^0\gamma^1,\quad (\gamma_5)^2=1\,. \tag{A5}$$

The two-dimensional charge conjugation matrix C has the properties

$$C\gamma_\mu C^{-1}=-\gamma_\mu^{\text{transpose}},\quad C=-C^{\text{transpose}}. \tag{A6}$$

An explict realization of these matrices is

$$\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad C=\gamma^0=C^{-1}. \tag{A7}$$

By definition, Majorana spinors satisfy the constraint

$$\bar\psi\equiv\psi^\dagger\gamma^0=\psi^{\text{transpose}}C\,. \tag{A8}$$

Thus, if ψ and χ are both Majorana, the three types of spinor bilinears have the following properties:

$$\bar\psi\chi=+\bar\chi\psi,\quad \bar\psi\gamma_\mu\chi=-\bar\chi\gamma_\mu\psi,\quad \bar\psi\gamma_5\chi=-\bar\chi\gamma_5\psi. \tag{A9}$$

Spinor trilinears may be rearranged using the usual completeness properties of $\{1,\gamma_\mu,\gamma_5\}$. We have

$$\begin{pmatrix}\psi(\bar\chi\phi)\\ \gamma_\mu\psi(\bar\chi\gamma^\mu\phi)\\ \gamma_5\psi(\bar\chi\gamma_5\phi)\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}-1 & -1 & -1\\ -2 & 0 & 2\\ -1 & 1 & -1\end{pmatrix}\times\begin{pmatrix}\phi(\bar\chi\psi)\\ \gamma_\mu\phi(\bar\chi\gamma^\mu\psi)\\ \gamma_5\phi(\bar\chi\gamma_5\psi)\end{pmatrix}. \tag{A10}$$

These rearrangement relations immediately give the identity

$$\psi^a(\bar\psi^b\psi^b)-\gamma_5\psi^a(\bar\psi^b\gamma_5\psi^b)=-\gamma_\mu\psi^b(\bar\psi^b\gamma^\mu\psi^a)\,, \tag{A11}$$

where a sum over $b=1,\ldots,N$ is understood. The pseudoscalar term on the LHS of this equation vanishes by (A9) if the ψ^a are Majorana spinors. The identify in (A11) is useful in computing $\epsilon^{\mu\nu}\partial_\mu B_\nu$ for the O(N) and CP($N-1$) models, as discussed in Secs. II and IV of the text.

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[26]It is amusing to note that "trivial" nonlocal currents may always be constructed in two-dimensional theories. By "trivial," we mean the corresponding charges are "disconnected" products of previously known charges and give no additional dynamical information. For example, $J_\mu^\theta(x,t)=\theta_{\mu\nu}(x,t)\int_{-\infty}^{x}dy\,\theta_0^\nu(y,t)$ is conserved when $\theta_{\mu\nu}$ is the conserved energy-momentum tensor. The charge is $Q^\theta=\frac{1}{2}P_\mu P^\mu$. Similar trivial conserved bilinears may be constructed from supercurrents , or noncommuting vector currents. Higher-degree disconnected charges may also be constructed using nontrivial nonlocal currents.

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