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Functional integration through inverse scattering variables

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We develop here inverse scattering techniques to compute functional integrals. The usual functional integration variables are taken as "potentials" of an auxiliary linear problem. This linear differential problem defines the new integration variables through its scattering data and bound states. The functional integral becomes an infinite sum over the number of bound states and a continuous integral over all scattering variables. The integration bounds on these new integration variables are found. The functional measure in the new variables (scattering data and bound states) is found semiclassically in general and more complete results are given for specific systems. We apply this approach to the N-component anharmonic oscillator. By computing the imaginary-time path integral, an analytic expression is found for the ground-state energy. This expression accurately exhibits some of the known properties of the ground state. In particular, the three-sheeted structure in the coupling-constant plane (g) and the large-order behavior in the N^{-1} expansion are well reproduced by our formula. We compute from it the discontinuity across the cut on the negative g axis, finding the exact leading behavior for $g\to 0^-$ and arbitrary N. We also obtain the power series in g of corrections to this leading behavior for large N.

I. INTRODUCTION

Since the discovery by Gardner, Greene, Kruskal, and Miura[1] of the inverse scattering method, important work has been done in two-dimensional classical fields in connection with it.[2] Within this approach Faddeev, Zajarov, and others have found angle-action variables for several field models.[3-6] A natural way to extend this method to quantum field theory appears to take the canonical variables that completely solve the classical theory as integration variables in the functional-integral approach. In this way, one can hope to compute functional integrals other than Gaussian integrals. However, little has been done in this direction.

Inverse scattering techniques have been used to find stationary points of functional integrals in several problems.[7,8] Recently Bergknoff and Thacker,[9,10] Faddeev, Sklianin, and Tajtadyan,[11] and Honerkamp *et al.*[12] in a series of beautiful papers find eigenvalues and eigenstates of the sine-Gordon–massive-Thirring model and of the nonlinear Schrödinger equation by diagonalizing a complete set of operators. They also compute the S matrix. In these works, inverse scattering techniques are used whether implicitly through the Bethe ansatz[9] or explicitly[10-12] in an operational approach.

In this paper we deal with the problem of computing functional integrals by integrating over suitable scattering variables (SV). Two main difficulties arise in this problem: (i) What are the integration bounds for the SV? (ii) What is the functional measure in SV?

In this paper, problem (i) is essentially solved. In what concerns (ii), we have general expressions for the measure in the semiclassical regime and we have more complete results for specific cases (the N-component anharmonic oscillator and the quantum pendulum).

We consider a functional integral like

$$\int Dv(\,)e^{-S[v(.)]/\hbar}. \tag{1.1}$$

New variables (SV) are defined in terms of the scattering data and eigenvalues of an auxiliary linear problem where v plays the role of a potential. Of course, these new variables are useful only if $S[v]$ completely separates when expressed in terms of them.

Moreover, Gelfand-Levitan-type equations[2] tell us what are the independent magnitudes that we should take as SV and over what we should then integrate. Typically, the SV consist of the eigenvalues of the auxiliary linear problem plus the normalization constants of their respective wave functions and the modulus and phase of the reflection coefficient as a function of the wave number.

It is, in principle, hard to compute the functional Jacobian associated with the change from $v(.)$ to SV. However, in several cases we can do better by noting that the transition from the original field variables to SV can be recast as a canonical transformation.[3-6] Hence, the Jacobian of this transformation is rigorously equal to one at the classical level.

In this paper we shall consider quantum-mechanical problems, i.e., $v(x)$ in (1.1) depends only on one continuous variable. It is also possible to use SV in two-dimensional field theories. In this case, the SV will have an additional dependence on time.

The integration bounds in SV turn out to be as follows (Sec. II). The path integral becomes a discrete sum over the number N_B of bound states of the auxiliary linear system ($0 \leq N_B < \infty$) because this number is arbitrary for a generic shape $v(x)$. [The number of eigenvalues N_B plays the role of a "topological index" labeling the field configurations $v(x)$.] For each N_B one must integrate over all values of the SV compatible with the boundary conditions on $v(x)$. For example, each bound-state eigenvalue varies from zero up to $+\infty$ if $v(\pm\infty)=0$. The phase of the reflection coefficient goes in this case from 0 to 2π and its modulus from 0 to 1, and so on. The normalization constants of wave functions can vary in principle from zero to infinity. A closer analysis shows that this infinite variation is proportional to the infinite length of the x axis, the proportionality coefficient being calculable [Eq. (2.13)]. In this way, "volume factors" appear in the integration over SV.

As is well known, functional integrals can be defined as the continuous limit of a multiple integral on a lattice.[13] That is, an integral associated with each point of a discrete lattice that replaces the space-time or the Euclidean-space continuum. These integrations should be performed *before* letting the lattice spacing approach zero. So, rigorously speaking, one should introduce SV for a discretized system, only taking the continuum limit after integration. In other words, one is not allowed to use the value one of the Jacobian. Classically, the Jacobian is equal to one, but quantum mechanically one can consider that it fluctuates around its classical value. In the semiclassical regime ($\hbar \to 0$), the functional integral is dominated by the factor $e^{-S/\hbar}$; the Jacobian, being of order $\hbar^0$, we can approximate by some constant. In this approximation, the evaluation of the functional integral reduces to a finite number of simple integrals. In this paper we do slightly better by assuming that the Jacobian is nonconstant, but factorizable when expressed in SV. Therefore, we can take into account some quantum effects. A number of nontrivial checks show that the formulas obtained in this way are correct with order estimates for their validity.

We deal in this paper with the N-component anharmonic oscillator. Its action, for imaginary time x, reads

$$S[\vec{\phi}] = \int dx\left[\frac{1}{2}\left(\frac{d\vec{\phi}}{dx}\right)^2 + \tfrac{1}{2}\vec{\phi}^2 + \frac{g}{N}(\vec{\phi}^2)^2\right], \qquad (1.2)$$

where $\vec{\phi} = (\phi_1, \ldots, \phi_N)$.

