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Nonlocal conserved currents for two-dimensional chiral theories

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We derive an infinite set of conserved nonlocal currents for two-dimensional chiral models with fields assuming values in an arbitrary Lie group G. Explicit formulas are presented for the Minkowski metric, but with slight changes described in the text the analogous results are valid for the Euclidean case as well. The Poisson-bracket algebra of conserved charges is further analyzed for G = SU(2). We expose the geometric structure of chiral models, and outline the relation between models described in this paper and chiral theories corresponding to lower-dimensional homogeneous spaces.

I. INTRODUCTION

The rich geometric structure of nonlinear chiral theories is mathematically well understood. In two spacetime dimensions, at least, it leads in a natural way to the linear equations associated with equations of motion, thus leading to the conjecture that all two-dimensional chiral models are solvable by means of the inverse-scattering method.[1] By chiral theories we mean here model field theories with fields taking values in a Riemannian homogeneous space of a Lie group G, with dynamics determined by the Lagrangian which is proportional to the G-invariant Riemannian metric. The prototype for all such models is the principal chiral field assuming values in the Lie group itself[2] with the Lagrangian given by the bi-invariant metric on the group: Any G-homogeneous space can be realized as a quotient G/H, where H is a subgroup of G, and the relevant mathematical structures on G can be mapped into G/H.

The basic objects in chiral theories are currents corresponding to the group G. Geometrically, they describe the integrable G connection on the field manifold G/H treated as an associated fiber bundle with spacetime as a base space. More importantly they give rise to a one-parameter family of integrable connections, and the associated linear problem is then interpreted as equations for a corresponding one-parameter family of parallel-transport operators. The latter belong to (a representation of) the Lie group G. Therefore, each solution of the chiral theory, say $\phi(x)$, determines a one-parameter family of G-valued functions $g^{(\lambda)}(x)$, and it turns out that by means of the group action on the field space we obtain a one-parameter family of solutions to the field equations, $\phi^{(\lambda)}(x)=g^{(\lambda)}(x)\,\phi(x)$. This has been shown for a particular case of n fields [corresponding to the homogeneous space $S^{N-1}=O(N)/O(N-1)$] by Pohlmeyer[3]; below we prove this for principal chiral fields, but for the reasons explained above we believe that this is a general statement.

The distinguishing feature of theories solvable by the inverse-scattering method is the existence of an infinite number of nontrivial conservation laws. In this paper we relate them to the one-parameter family of connections described above, and we derive their explicit form (in the sense of the recursion relation) for principal chiral theories. The construction proceeds as follows.

Given a one-parameter family of solutions, we can form the parametric family of currents out of the known currents corresponding to the Noether-type symmetries of the theory. In $d=2$ chiral theories they are the already discussed internal-symmetry currents and the conformal currents, expressible in terms of the energy-momentum tensor $\Theta_{\mu\nu}$. We can then expand the currents in (formal) power series in the parameter, thus producing new conservation laws. In this paper we show that power-series expansion of the internal-symmetry currents does indeed give rise to an infinite set of nonlocal conserved currents, while the parameter dependence of the energy-momentum tensor is trivial.

The plan of the paper is as follows. In Sec. II we describe the model and we derive the recursion relation for the coefficients of the formal power-series expansion of the solution matirx of the associated linear problem. The calculations are done in Minkowski spacetime but at the end of the section we give the Euclidean version of the linear problem leading directly to the Euclidean recursion relations. This is used in Sec. III to construct the infinite set of nonlocal conserved currents. Next we set up the canonical formalism to analyze the algebra of conserved charges.

Although we have stressed the differential geometric nature of the chiral theories in the introduction above, because it provides a basis for generalizations, in the body of the paper we specialize to principal chiral fields. The method also can be applied to other homogeneous spaces described as submanifolds of G by imposing additional constraints on the fields (i.e., using the reduction procedure of Ref. 1).

