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Multiple scattering of electromagnetic waves by randomly distributed and oriented dielectric scatterers

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The coherent wave propagation and attenuation of electromagnetic waves in an inhomogeneous medium containing randomly distributed and randomly oriented nonspherical dielectric scatterers is studied using statistical averaging procedures and a self-consistent multiple-scattering theory. The specific geometry and orientation of the inhomogeneities are incorporated into the T matrix of the scatterer thus making the formalism a convenient and computationally efficient scheme to study randomly oriented dielectric scatterers for a range of frequencies. The T matrix of identical scatterers evaluated with respect to axes natural to the scatterers is then transformed to the arbitrary coordinate system by introducing rotation matrices that contain information about the orientation of individual scatterers. The rotation matrices can be integrated conveniently for random orientations of the scatterers.

I. INTRODUCTION

In a previous paper, Varadan, Bringi, and Varadan[1] gave a multiple-scattering formalism for coherent electromagnetic wave propagation through a medium containing an ensemble of dielectric scatterers that had a particular preferred orientation. The frequency dependence of the phase velocity and attenuation in such a medium was studied for spheroidal scatterers of various aspect ratios and scatterer concentrations, when the wave was incident along the symmetry axes of the spheroids which were all parallel.

In this paper, we extend the treatment to randomly oriented scatterers. The concept of a T matrix as defined by Waterman[2] for single scatterers is generalized to relate the scattered field from a scatterer in the presence of several other scatterers to the field that excites or is incident on the scatterer. Ensemble averaging over the position of the scatterers taking into account the hole correction results in a set of homogeneous equations for the average scattered field coefficients. The determinant of the coefficient matrix then yields the dispersion relation for the average propagation vector $\vec{K}$ in the macroscopically homogeneous medium. The vector $\vec{K}$ is complex, and the real part is related to the coherent phase velocity in the medium whereas the imaginary part gives the attenuation in the medium due to geometric dispersion as well as absorption, if any, by the scatterers.

The advantage of using the concept of a T matrix is that it allows us to conveniently include all factors relating to the geometry, dielectric properties, and orientation of the scatterer within the T matrix itself leaving the rest of the formalism uncluttered by the specific features of a particular set of scatterers. To this end, the formalism given in Ref. 1 can be used as is for the case of random orientation of the scatterers and the only change will be a reinterpretation of the T matrix as it appears in Ref. 1. This time, two averages have to be performed, one an ensemble average over the positions and the second an average over all possible orientations of the scatterers. Once again we can obtain the dispersion relation that can be solved to obtain the average propagation vector $\vec{K}$ in a macroscopically homogeneous and, in this case, isotropic medium. Recently, a similar approach[3] has been used for scalar coherent wave propagation in two dimensions through a distribution of randomly distributed and oriented scatterers.

Since we are discussing the coherent wave propagation, it is enough to average the T matrix of a particular scatterer over orientations. If we are interested in the incoherent field or the average intensity, then the intensity or scattering cross section must be averaged over all possible orientation of the scatterers.

Coherent phase velocity and attenuation are studied for both real and complex frequency-dependent dielectric scatterers in free space for different scatterer concentrations and range of frequencies. The results are compared with those obtained for aligned scatterers. At higher frequencies, the results indicate that there is a significant difference in phase velocity and attenuation between aligned and randomly oriented scatterers. The analysis is important in many fields of engineering and science such as microwave communications through hail-, ice-, and dust-filled medium and radar meteorological applications, etc.

II. MULTIPLE-SCATTERING FORMALISM

Consider N number of dielectric scatterers randomly distributed and oriented in free space which are referred to a coordinate system as shown in Fig. 1. O_i and O_j refer to the centers of the ith and jth scatterers, and $\vec{r}_i$ and $\vec{r}_j$ their position with respect to the origin 0 of the x, y, z coordinate system. Let ϵ_r be the dielectric constant of the homogeneous scatterers.

A plane electromagnetic wave of unit amplitude, frequency ω, $e^{-i\omega t}$ time dependence, and wave vector $\vec{k}$ is incident along the z axis of the coordinate system and is represented by

$$\vec{E}^0(\vec{r}) = \hat{e} e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{1}$$

where $k=\omega/c$ is the free-space wave number, $\hat{e}$ is the unit polarization vector, and $\hat{e}\cdot\vec{k}=0$.