By introducing a Lagrange multiplier $\alpha(x)$ conjugate to $\vec{\phi}^2$, the generating functional for this system can be rewritten as[14]

$$\iint D\alpha(\;)\exp\left\{-\frac{N}{2}\log\det\left[-\frac{d^2}{dx^2}+1+4i\left(\frac{g}{N}\right)^{1/2}\alpha(\;)\right] - \int dx\,\alpha(x^2)\right\}. \qquad (1.3)$$

We recognize in it the Fredholm determinant of the linear Schrödinger operator. This already suggested the application of SV to the problem of large orders in the N^{-1} expansion.[8] Here we use integration over these SV to compute the ground-state energy of this O(N)-symmetric anharmonic oscillator. The final result reads (see Secs. II, III, and IV)

$$E_G(N,g) = \frac{N}{2}\left[[1+z(g)]^{1/2} - \frac{z^2(g)}{8g}\right] + C_N(g) - \frac{4N}{\pi g}\int_0^{\infty\times e^{i\pi/3}} \frac{S^5 dS}{\{S+[1+z(g)]^{1/2}\}^2} \times \exp\left[\frac{N}{g}\left(\frac{S^3}{3}-\frac{zS}{2}\right)\right] \times\left[\frac{(1+z)^{1/2}+S}{(1+z)^{1/2}-S}\right]^{N/2}\left[1+O\left(\frac{g}{N}\right)\right]. \qquad (1.4)$$

Here, $z(g)$ [Eq. (3.7)] is related to the constant stationary point of the path integral (1.3) through Eq. (3.5). The simple integral over S in Eq. (1.4) comes from the integration over the bound-state eigenvalues of the auxiliary Schrödinger equation. The factors of order N^0 and g^0 in the S integrand has been adjusted by using the large orders in N^{-1} of $E_G(N,g)$.[8,15] $C_N(g)$ stands for the contribution of the "scattering" part of SV (essentially, the reflection coefficient). It is of order N^0 for large N and it does not contribute, at least for leading behavior, to the large orders of the N^{-1} expansion.

Several results follow from Eq. (1.4). First, we find the correct analytic structure for $E_G(N,g)$ as a function of g. That is a three-sheeted Riemann surface with cuts from $g=0$ to $g=\infty$ (see Sec. IV and Figs. 1 and 2). Second, for $g\to\infty$ and fixed N, we get $E_G \sim g^{1/3}$ as it should be.

For large N and fixed g the dominant term (of order N^{+1}) is the correct one in Eq. (1.4). The next term (of order N^0) may come from $C_N(g)$ because the integral over S is of order N^{-1} for large N.

The better results that we have extracted from Eq. (1.4) concern the discontinuity of $E_G(N,g)$ across its cut in the g plane. We find for small negative $g \equiv -h$ and fixed N (see Sec. V)

$$\mathrm{Im}E_G(N, -h+i0) = \left(\frac{2N}{h}\right)^{N/2} \frac{e^{-N/3h}}{\Gamma(N/2)}\left[1+\sum_{K>1} T_K(N)\left(\frac{h}{N}\right)^K\right],$$

$$(27)^{-1/2} > h > 0\,. \quad (1.5)$$

The leading factor (in front of the brackets) is exact.[16,17] Moreover, we have computed the coefficients $T_K(N)$ for large N and arbitrary K. We obtain that the $T_K(N)$ are polynomials in N of degree $2K$ with rational coefficients. This is precisely what Zinn-Justin recently found by numerical computation.[18] Explicitly, we get

$$T_K(N) = \frac{(-1)^K}{K!}\left(\tfrac{1}{8}\right)^K N^{2K}\left[1 - \frac{41K-104}{49}\frac{2K}{N} + O\left(\frac{1}{N^2}\right)\right]. \quad (1.6)$$

This is in exact agreement with the available numerical values $(1 \le K \le 4)$.[18]

Finally, by assuming that the last factor on the S integrand in Eq. (1.4) is of the form

$$1 + \frac{g}{N}D(S) + O\left(\frac{g^2}{N^2}\right), \quad (1.7)$$

we find that the correct $T_1(N)$ (Refs. 18, 19) is reproduced if $D(1)=\frac{17}{6}$ [see Eqs. (5.15) and (5.16)]. By using this value we obtain as output the coefficients of N^2 and N^4 in $T_2(N)$ and $T_3(N)$, respectively, in agreement with their numerical values obtained in Ref. 18.

Inverse scattering techniques can also be applied to the quantum pendulum and to the anharmonic oscillator for $N=1$ and $N=2$. In the first case, the appropriate CSV are those associated with the sine-Gordon model.[4] In the second case, the CSV of the nonlinear Schrödinger equation[5] can be used to compute the functional integral without appealing to the α representation (1.3). We obtain a simple expression for the ground state of the simple pendulum (Mathieu's differential equation). It correctly reproduces the tunneling contributions (nonanalytic in $\hbar$) for $\hbar \to 0$, and also the large orders of perturbation theory.[20]

Future progress may arise from a more careful treatment of the space discretization. In particular, it will be very interesting to discretize in such a way that the complete separability of the Euclidean action continues to hold on the lattice. The present paper is a first step in this line. Work in this direction is in progress.

II. INTEGRATION MEASURE AND INTEGRATION BOUNDS FOR SCATTERING VARIABLES

Let us consider a path integral over a single functional variable $v(x)$ like

$$\iint \prod_x dv(x) e^{-S[v(.)]/\hbar}\,. \quad (2.1)$$

These kind of integrals appear in quantum mechanics, $v(x)$ being the particle position for an imaginary time x. Our aim is to solve it by using a new set of integration variables that completely separates $S[v(.)]$. In that case, the functional integral will essentially factorize into an infinite product of simple integrals.

Such types of variables can be found in some cases through an auxiliary linear problem where $v(x)$ plays the role of a potential. They are typically the scattering data and the bound states of this linear problem.