II. PRINCIPAL CHIRAL FIELDS AND ASSOCIATED LINEAR EQUATIONS

We shall deal with fields $g(x^0, x^1)$ defined on the two-dimensional spacetime, and assuming values in a real Lie group G realized by matrices in some fixed representation. Since we wish to discuss the Euclidean and Minkowski cases simultaneously whenever possible, we use the summation convention throughout. In physical applications a restriction on the choice of G or a representation thereof is usually made to guarantee the reality and the existence of a lower bound for Euclidean action or energy, respectively. Our procedure, however, is general, and deliberately we do not choose any particular example before discussing the canonical formalism, when independent variables must be specified. Our Minkowski metric is $g_{00}=-g_{11}=1$, and $\epsilon_{01}=\epsilon^{01}=1$. The dynamics is specified by the action integral

$$S=\mathrm{tr}\int d^2x\,\partial_\mu g\partial^\mu g^{-1}. \qquad (2.1)$$

Correspondingly, the field equations written in the matrix form read

$$\partial^2 g+\partial_\mu g\partial^\mu g^{-1}g=0 \qquad (2.2)$$

together with the accompanying equation for g^{-1}.

The model (2.1) has a global $G\times G$ invariance, corresponding to left and right translations of g:

$$g\to hg, \qquad (2.3a)$$

$$g\to gh \qquad (2.3b)$$

with $h\in G$. Correspondingly, there exist conserved currents—here written in a matrix form as elements of the Lie algebra of G

$$A_\mu=\partial_\mu g g^{-1}, \qquad (2.4a)$$

$$\tilde{A}_\mu=g^{-1}\partial_\mu g. \qquad (2.4b)$$

The appealing feature of chiral theories—from the point of view of geometric interpretation and physical applications as well—is that they can be entirely reformulated in terms of currents (2.4). The system of equations satisfied by the currents

$$\begin{aligned}&\partial_\mu A^\mu=0,\\ &\partial_\mu A_\gamma-\partial_\gamma A_\mu-[A_\mu, A_\gamma]=0\end{aligned} \qquad (2.5)$$

or the corresponding system for $\tilde{A}_\mu$ has the same content as Eq. (2.2).

In recent papers[1,2] Zakharov and Mikhailov have found that Eqs. (2.5) can be treated as integrability conditions of a one-parameter family of overdetermined systems of linear partial differential equations. They restrict the discussion to the Minkowskian case, using the light-cone coordinates; we will treat both Minkowskian and Euclidean cases, using the Cartesian coordinates. We will treat the Minkowskian case in detail, and the corresponding Euclidean formulas will be presented at the end of this section. We find that the associated linear problem can be written as (in matrix notation)

$$\partial_\mu\Psi=(1-\lambda^2)^{-1}(A_\mu-\lambda\epsilon_{\mu\nu}A^\nu)\Psi. \qquad (2.6)$$

For real values of λ the matrix Ψ is an element of G. Following the reasoning outlined in the Introduction, we would like to use it to construct a one-parameter family of solutions with the group action defined by (2.3). It turns out, however, that one cannot generate new solutions with the help of $\Psi(\lambda)$ only, and one should take into account also another linear problem, corresponding to the right group action:

$$\partial_\mu\tilde{\Psi}=-(1-\lambda^2)^{-1}(\tilde{A}_\mu-\lambda\epsilon_{\mu\nu}\tilde{A}^\nu)\tilde{\Psi}. \qquad (2.7)$$

Recalling now that the full symmetry group of the model (2.2) is a direct product $G\times G$, one finds after a straightforward calculation that for any solution $g(x)$, the quantity

$$g^{(\lambda)}=\Psi(\lambda)^{-1}g\tilde{\Psi}(\lambda) \qquad (2.8)$$

satisfies field equation (2.2) for any real λ.

Now the parametrized solutions $g^{(\lambda)}$ can be substituted into the expression for the currents (2.4). We get then a parametric family of conserved currents

$$\begin{aligned}A_\mu^{(\lambda)}&=\partial_\mu\Psi^{-1}\Psi+\Psi^{-1}A_\mu\Psi+\Psi^{-1}g\partial_\mu\tilde{\Psi}\tilde{\Psi}^{-1}g^{-1}\Psi\\ &=-(1-\lambda^2)^{-1}\Psi^{-1}(\lambda^2A_\mu-2\lambda\epsilon_{\mu\nu}A^\nu+A_\mu)\Psi,\end{aligned} \qquad (2.9)$$

which will be treated as a generating function for the infinite number of conservation laws associated with the linear problem (2.6). All we need to know is the λ dependence of Ψ, and now we shall proceed to determine it in the sense of a formal expansion of Ψ in a power series in λ at $\lambda=0$. We set

$$\Psi(\lambda)=\sum_{n=0}^{\infty}\lambda^n\Psi_n. \qquad (2.10)$$

Substitution of (2.10) into (2.6) yields the differential recursion relation, immediately signaling that the coefficient matrices Ψ_n will be represented nonlocally in terms of $g(x)$.