Let $\vec{E}_i^s$ be the field scattered by the ith scatterer, so that the total electric field at any point $\vec{r}$ is given by

$$\vec{E}(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{i=1}^{N} \vec{E}_i^s(\vec{r}-\vec{r}_i) . \tag{2}$$

The field $\vec{E}_i^e$ that excites the ith scatterer is the incident field $\vec{E}^0$ plus the field scattered by all other scatterers. Thus, at a point $\vec{r}$ in the neighborhood of the ith scatterer

$$\vec{E}_i^e(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{j\neq i}^{N} \vec{E}_j^s(\vec{r}-\vec{r}_j), \quad a \leq |\vec{r}-\vec{r}_i| \leq 2a . \tag{3}$$

In writing Eq. (3), the auxiliary condition on $|\vec{r}-\vec{r}_i|$ implies that there is no interpenetration of the scatterers. The incident field is assumed to be produced by sources at infinity.

The fields $\vec{E}^0$, $\vec{E}_i^e$, and $\vec{E}_j^s$ are expanded in vector spherical functions $\vec{M}$ and $\vec{N}$ as in Ref. 1:

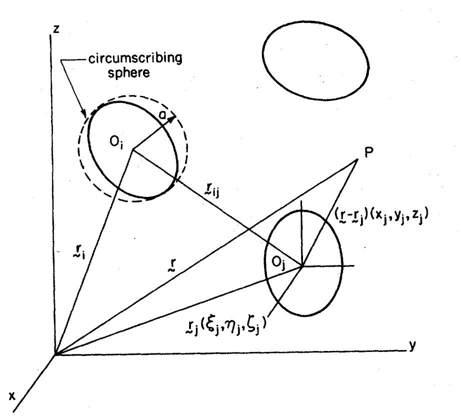

FIG. 1. Randomly distributed and oriented scatterers.

$$\vec{E}^0(\vec{r}) = e^{i\vec{k}\cdot\vec{r}_i} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\alpha_n^m \operatorname{Re}\vec{M}_{mn}(\vec{r}-\vec{r}_i) + \beta_n^m \operatorname{Re}\vec{N}_{mn}(\vec{r}-\vec{r}_i)] , \tag{4}$$

$$\vec{E}_i^e(\vec{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [b_n^{m(i)} \operatorname{Re}\vec{M}_{mn}^i(\vec{r}-\vec{r}_i) + c_n^{m(i)} \operatorname{Re}\vec{N}_{mn}^i(\vec{r}-\vec{r}_i)] , \tag{5}$$

$$\vec{E}_j^s(\vec{r}-\vec{r}_j) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [B_n^{m(j)} \vec{M}_{mn}^j(\vec{r}-r_j) + C_n^{m(j)} \vec{N}_{mn}^j(\vec{r}-\vec{r}_j)] , \tag{6}$$

where, from Eq. (1), if $\hat{e}$ is taken parallel to the z axis (Stratton[4])

$$\alpha_n^m = (2n+1)i^n[\delta_{m,1} + n(n+1)\delta_{m,-1}]/2in(n+1) , \tag{7a}$$

$$\beta_n^m = (2n+1)i^n[\delta_{m,1} - n(n+1)\delta_{m,-1}]/2in(n+1) , \tag{7b}$$

and the wave functions M_{mn} and N_{mn} are given by

$$\vec{M}_{mn}(\vec{r}) = \nabla\times[\vec{r} h_n(kr) y_{mn}(\theta,\phi)] , \tag{8a}$$

$$\vec{N}_{mn}(\vec{r}) = (1/k)\nabla\times\vec{M}_{mn}(\vec{r}) , \tag{8b}$$

with

$$y_{mn}(\theta,\phi) = P_n^m(\cos\theta)e^{im\phi} . \tag{8c}$$

In Eqs. (4) and (5), Re denotes the regular meaning that h_n is replaced by j_n.