To begin with, we consider as auxiliary linear problem, the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + v(x)\right]\psi(x;k) = k^2\psi(x;k)\,. \quad (2.2)$$

The scattering variables in this case are[2,3]

$$\mathrm{SV} = \{r(k), k \in R^+; \kappa_l, C_l,\ l = 1, \ldots, N_B\ (N_B \ge 0)\}\,. \quad (2.3)$$

Here, $r(k)$ stands for the reflection coefficient, $-\kappa_l^2$ is the eigenvalue of the lth bound state, and C_l is the corresponding normalization coefficient of the wave function. N_B stands for the total number of bound states. That is, we define the lth bound-state wave function such that

$$\psi_l(x) \underset{x\to-\infty}{\sim} e^{\kappa_l x}\,, \quad (2.4)$$

Then,

$$C_l^{-1} = \int_{-\infty}^{+\infty} dx\, \psi_l(x)^2\,.$$

One can consider directly the SV as new integration variables because they are in one-to-one correspondence with $v(x)$ under some regularity hypothesis.[2,3] Then, the corresponding Jacobian should be computed. However, one can do better by noting that the change from $v(x)$ to SV given by Eq. (2.3) can be recast as a canonical transformation. This is shown in Ref. 3, where the canonically conjugated variables

$$v(x),\ \ \pi(x) = \int_{-\infty}^{x} v(y)dy \quad (2.5)$$

are considered together with the associated Poisson bracket

$$\{\alpha, \beta\} = \int dx\left[\frac{\delta\alpha}{\delta v(x)}\frac{d}{dx}\left(\frac{\delta\beta}{\delta v(x)}\right) - (\alpha \longleftrightarrow \beta)\right]. \quad (2.6)$$

We can trivially introduce the variable $\pi(x)$ by rewriting our functional integral (2.1) as

$$\int \prod_x \left[dv(x)\, d\pi(x)\, \delta\left(\pi(x) - \int_{-\infty}^{x} v(y)dy\right)\right] e^{-S(v)} = \int d\mu\, e^{-S(v)}\,. \quad (2.7)$$

Then, we change from the canonical pair (π, v) to the set defined by Faddeev and Zajarov[3]

$$p_j = \kappa_j^2, \quad q_j = 2\ln[ic_j\dot{F}(i\kappa_j)],$$
$$P(k) = -\frac{k}{\pi}\ln[1-|r(k)|^2], \quad Q(k) = \arg[r(k)F(k)]. \tag{2.8}$$

Here $F(k)$ stands for the Jost function of the Schrödinger equation (2.2) and

$$\dot{F}(i\kappa) \equiv \left.\frac{dF(k)}{dk}\right|_{k=i\kappa}.$$

These new variables are canonical for the Poisson bracket (2.6), i.e.,

$$\{Q(k), P(k')\} = \delta(k-k'), \quad \{q_j, p_l\} = \delta_{jl}, \tag{2.9}$$

all other Poisson brackets vanishing. Then, the Jacobian of the transformation

$$[\pi(x), v(x); x\in R] \to \mathrm{CSV} \equiv [P(k), p_j, Q(k), q_j, k\in R^+, 1\le j\le N_B, N_B \in \mathfrak{N}] \tag{2.10}$$

has unit value. Here CSV stands for "canonical scattering variables." The Jacobian is equal to one only for $v(x)$ sufficiently smooth which is not necessarily the case for the integration variable in a functional integral. Consequently, one should expect that quantum fluctuations will modify the classical unit value of the Jacobian associated to (2.10).

This problem can be overcome by introducing an ultraviolet regularization (e.g., by defining the system on a lattice). A finite volume regularization can also be necessary. Of course a discretization procedure is not unique. One can write many different discrete systems all of which have the same continuous limit. The best thing to do would be to define the discretization such that $\exp[-S(v)]$ continues to be completely factorizable when expressed in terms of the discrete variables analogous to the CSV. One cannot know *a priori* if such discretization exists. However, completely integrable systems defined over infinite lattices are known.

We wish to note that we are using CSV in a different way than they have been used in the literature.[3-7,10-12] For us, they replace $v(x)$ in the functional integral, where x is the imaginary time. Usually, scattering variables depending implicitly on the (real) time are associated with fields that are functions of space and time.

Let us now discuss the integration bounds on the CSV corresponding to a $v(x)$ varying between $-\infty$ and $+\infty$ for all finite x and subject to the boundary condition

$$v(\pm\infty) = 0. \tag{2.11}$$

In this case the eigenvalues κ_j^2 can take any positive value and the reflection coefficient $r(k)$ any complex value within the unit circle. Then, from Eq. (2.8) it follows that

$$0 \le p_j < +\infty, \quad j = 1, \ldots, N_B$$
$$0 \le P(k) < +\infty, \quad 0 \le Q(k) < 2\pi. \tag{2.12}$$

The number of bound states N_B being arbitrary, one must sum over N_B from zero to infinity. The variable q_j can take, in principle, any real value from $-\infty$ to $+\infty$. In fact, this infinity can be shown to be proportional to the infinite length of the imaginary-time axis as follows. Suppose one does a translation $x \to x+X$. This changes $v(x)$ into $v(x+X)$ and does not affect p_j, $P(k)$, and $Q(k)$. On the other hand, the coefficients C_j transform like

$$C_j \to C_j e^{-\kappa_j X}$$

because of the asymptotic behavior of the bound-state wave functions (2.4). Hence q_j changes as

$$q_j \to q_j - 2\sqrt{p_j}X.$$

If we put the system into a large box of length $2L$ we have

$$2\sqrt{p_j}L > q_j > -2\sqrt{p_j}L. \tag{2.13}$$

We shall now proceed to discretize our functional integral (2.1) as well as the SV variables without attempting to preserve the canonical nature of the mapping from (π, v) to the CSV on the lattice. This will introduce a nonunit Jacobian. Their effects will be taken into account *a posteriori* (see Sec. IV).

In the semiclassical regime $(\hbar\to 0)$, the functional integral is dominated by the factor $e^{-S/\hbar}$ while the Jacobian is of order $\hbar^0$. Then the integration procedure over CSV will work semiclassically as will be explicitly confirmed later.

We discretize the imaginary-time axis as a lattice of $2M$ points $(2M \gg 1)$ over a length $2L$. That is, $\Delta \equiv L/M$ is the lattice spacing. The scattering variables become

$$Q_\alpha \equiv (\pi/L)^{1/2}Q(k_\alpha), \quad P_\alpha \equiv (\pi/L)^{1/2}P(k_\alpha) \tag{2.14}$$

together with the (q_j, p_j). Here $k_\alpha = \pi\alpha/L$ and α is an integer between $-M$ and $+M$. Here the distinction between "continuum" and "bound-state" variables disappears. One should integrate over $2M - N_B$ pairs (Q_α, P_α) for each N_B, because the total number of independent variables is $2M$.