Explicitly, we get

$$\begin{aligned}&(\partial_\mu-A_\mu)\Psi_0=0,\\ &(\partial_\mu-A_\mu)\Psi_1=-\epsilon_{\mu\nu}A^\nu\Psi_0,\\ &(\partial_\mu-A_\mu)\Psi_n=\partial_\mu\Psi_{n-2}-\epsilon_{\mu\nu}A^\nu\Psi_{n-1},\ n\geq 2.\end{aligned} \qquad (2.11)$$

These expressions can be simplified if one introduces matrices χ_n by

$$\Psi_n = g\chi_n, \tag{2.12}$$

for which relations (2.10) transform into

$$\begin{aligned}
&\partial_\mu \chi_0 = 0,\\
&\partial_\mu \chi_1 = -\epsilon_{\mu\nu}\tilde{A}^\nu \chi_0,\\
&\partial_\mu \chi_n = (\partial_\mu + \tilde{A}_\mu)\chi_{n-2} - \epsilon_{\mu\nu}\tilde{A}^\nu \chi_{n-1}, \quad n \geq 2.
\end{aligned} \tag{2.13}$$

These equations are integrable, and the integrability conditions for the equation for χ_n are given by the field equations (2.5) and the preceding equations for χ_k, $k=n-1, n-2, \ldots, 0$. The solutions are given in terms of n-fold line integrals, and are path-independent. We can normalize the matrix Ψ to be set equal to unity at $x^1 = -\infty$ so that $\chi_0 = I$, and we evaluate the line integrals over straight lines parallel to the x^1 axis extending from $-\infty$ to the point $x = (x^0, x^1)$. For $n = 1, 2, 3$ we have

$$\chi_1(x) = \int_{-\infty}^{x^1} ds\, \tilde{A}_0(x^0, s),$$

$$\chi_2(x) = \int_{-\infty}^{x^1} ds\, \tilde{A}_1(x^0,s) + \int_{-\infty}^{x^1} ds_1 ds_2 \Theta(s_1 - s_2)\tilde{A}_0(x^0, s_1)\tilde{A}_0(x^0, s_2),$$

i.e.,

$$\chi_2(x) = \int_{-\infty}^{x^1} ds\, \tilde{A}_1(x^0, s) + \frac{1}{2!}\int_{-\infty}^{x^1} ds_1 ds_2 P\left\{\tilde{A}_0(x^0, s_1)\tilde{A}_0(x^0, s_2)\right\} \tag{2.14}$$

and

$$\chi_3(x) = \int_{-\infty}^{x^1} ds\, \tilde{A}_0(x^0, s) + \int_{-\infty}^{x^1} ds_1 ds_2 P\left\{\tilde{A}_1(x^0, s_1)\tilde{A}_0(x^0, s_2)\right\} + \frac{1}{3!}\int_{-\infty}^{x^1} ds_1 ds_2 ds_3 P\left\{\tilde{A}_0(x^0, s_1)\tilde{A}_0(x^0, s_2)\tilde{A}_0(x^0, s_3)\right\}.$$

The P ordering here refers to the spatial variables.

We also need the coefficients of the power-series expansion of Ψ^{-1} [cf. Eq. (3.1) below]. Substituting

$$\Psi^{-1}(\lambda) = \sum_{n=0}^{\infty} \lambda^n \phi_n \tag{2.15}$$

into the differential equation satisfied by Ψ^{-1} we obtain the recursion formula for coefficient matrices ϕ_n. Further simplification is obtained by substitution $\phi_n = \Theta_n g^{-1}$, and finally we get

$$\partial_\mu \Theta_n = \partial_\mu \Theta_{n-2} - \Theta_{n-2}\tilde{A}_\mu + \Theta_{n-1}\epsilon_{\mu\nu}\tilde{A}^\nu. \tag{2.16}$$

These relations can be integrated similarly as above.

If the field matrix g is in a unitary representation, there is a simple relation between the coefficients of expansions (2.10) and (2.15): $\Theta_n = \chi_n^\dagger$, and one need not solve the recursion relations (2.16) separately.