If the expansions in Eqs. (4)–(6) are substituted in (3), we can get a relationship between the scattered field coefficients B and C and the exciting field coefficients b and c. However, the wave functions in the expansion of $\vec{E}_i^e$ and $\vec{E}$ are referred to a coordinate system at O_i whereas $\vec{E}_j^s$ is referred to an origin at O_j. Further, it is important to make the coordinate axes at O_i and O_j parallel to the xyz axes so that we can use the translation theorem for the vector spherical functions, as in Ref. 1, and the orthogonality of the vector spherical functions to obtain the following equations:

$$b_n^{m(i)} = \psi_{mn}^i = \frac{2n+1}{n(n+1)} i^n \frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i} [\delta_{m,1} + n(n+1)\delta_{m,-1}] + {\sum_{j=1}^{N}}' \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} [B_{n_1}^{m_1(j)} B_{mn}^{m_1 n_1}(\vec{r}_i-\vec{r}_j) + C_{n_1}^{m_1(j)} C_{mn}^{m_1 n_1}(\vec{r}_i-\vec{r}_j)] , \tag{9a}$$

$$c_n^{m(i)} = \chi_{mn}^i = \frac{2n+1}{n(n+1)} i^n \frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i} [\delta_{m,1} - n(n+1)\delta_{m,-1}] + {\sum_{j=1}^{N}}' \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} [B_{n_1}^{m_1(j)} C_{mn}^{m_1 n_1}(\vec{r}_i-\vec{r}_j) + C_{n_1}^{m_1(j)} B_{mn}^{m_1 n_1}(\vec{r}_i-\vec{r}_j)] , \tag{9b}$$

where $\sum'$ denotes $j \neq i$ and δ_{mn} is the Kronecker delta. The functions $B_{mn}^{m_1 n_1}$ and $C_{mn}^{m_1 n_1}$ resulting from the translation theorem of the vector spherical functions are given in Eqs. (11) and (12) of Ref. 1.

At this stage, a T matrix has to be introduced to relate the scattered field coefficients B and C to the exciting field coefficients b and c. If the scatterers are all parallel to each other and if the T matrix is computed with respect to a set of axes parallel to xyz, then the T matrix of all the N scatterers is the same. However, if the orientation of each scatterer with respect to the xyz axes is defined by the Euler angles $\alpha_i, \beta_i, \gamma_i$ (see Fig. 2), then the T matrix of the ith scatterer will be a function of the Euler angles.

III. T MATRIX FOR RANDOMLY ORIENTED SCATTERERS

Let XYZ be the set of coordinate axes natural to the scatterer. For spheroidal scatterers, the XYZ axes coincide with the symmetry axes of the spheroid. Let α, β, γ be the Euler angles of the XYZ axes with respect to the xyz axes. All quantities that are referred to the XYZ system are distinguished by a caret. Thus,

$$\vec{E}_j^s(\vec{r}-\vec{r}_j) = \sum_n \sum_m [\hat{B}_n^{m(j)} \hat{\vec{M}}_{mn}^j(\vec{r}-\vec{r}_j) + \hat{C}_n^{m(j)} \hat{\vec{N}}_{mn}^j(\vec{r}-\vec{r}_j)]$$
$$= \sum_n \sum_m [B_n^{m(j)}(\vec{r}-\vec{r}_j) \vec{M}_{mn}^j(\vec{r}-\vec{r}_j) + C_n^{m(j)}(\vec{r}-\vec{r}_j) \vec{N}_{mn}(\vec{r}-\vec{r}_j)] . \quad (10)$$

The spherical harmonics $y_{mn}(\theta,\phi)$ are eigenfunctions of the rotation operator. From the quantum-mechanical theory of angular momentum (see Edmonds[5]),

$$y_{mn} = D(\alpha,\beta,\gamma)\, \hat{y}_{mn} = \sum_{m'=-n}^{n} D_{mm'}^n(\alpha,\beta,\gamma)\, \hat{y}_{m'n} , \quad (11)$$

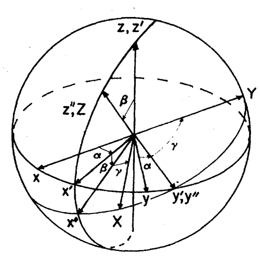

FIG. 2. Euler angles.