In conclusion, we write our integration measure as

$$d\mu = \frac{1}{N_B!}\prod_{\alpha=1}^{2M-N_B} dQ_\alpha dP_\alpha \prod_{j=1}^{N_B} dp_j dq_j \tag{2.15}$$

times an unknown Jacobian that tends to one in the $\Delta\to 0$, $L\to\infty$ limit. The integration over all $v(x)$

and $\pi(x) \equiv \int^x v(y)dy$ corresponds in the CSV to sum over N_B from zero to $2M$ ($\gg 1$) and to integrate over (P, Q, p_j, q_j) in the intervals (2.12) and (2.13). We are integrating independently over N_B different eigenvalues $p_j = \kappa_j^2$ and then we must divide by $N_B!$ in order to avoid double counting of configurations.

III. THE N-COMPONENT ANHARMONIC OSCILLATOR IN IMAGINARY-TIME SCATTERING VARIABLES

We apply in this section the imaginary-time scattering variables considered in Sec. II to the specific problem of the N-component anharmonic oscillator with O(N) symmetry. Its generating functional for imaginary time x reads

$$Z(N,g) = \iint \mathfrak{D}\vec{\phi}\, \exp\{-S[\vec{\phi}]\}/Z_0\,, \tag{3.1}$$

$$S[\vec{\phi}] \equiv \int_{+\infty}^{-\infty} dx\left[\frac{1}{2}\left(\frac{d\vec{\phi}}{dx}\right)^2 + \tfrac{1}{2}\vec{\phi}^2 + \frac{g}{N}(\vec{\phi}^2)^2\right], \tag{3.2}$$

where $\vec{\phi}(x) = (\phi_1, \ldots, \phi_N)$ and g is the coupling constant. Z_0 is a normalization constant fixed by $Z(N,0) \equiv e^{-(N/2)(2L)}$.

The generating function (3.1) can be recast as a functional integral over a single function $\alpha(x)$ by using the identity

$$\exp\left[-\frac{g}{N}\int dx(\vec{\phi}^2)^2\right] = \iint \mathfrak{D}\alpha(.)\exp\left[-\int dx\, \alpha(x)^2 - 2i\left(\frac{g}{N}\right)^{1/2}\int \alpha(x)\vec{\phi}^2(x)dx\right]. \tag{3.3}$$

Upon placing Eq. (3.3) into (3.1) one gets, after performing the integration over $\vec{\phi}$, which becomes Gaussian,

$$Z(N,g) = \frac{1}{Z_0}\int \mathfrak{D}\alpha(.)\exp\left\{-\frac{N}{2}\ln\det\left[-\frac{d^2}{\alpha x^2} + 1 + 4i\left(\frac{g}{N}\right)^{1/2}\alpha(.)\right] - \int dx\,\alpha(x)^2\right\}. \tag{3.4}$$

This functional integral has a stationary point at

$$\alpha(x) = \alpha_0 = \frac{1}{4i}\left(\frac{N}{g}\right)^{1/2} z(g)\,, \tag{3.5}$$

where $z(g)$ verifies

$$z^2(1+z) = 4g^2\,. \tag{3.6}$$

Hence $z(g)$ is a three-valued function of g. In the physical sheet

$$z(g) = -\tfrac{1}{3} + [g + (g^2 - \tfrac{1}{27})^{1/2}]^{2/3} + [g - (g^2 - \tfrac{1}{27})^{1/2}]^{2/3} \tag{3.7}$$

and

$$\lim_{g\to 0}\frac{z(g)}{2g} = +1\,. \tag{3.8}$$

The systematic expansion of the exponent in the integrand of (3.4) around this extremum gives the $1/N$ series.[14] For the ground-state energy we get

$$\begin{aligned} E_G(N,g) &= -\lim_{L\to\infty}\frac{1}{2L}\ln Z(N,g) \\ &= \frac{N}{2}\left[(1+z)^{1/2} - \frac{z^2}{8g}\right] - \lim_{L\to\infty}\frac{1}{2L}\ln I(N,g)\,, \end{aligned} \tag{3.9}$$

where

$$I(N,g) = \iint \mathfrak{D}\alpha(.)\exp\{-S_{\text{eff}}[v(.)]\}\,, \tag{3.10}$$

$$S_{\text{eff}}[v] = \frac{N}{2}\left[\ln\det\left(\frac{-\dfrac{d^2}{dx^2} + 1 + z + v(.)}{-\dfrac{d^2}{dx^2} + 1 + z}\right) - \frac{1}{8g}\int dx\, v(x)^2 - \frac{z}{4g}\int dx\, v(x)\right] \tag{3.11}$$

and where the following shift of the integration variable has been done:

$$\alpha(x) = \alpha_0 + \frac{1}{4i}\left(\frac{N}{g}\right)^{1/2} v(x), \quad v(\pm\infty) = 0\,. \tag{3.12}$$

We identify this $v(x)$ with the potential of the auxiliary linear problem of Sec. II. The crucial advantage of this representation lies in the fact that the $S_{\rm eff}[v]$ expresses in a separable way in terms of the SV associated with $v(x)$. Thus, explicit computation of the integral (3.10) will be possible in the CSV if we disregard the Jacobian problem. This Jacobian will be clearly N independent, that is of order N^0 for large N, and consequently much smaller than $S_{\rm eff}[v]$ which is proportional to N. N/g plays the role of $\hbar^{-1}$ here.

The effective action can be written in terms of the CSV[8] by using the identity between the Fredholm determinant and the Jost function for the Schrödinger equation and the trace identities.[3] One finds a completely separated expression:

$$S_{\rm eff}[v]=-N\sum_{j=1}^{N_B}F(p_j)+N\int_{-\infty}^{+\infty}dk\,f(k)P(k)\,, \tag{3.13}$$

where

$$F(p)=\arg\tanh\left(\frac{1+z(g)}{p}\right)^{1/2}+\frac{i\pi}{2}+\frac{p^{3/2}}{3g}-\frac{z\sqrt{p}}{2g}\,, \tag{3.14}$$

and

$$f(k)=-\frac{k}{4g}+\frac{1}{4k[(1+z)^{1/2}+ik]}-\frac{z}{8gk}\,. \tag{3.15}$$

For future reference we note that

$$F(p)=\frac{p^{3/2}}{3g}\,\frac{1+\frac{3}{2}z}{1+z}+\sum_{n=2}^{\infty}\left(\frac{z\sqrt{p}}{2g}\right)^{2n+1}\frac{1}{2n+1}\,. \tag{3.16}$$