Now let us turn to the Euclidean case. To account for the sign changes in the field equations, we have to replace the linear equations (2.6) by

$$\partial_\mu \Psi = (1+\lambda^2)^{-1}(A_\mu + \lambda\epsilon_{\mu\nu}A_\nu)\Psi,$$

i.e., (2.17)

$$\begin{aligned}
&\partial_0 \Psi = (1+\lambda^2)^{-1}(A_0 + \lambda A_1)\Psi,\\
&\partial_1 \Psi = (1+\lambda^2)^{-1}(A_1 - \lambda A_0)\Psi.
\end{aligned}$$

With analogous sign changes in (2.7), one can follow the same procedure to derive Euclidean counterparts of formulas (2.9)—(2.14) without much difficulty.

III. CONSERVATION LAWS

With the expansions (2.10) and (2.15) we can proceed to the corresponding expansion of the conserved current (2.9) in powers of λ. The factor $(1-\lambda^2)^{-1}$ plays no role in the continuity equation, hence we omit it, and the generating function for the infinite set of conservation laws is rewritten as

$$\mathcal{J}_\mu^{(\lambda)} = \Psi(\lambda)^{-1}(A_\mu - 2\lambda\epsilon_{\mu\nu}A^\nu + \lambda^2 A_\mu)\Psi(\lambda). \tag{3.1}$$

Now we make the formal power-series expansion

$$\mathcal{J}_\mu^{(\lambda)} = \sum \lambda^n \mathcal{J}_\mu^{(n)}, \tag{3.2}$$

and inserting into (3.1) the expansion of $\psi(\lambda)$ found in Sec. II and collecting the terms at the same power of λ we obtain

$$\begin{aligned}
\mathcal{J}_\mu^{(0)} &= \tilde{A}_\mu,\\
\mathcal{J}_\mu^{(1)} &= \tilde{A}_\mu\chi_1 + \Theta_1\tilde{A}_\mu - 2\epsilon_{\mu\nu}\tilde{A}^\nu,\\
\mathcal{J}_\mu^{(n)} &= \sum_{k=0}^{n}\Theta_k\tilde{A}_\mu\chi_{n-k} + \sum_{k=0}^{n-2}\Theta_k\tilde{A}_\mu\chi_{n-2-k} - 2\sum_{k=0}^{n-1}\Theta_k\epsilon_{\mu\nu}\tilde{A}^\nu\chi_{n-1-k}.
\end{aligned} \tag{3.3}$$

If the field g is in a unitary representation the explicit form of the first two currents is

$$\begin{aligned}
\mathcal{J}_\mu^{(1)} &= [\tilde{A}_\mu, \chi_1] - 2\epsilon_{\mu\nu}\tilde{A}^\nu,\\
\mathcal{J}_\mu^{(2)} &= \tilde{A}_\mu - 2\epsilon_{\mu\nu}[\tilde{A}^\nu, \chi_1] + \tilde{A}_\mu\chi_2 + \chi_2^\dagger\tilde{A}_\mu + \chi_1^\dagger\tilde{A}_\mu\chi_1.
\end{aligned} \tag{3.4}$$

The first nontrivial conservation law is a continuum limit of the corresponding equation for a lattice chiral theory which was stated in Ref. 4.

Another set of conserved Noether currents in this theory is constructed with the energy-momentum tensor

$$\Theta_{\mu\nu} = \mathrm{tr} A_\mu A_\nu - \tfrac{1}{2} g_{\mu\nu} \mathrm{tr} A_\alpha A^\alpha . \tag{3.5}$$

However, the method described above does not produce any new conservation laws here because of the trace: Upon insertion of (2.9) into (3.5) the term $(1-\lambda^2)^{-2}$ factorizes, and the remainder is a fourth-order polynomial in λ with coefficients proportional to the energy-momentum tensor itself.

In order to analyze the Poisson-bracket (Pb) algebra of the charges corresponding to conserved currents (3.3), one must first solve the constraint $gg^\dagger = I$ by introducing group parameters as independent variables. For instance, taking $G = \mathrm{SU}(2)$ and using Euler's angles as independent variables, the unconstrained form of Lagrangian (2.1) is

$$\mathcal{L} = \tfrac{1}{2}(\alpha_\mu \alpha^\mu + \beta_\mu \beta^\mu + \gamma_\mu \gamma^\mu) + \alpha_\mu \beta^\mu \cos 2\gamma. \tag{3.6}$$

Then one can verify that Poisson brackets of time and space components of A_μ and $\tilde{A}_\mu$ give the usual $\mathrm{SU}(N) \times \mathrm{SU}(N)$ chiral current algebra. This is sufficient to calculate the Pb's of the nonlocal currents (3.3). The calculations are tedious, and care must be taken of the boundary terms. Using the reasoning similar to that of Lüscher and Pohlmeyer[3] we find that

(i) the multiplet of charges $Q_a^{(n)}$ corresponding to the matrix current $\mathcal{J}^{(n)}$ of Eq. (3.3) transforms under the adjoint representation of the global symmetry group, and

(ii) $\{Q_a^{(n)}, Q_b^{(m)}\}$ is expressed by $Q_c^{(l)}$, $l \leq n+m+1$.