By using the integration measure $d\mu$ given by (2.15), $I(N,g)$ reads

$$I(N,g)=\lim_{M\to\infty}\frac{1}{Z_0}\sum_{N_B=0}^{2M}\frac{1}{N_B!}\int_0^{\infty}\int_0^{2\pi}\prod_{\alpha=1}^{2M-N_B}\left(\frac{\pi}{L}\right)dP(k_\alpha)\,dQ(k_\alpha)e^{-NP(k_\alpha)f(k_\alpha)}\int_0^{\infty}\int_{-2\sqrt{p_j}L}^{+2\sqrt{p_j}L}\prod_{j=1}^{N_B}dp_j\,dq_j\exp[NF(p_j)]\,. \tag{3.17}$$

The integrals in (3.17) are elementary except the one over p_j. Then, we obtain

$$I=\lim_{M\to\infty}Z_0^{-1}\left(\frac{2\pi}{NL}\right)\sum_{N_B=0}^{2M}\frac{1}{N_B!}\left(+\frac{2NL}{\pi}\right)^{N_B}\left[\int_0^{\infty}dp\,\sqrt{p}\,e^{NF(p)}\right]^{N_B}\prod_{\alpha=1}^{2M-N_B}\frac{1}{f(k_\alpha)}\,.$$

Furthermore, in the $\Delta\to0$, $M\to\infty$ limit

$$\prod_{\alpha=1}^{2M-N_B}\frac{1}{f(k_\alpha)}=\exp\left\{-\frac{L}{\pi}\int_{-\pi/\Delta}^{\pi/\Delta}dk\,\ln f(k)\left[1+O\left(\frac{1}{L}\right)\right]\right\}, \tag{3.18}$$

and we get for the ground-state energy

$$E_G(N,g)=\frac{N}{2}\left[(1+z)^{1/2}-\frac{z^2}{8g}\right]+\lim_{\Delta\to0}\left\{\int_{\pi/\Delta}^{+\pi/\Delta}\frac{dk}{2\pi}\ln f(k)-\lim_{L\to\infty}\frac{1}{2L}\ln\left[Z_0\left(\frac{NL}{2\pi}\right)^{2M}\right]\right\}+\frac{N}{\pi}\int_0^{\infty}\sqrt{p}\,dp\,B(p,N,g)e^{NF(p)}\,, \tag{3.19}$$

where we have inserted $B(p,g,N)$ to try to take into account the effects of quantum fluctuations of the Jacobian. $B(p,g,N)$ should be of order N^0 for large N as we shall see explicitly in the next section. The limit of the expression within braces in Eq. (3.19) is easily seen to be zero from the relation

$$\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\ln\frac{f(k)}{f_0(k)}=0\,. \tag{3.20}$$

Here $f_0(k)\equiv-k/4g$. We then get

$$E_G(N,g)=\frac{N}{2}\left[(1+z)^{1/2}-\frac{g}{2(1+z)}\right]+C_N(g)+\frac{2N}{\pi}\int_0^{\infty}s^2ds\,B_N(s,g)e^{NF(s)}\,, \tag{3.21}$$

where we have used the definition of Z_0 and we have set $s=\sqrt{p}$ in the integral. Also, we added a contribution $C_N(g)$ for the same reason we introduced $B_N(s,g)$. $C_N(g)$ must be of order N^0 for large N.

The integration variable s in Eq. (3.21) corresponds to an eigenvalue of the auxiliary Schrödinger problem. This suggests the following interpretation. The functional integral (3.17) is formally similar to the grand partition function of a gas of free solitons at temperature $1/N$ in one dimension. $-F(p)$ being the "kinetic energy" of a soliton with momentum p and at position q, $N^{-1}\ln(N/\pi)$ can be identified with the chemical potential. We label "soliton" the configuration defined in SV by

$N_B=1$, $\kappa_1=s$, $c_1=c$, and $r(k)\equiv 0$ which takes the form

$$v(x)=-\frac{2s^2}{\cosh^2(sx-\delta)} \tag{3.22}$$

in the v representation.[2] Here $\delta=\frac{1}{2}\ln(c/2s)$. In this analogy with classical statistical mechanics the ground-state energy $E_G(N,g)$ corresponds to the gas pressure divided by its temperature (i.e., times N). In this language, the quantum effects contained in the Jacobian of the change of variable (2.10) can be perhaps treated as "interactions" between the solitons.

IV. THE GROUND-STATE ENERGY AND ITS PROPERTIES

The evaluation of the ground-state energy of the N-component anharmonic oscillator by an imaginary-time inverse scattering method led to Eq. (3.21).

In this section we shall use the known large-order behavior of the N^{-1} expansion of this ground-state energy to determine as far as possible the unknown functions B_N and C_N.

The N^{-1} expansion of $E_G(N,g)$ has the form

$$E_G(N,g)=\sum_{K=-1}^{+\infty}\frac{A_K(g)}{N^K}. \tag{4.1}$$

The coefficients $A_K(g)$ behave for large K as[8,15]

$$A_K(g)=\left[\frac{\Gamma(K+\frac{1}{2})}{[\rho_+(g)]^{K+1/2}}\left(\frac{\sqrt{2}}{\pi}\right)^{3/2}\times\frac{(3z+2)^{7/4}\sqrt{z}}{\sqrt{g}[(3z+2)^{1/2}+(2+2z)^{1/2}]^2}+\text{c.c.}\right]\times[1+O(1/K)], \tag{4.2}$$

where

$$\rho_+(g)=\tfrac{1}{2}\ln z(g)-\ln[(2z+2)^{1/2}+(3z+2)^{1/2}]-\frac{1}{3g}(1+\tfrac{3}{2}z)^{1/2}+\frac{i\pi}{2}. \tag{4.3}$$

On the other hand, we get from Eq. (3.21)

$$A_K(g)=\int\frac{dN}{2\pi i}N^{K-1}E_G(N,g)\underset{K\gg1}{=}-\frac{2}{\pi}\int\frac{dN}{2\pi i}N^K\int_0^\infty s^2ds\,B_N(s,g)e^{NF(s)}, \tag{4.4}$$

where we assume that $C_N(g)$ does not contribute for large K. The results will confirm this assumption.

The integral over s has a stationary point where

$$F'(S_0)=0.$$

This gives

$$S_0=+(1+\tfrac{3}{2}z)^{1/2}. \tag{4.5}$$

This is precisely the bound-state eigenvalue that defines in SV the instanton for the large orders on N^{-1} in the original functional integral (3.9) and (3.10).[8]

We obtain from (4.4) by the steepest-descent method

$$A_K(g)=-\frac{2}{\pi}\frac{B_\infty(S_0,g)}{\sqrt{2\pi}}\times(gzS_0)^{1/2}\frac{\Gamma(K+\frac{1}{2})}{F(S_0)^{K+1/2}}\left[1+O\left(\frac{1}{K}\right)\right]. \tag{4.6}$$

We note that the leading factor $\Gamma(K+\frac{1}{2})F(S_0)^{-K-1/2}$ in Eq. (4.2) is correctly reproduced by this expression. The identification of (4.6) with the actual behavior (4.2) of $A_K(g)$ gives us the constraint

$$B_\infty(S_0,g)=\frac{2S_0^3}{g}\frac{1}{[S_0+(1+z)^{1/2}]^2}. \tag{4.7}$$

In other words, by fixing the function $B_\infty(S,g)$ on the curve $S=S_0(g)$ we obtain that Eq. (3.21) for $E_G(N,g)$ also reproduces the determinant of small fluctuations around the instanton of Ref. 8. Moreover, Eq. (4.7) suggests

$$B_\infty(s,g)=+\frac{2s^3}{g}\frac{1}{[s+(1+z)^{1/2}]^2} \tag{4.8}$$

for *all* s and g. In the following section we shall see that this last conjecture is correct.

As we see, the assumption that $C_N(g)$ does not contribute to the large-order behavior of the N^{-1} series is consistent.

Let us now consider the small orders in the N^{-1} expansion of $E_G(N,g)$. The first term can be easily computed from the functional integral [(3.9)–(3.11)] by the stationary-point expansion[8] or by the Rayleigh-Schrödinger perturbation theory in radial variables[15] with the result

$$E_G(g,N)=\frac{N}{2}\left[(1+z)^{1/2}-\frac{g}{2(1+z)^{1/2}}\right]+(\tfrac{3}{2}z+1)^{1/2}-(1+z)^{1/2}+\frac{1}{N}\frac{3z(1+z)}{2(1+\frac{3}{2}z)^2}\left[\frac{1+\frac{25}{24}z}{(1+z)^{1/2}}-(1+\tfrac{3}{2}z)^{1/2}\right]+O\left(\frac{1}{N^2}\right). \tag{4.9}$$

On the other hand, we see that in Eq. (3.21) the integral over s is dominated by the origin for large N. This is precisely what one should expect because $s=0$ corresponds to a null configuration $v(x)\equiv 0$, i.e., $s=0$ is associated with the stationary point $\alpha(x)=\alpha_0$, previously discussed. Then from Eqs. (3.16) and (3.21), it follows that

$$\frac{2N}{\pi}\int_0^\infty s^2ds\,B_\infty(s,g)e^{NF(s)}\underset{N\to\infty}{\sim}-\frac{12g}{\pi N}\frac{(1+z)B_\infty(0,g)}{(1+\frac{3}{2}z)^2}. \tag{4.10}$$

This shows that Eqs. (4.4) and (4.8) are consistent with the $1/N$ expansion (4.9) of $E_G(N,g)$ although there is too much freedom in the functions $C_N(g)$ and $B_N(s,g)$ to make more precise statements. Equations (4.9) and (4.10) suggest, however, that one could have

$$C_\infty(g)=(\tfrac{3}{2}z+1)^{1/2}-(1+z)^{1/2}. \tag{4.11}$$

Equation (4.10) also gives the limiting behavior of $E_G(N,g)$ for $g\to 0$.

Concluding the preceding discussion, we have from Eqs. (3.14) and (4.8)

$$E_G(N,g)=\frac{N}{2}\left[(1+z)^{1/2}-\frac{z^2}{8g}\right]+C_N(g)+\frac{4N}{\pi g}\int_0^\infty \frac{s^5ds}{[s+(1+z)^{1/2}]^2}\left[1+O\left(\frac{g}{N}\right)\right]\times\left[\frac{(1+z)^{1/2}+s}{(1+z)^{1/2}-s}\right]^{N/2}\exp\left[\frac{N}{g}\left(\frac{s^3}{3}-\frac{z}{2}s\right)\right]. \tag{4.12}$$

Rigorously speaking, the $O(g/N)$ in the integrand may be nonuniform on s for small s. For this reason we cannot be sure that Eq. (4.11) is true.

The integral in (4.12) converges for $\mathrm{Re}g<0$ as the original functional integral [(3.10) and (3.11)] does for real $v(x)$. Let us analytically continue $E_G(N,g)$ on the g plane for real positive N. We set

$$g=|g|e^{i\varphi}$$

and assume that the large-s behavior of the integrand of (4.12) is dominated by the exponential factor $\exp(Ns^3/3g)$.

Then, if we distort the integration path in the s plane such that it approaches infinity as

$$s=|s|e^{i\alpha},\quad |s|\to+\infty,$$

the integral will converge for

$$0>\mathrm{Re}\left(\frac{s^3}{g}\right)=-\left(\frac{s^3}{g}\right)\cos(3\alpha-\varphi). \tag{4.13}$$

This condition holds if

$$\frac{\pi}{2}<3\alpha-\varphi<\frac{3\pi}{2}\ (\mathrm{mod}\ 2\pi). \tag{4.14}$$

In other words, for each complex g, we find three inequivalent integration paths for which the integral converges. This is sketched in Fig. 1. This three-sheeted Riemann structure is also exhibited by the functions $z(g)$ and $[1+z(g)]^{1/2}=2g/z(g)$ that appear in Eq. (4.12) [see Eqs. (3.6) and (3.7) and Fig. 2]. Then Eq. (4.12) possesses the correct three-sheeted Riemann structure of the ground-state energy of the anharmonic oscillator.[21] It must be noted that this property is not shared by the usual $1/N$ expansion (4.9). The terms of this series exhibit a cut in the g plane beginning at $g=-(27)^{-1/2}$ and not at the origin.