IV. FINAL REMARKS

We have shown that the infinite set of conserved nonlocal currents in two-dimensional chiral models is essentially generated by a one-parameter transformation group (2.8) of the solutions of field equations. This transformation is in turn determined by the parametric family of associated linear problems. The conserved charges $Q_a^{(n)}$ can be deduced directly from the linear equations (Lüscher and Pohlmeyer[3]); however, the currents are more fundamental objects, and only currents have meaning in the Euclidean formulation.

Now let us turn to chiral theories defined on homogeneous spaces different from the group G itself. They can be described as submanifolds of G by imposing additional constraints,[1] and correspondence with the usual description is achieved by using explicit parametrization of this submanifold. However, if we keep the matrix notation, then the whole procedure of finding the conservation laws is virtually unchanged. For instance, the theory of unit vector field $\vec{q}(x) = (q_1, \ldots, q_N)$(Ref. 3) corresponding to the homogeneous space $\mathrm{O}(N)/\mathrm{O}(N-1) = S^{N-1}$ can be cast in the form (2.1) by taking $g \in \mathrm{O}(N)$ and imposing the additional constraint $g^2 = I$. Then g can be represented as $g = I - 2P$, with P a projection operator.[1] Requiring the projection to be one-dimensional, g can be parametrized as $g_{ik} = \delta_{ik} - 2q_i q_k$. It is easy to see then that the matrix currents (2.4b) reduce to the corresponding formula for the $\vec{q}$ field, giving the $\mathrm{O}(N)$ currents $J_\mu^{ik} = 2q^i \overleftrightarrow{\partial}_\mu q^k$. It follows that in this case the nonlocal charges obtained from the currents of form (3.3) are directly related to those obtained straight from the linear problem.[3]

In this paper we have concentrated on particular nonlocal differential conservation laws simply related to the associated linear problems. This choice is not unique, and one can find different systems of conservation laws still before expressing the theory in terms of action-angle variables, which may be difficult in constrained systems The other method of construction of conservation laws is based on the Bäcklund transformation, and has been applied to many systems [see, e.g.,[5] the nonlinear $\mathrm{O}(N)$ σ model has been treated in this way in Ref. 3]. The Bäcklund transformation is rather trivial from the point of view of the linear equations of the inverse-scattering method and corresponds to a simple change of scattering data.[6] The Bäcklund transformations leading to soliton solutions are characterized by the addition of one eigenvalue in the inverse-scattering data.

The relevant Bäcklund transformation for the principal chiral field theory (2.2) can be easily extracted from the results of Zakharov and Mikhailov.[1,2] These authors reformulate the problem of finding the soliton solutions for systems (2.2) from the inverse-scattering equations as a particular version of the Riemann-Hilbert problem for matrices on the complex λ plane, when the spacetime variables are treated as deformation parameters. The addition of one eigenvalue in the inverse-scattering method is replaced here by changing the number of zeros of the Riemann-Hilbert problem by one. The two different solutions of Eqs. (2.2) are then related by

$$g' = \left(I + \frac{\bar{\lambda} - \lambda}{\lambda} P \right) g, \tag{4.1}$$

where P is a Hermitian projection operator satisfying an integrable system of first-order differential equations, which allows us to express P in terms of the solutions of the linear equations (2.6) with coefficients corresponding to the "old" solution g. Equation (3.1) is nothing but an integrated Bäcklund transformation and its differentiation gives a system of first-order equations. Then it can be analyzed in a way almost identical to that used by Pohlmeyer.[3] It suffices to note that our matrix $\Psi^{(\lambda)}$ corresponds to $R^{(\gamma)}$ of Ref. 3, and (2.6) can be brought to this form using the above-described relation between currents and the substitution $\lambda = (1+\gamma)/(1-\gamma)$.

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