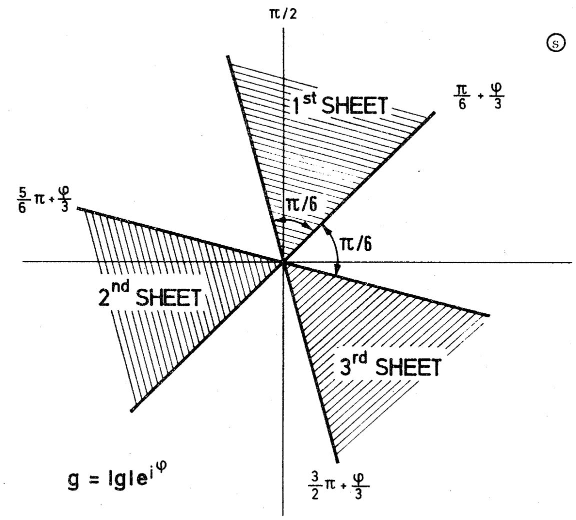

FIG. 1. The integral in the ground-state formula converges when the integration path goes to infinity through one of three hatched regions.

The strong-coupling limit $g\to\infty$, for fixed N, can also be studied. It is easy to see that Eq. (4.12) behaves like $g^{1/3}$ in this limit. For the same reasons as before, for the coefficient of N^{-1}, we cannot compute here the coefficient of $g^{1/3}$.

We can consider Eq. (4.12) as the leading term of a new kind of $1/N$ expansion. The study of the discretized system in terms of scattering variables could provide a way to compute systematically the high-order terms. Work in this direction is in progress.

V. THE DISCONTINUITY OF THE ENERGY IN THE NEGATIVE g AXIS

In this section we compute from Eq. (4.12) the discontinuity of $E_G(N,g)$ across its cut on $\mathrm{Re}(g)<0$. As will be seen below, Eq. (4.12) results are particularly successful in the regime $h\equiv -g\to 0^+$.

We assume $0>g>-(27)^{-1/2}$. The discontinuity in this interval comes solely from the singularity $[(1+z)^{1/2}-s]^{-N/2}$ in the integrand (see Fig. 2). Thus,

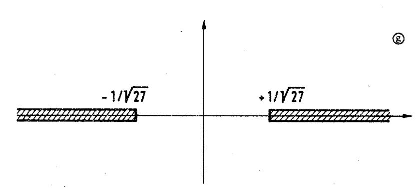

FIG. 2. The function $z(g)$, solution of Eq. (3.6), has three Riemann sheets. They are connected pairwise by square-root-type cuts. The cut from $-3^{-3/2}$ to $-\infty$ connects the 1st sheet (physical sheet) with the second one and the cut from $+3^{-3/2}$ to $+\infty$ connects the 2nd and 3rd sheets between them.

$$\operatorname{Im}\underset{0<h<(27)^{-1/2}}{E_G(N,-h+i0)}=\frac{4N}{\pi h}\sin\left(\frac{N\pi}{2}\right)\int_{[1+z(-h)]^{1/2}}^{+\infty}\frac{s^5ds}{[s+(1+z)^{1/2}]^2}\,\frac{s+[1+z(-h)]^{1/2}}{s-[1+z(-h)]^{1/2}}^{N/2}$$
$$\times\exp\left\{-\frac{N}{h}\left[\frac{s^3}{3}-\frac{s}{2}z(-h)\right]\right\}\left[1+O\left(\frac{h}{N}\right)\right], \tag{5.1}$$

where we assume that $C_N(g)$ does not contribute.

This integral is dominated for $h\to 0^+$ by its lower bound. Then, we set

$$s=[1+z(-h)]^{1/2}(1+ht), \tag{5.2}$$

where t is a new integration variable. We obtain in this way

$$\operatorname{Im}E_G(N,-h+i0)=\frac{N}{\pi}\sin\left(\frac{N\pi}{2}\right)\left(\frac{2}{h}\right)^{N/2}$$
$$\times\int_0^\infty\frac{dt}{t^{N/2}}\epsilon(h,ht)e^{-N\sigma(h,ht)}\left[1+O\left(\frac{h}{N}\right)\right], \tag{5.3}$$

where

$$\epsilon(h,x)=[1+z(-h)]^2\frac{(1+x)^5}{(1+\frac{1}{2}x)^2}, \tag{5.4}$$

$$\sigma(h,x)=\frac{1}{3h}[1+z(-h)]^{3/2}(1+x)^3$$
$$+1+x-\tfrac{1}{2}\ln(1+\tfrac{1}{2}x). \tag{5.5}$$

We have the following expansions in h:

$$\sigma(h,ht)=\frac{1}{3h}+t+h(t^2-\tfrac{9}{4}t-\tfrac{1}{2})$$
$$+h^2(\tfrac{1}{3}t^3-\tfrac{47}{16}t^2-\tfrac{3}{2}t-\tfrac{4}{3})+O(h^3), \tag{5.6}$$

$$E(h,ht)=1+4h(t-1)+h^2(\tfrac{23}{4}t-16)t+O(h^3). \tag{5.7}$$

Then, from (5.3)–(5.7) we obtain the discontinuity of $E_G(N,g)$ as a power series in h/N,

$$\operatorname{Im}E_G(N,-h+i0)=\left(\frac{2N}{h}\right)^{N/2}\frac{e^{-N/3h}}{\Gamma(\frac{1}{2}N)}$$
$$\times\sum_{K\geq 0}T_K(N)\left(\frac{h}{N}\right)^K. \tag{5.8}$$

Here $T_0(N)\equiv 1$ and the higher-order coefficients follow from the integral

$$\int_0^\infty\frac{e^{-x}x^{-N/2}dx}{\Gamma(1-\frac{1}{2}N)}\epsilon\left(h,\frac{hx}{N}\right)e^{-N\lambda(h,hx/N)}\left[1+O\left(\frac{h}{N}\right)\right]$$
$$=\sum_{K\geq 0}T_K(N)\left(\frac{h}{N}\right)^K. \tag{5.9}$$

Here

$$\lambda(h,ht)\equiv\sigma(h,ht)-\frac{1}{3h}-t \tag{5.10}$$

is of order h for $h\to 0^+$.

Equation (5.8) is the more interesting result we have extracted from the expression (4.12) for $E_G(N,g)$. The factor in front of the series in (5.8) is exact.[16,17] We recall that its computation by functional integration involves the quantum determinant of small fluctuations around a classical instanton.[17] The action of this classical solution provides the exponential factor. It is remarkable that the simple integral (4.12) can reproduce all that.

Let us now consider the higher-order corrections in $\operatorname{Im}E_G$. Because we do not yet know the precise form of the $O(h/N)$ terms in the integrand of Eq. (4.12) or (5.9), we can compute the $T_K(N)$ only for large N (but K arbitrary) by expanding $e^{-N\lambda}$ and $\epsilon(h,ht)$. One sees in this way that the $T_K(N)$ are polynomials in N of degree $2K$. That is,

$$T_K(N)=\sum_{s=0}^{2K}m_{K,s}N^s. \tag{5.11}$$

All the numerical coefficients that we can compute result in rational numbers. This is precisely what Zinn-Justín found in recent numerical computations.[18]

Explicitly, we get from (5.9)

$$m_{K,2K}=\frac{(-1)^K}{K!}\frac{7^K}{2^{3K}}, \tag{5.12}$$

$$m_{K,2K-1}=\frac{(-1)^{K-1}}{(K-1)!}\frac{7^{K-2}}{2^{3K-1}}(41K-104), \tag{5.13}$$

where (5.6), (5.7) and (A4) were used. These formulas exactly reproduce the available numerical values ($K=1$ to 4)[18,19] for $m_{K,2K}$ and $m_{K,2K-1}$. It can be pointed out that these coefficients in the functional integral approach[17] correspond to diagrams with $(K+1)$ loops ($1<K<\infty$). The success in reproducing these higher-order quantum corrections encouraged us to determine the h/N correction in the integrand of (5.9) from the known $T_1(N)$.[18,19]

Assuming that

$$B_N(s,g)=B_\infty(s,g)\left[1+\frac{g}{N}D(s)+O\left(\frac{g^2}{N^2}\right)\right], \tag{5.14}$$

where $D(s)$ is analytic at $s=1$, we get from (5.9) after a short calculation

$$T_1(N)=-\frac{21N^2+54N+24D(1)-2}{24}. \tag{5.15}$$

The known value for this two-loop correction is

$$T_1(N) = -\frac{21N^2 + 54N + 20N}{24}. \quad (5.16)$$

Then, we conclude that

$$D(1) = \tfrac{17}{6}.$$

Now, by using this fitted value we get from (5.9) and (A4)

$$m_{2,2} = -\tfrac{305}{96} \text{ and } m_{3,4} = \tfrac{191}{64} \quad (5.17)$$

which exactly coincide with the numerical results.[18] This gives a consistency check of the assumed expansion (5.14). It must be noted that the explicit form assumed for $B_\infty(s,g)$ in Eq. (4.8) was crucial to obtain Eqs. (5.12), (5.13), (5.15), and (5.17).

VI. FINAL REMARKS

In the explicit computations of the preceding sections, we only dealt with the ground-state energy. As is clear, the integration bounds and integration measure in CSV given in Sec. II are valid for every functional integration. We shall now discuss Green's function and excited states. The two-point function of the N-component anharmonic oscillator reads

$$\int \mathfrak{D}\vec{\phi}\,\phi_a(x)\phi_b(x')e^{-S[\vec{\phi}]} = \delta_{ab}G(x-x'), \quad (6.1)$$

where $1 \leq a, b \leq N$, and $S[\vec{\phi}]$ is given by Eq. (3.2).

In the α representation we have

$$G(x-x') = \int \mathfrak{D}\alpha(.)g(x,x')e^{-S_{\text{eff}}[v(\,)]}, \quad (6.2)$$

where S_{eff} is given by Eq. (3.11) and

$$g(x,x') = \left\langle x \left| \frac{1}{\frac{-d^2}{dx^2} + 1 + v(.)} \right| x' \right\rangle. \quad (6.3)$$

On the other hand, the usual expansion of Green's function in the eigenstates of the Hamiltonian gives

$$\tilde{G}(E) = 2\sum_n \frac{E_n}{E^2 + E_n^{\,2}} |\langle 0|\phi_a|n\rangle|^2, \quad (6.4)$$

where

$$\tilde{G}(E) = \int_{-\infty}^{+\infty} e^{iEx}G(x)\,dx \quad (6.5)$$

and $|n\rangle$, E_n stand for the eigenstates and eigenvalues of the anharmonic oscillator. Hence, excited states of the system can be explicitly found if we succeed in computing the functional integral (6.2). In order to do that within the inverse scattering approach, we should express $g(x,x')$ in terms of the CSV. $g(x,x')$ is related to the Jost solutions $f_\pm(x,k)$ of the auxiliary Schrödinger equation (2.2) as follows:

$$g(x,x') = \frac{f_+(im, x_>)f_-(im, x_<)}{2m\,F(im)}. \quad (6.6)$$

Here $F(k)$ stands for the Jost function which is immediately expressable in terms of CSV. The Jost solutions can be related to the CSV through a linear integral equation following Marchenko's formalism.[2] The kernel of this integral equation has a simple expression in CSV. In this way, the computation of the Green's function of the anharmonic oscillator reduces to a linear problem.

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APPENDIX

In the calculations of Sec. V we need to expand for large N integrals like

$$I(f) = \int_0^\infty \frac{e^{-x}x^{-N/2}}{1-\frac{1}{2}N} f\left(\frac{x}{N}\right)dx. \quad (A1)$$

Here $f(t)$ admits an expansion like

$$f(t) = \sum_s \alpha_s t^s. \quad (A2)$$

We can explicitly integrate order by order with the result

$$I(f) = \sum_s \frac{\alpha^s}{N^s}\frac{\Gamma(1-\frac{1}{2}N+s)}{\Gamma(1-\frac{1}{2}N)}. \quad (A3)$$

Then, we can apply for large N Stirling's formula for the Γ function in (A3) and finally sum over s. This gives, after some calculations,

$$I(f) = f + \frac{1}{N}(f' - \tfrac{1}{4}f'') + \frac{1}{N^2}(f'' - \tfrac{5}{12}f''' + \tfrac{1}{32}f^{\text{iv}}) + O\left(\frac{1}{N^3}\right), \quad (A4)$$

where f, f', f'', f''', and f^{iv} are to be taken at $t = -\frac{1}{2}$.

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