where $D(\alpha,\beta,\gamma)$ is the operator associated with the finite rotations α,β,γ and $D_{mm'}^n$ is the rotation matrix associated with the rotation operator. The rotation operator leaves the length of the position vector $|\vec{r}|$ invariant. Further, the curl operation commutes with the rotation operator so that

$$\vec{M}_{mn} = D(\alpha,\beta,\gamma) \hat{\vec{M}}_{mn} = \sum_{m'=-n}^{n} D_{mm'}^n(\alpha,\beta,\gamma) \hat{\vec{M}}_{m'n} , \quad (12a)$$

$$\vec{N}_{mn} = \sum_{m'=-n}^{n} D_{mm'}^n(\alpha,\beta,\gamma) \hat{\vec{M}}_{m'n} . \quad (12b)$$

Using Eq. (12) and the orthogonality properties of the vector spherical functions in Eq. (10), we obtain the scattered field coefficients $\hat{B}$ and $\hat{C}$ in terms of B and C as given by

$$\hat{B} = D^t B, \quad \hat{C} = D^t C . \quad (13)$$

Similarly, one could express the exciting field coefficients $\hat{b}$ and $\hat{c}$ in terms of b and c through the rotation operator

$$\hat{b} = D^t b, \quad \hat{c} = D^t c , \quad (14)$$

where $\hat{B}$, $\hat{C}$, $\hat{b}$, and $\hat{c}$ are column vectors and D^t is the matrix transpose of D.

It has been shown in Refs. 1 and 2 that the scattered field expansion coefficients may be formally related to the exciting field expansion coefficients through the T matrix. Thus, we can write

$$\begin{pmatrix} \hat{B} \\ \hat{C} \end{pmatrix} = \begin{bmatrix} \hat{T}^{11} & \hat{T}^{12} \\ \hat{T}^{21} & \hat{T}^{22} \end{bmatrix} \begin{pmatrix} \hat{b} \\ \hat{c} \end{pmatrix} = \hat{T} \begin{pmatrix} \hat{b} \\ \hat{c} \end{pmatrix} \quad (15a)$$

and

$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = T \begin{pmatrix} b \\ c \end{pmatrix} . \quad (15b)$$

From Eqs. (13), (14), and (15), it can be shown that T is related to $\hat{T}$ as follows:

$$T = (D^t)^{-1} \hat{T} D^t . \quad (16)$$

Equation (16) gives the desired relation between the T matrices evaluated with respect to the two sets of coordinate axes. $\hat{T}$ is independent of position and orientation and is, hence, the same for identical scatterers. The matrix T, however, is different if the orientation of the scatterers is not the same.

The rotation matrix D has been given by Edmonds[5] as

$$D_{mm'}^n(\alpha,\beta,\gamma) = e^{im\alpha} d_{mm'}^n(\beta) e^{im'\gamma} , \quad (17)$$

where

$$d^n_{mm'}(\beta)=\left[\frac{(n+m)!\,(n-m)!}{(n+m')!\,(n-m')!}\right]^{1/2}\left(\cos\frac{\beta}{2}\right)^{m+m'} \times\left(\sin\frac{\beta}{2}\right)^{m-m'} P_{n-m}^{(m-m',\,m'+m)}(\cos\beta)\,. \quad (18)$$

In Eq. (18), $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial which can be expressed in terms of the associated Legendre polynomials. Since the T matrix is symmetric for general scatterers, Eq. (16) can be rewritten as

$$T=D\hat{T}D^{-1}\,. \quad (19)$$

Now the value of T averaged over all possible orientations of the scatterer may be written as

$$\langle T_{nm,n'm'}\rangle=\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\,\sin\beta\sum_{m_1m_2}[D^n_{mm_1}(\alpha,\beta,\gamma)\hat{T}_{nm_1,n'm_2}(D^{-1})^{n'}_{m_2m'}(\alpha,\beta,\gamma)]\,, \quad (20)$$

where $\langle\ \rangle$ denotes the average over orientations. From Edmonds,[5] we find that

$$\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\,\sin\beta D^n_{mm_1}(D^{-1})^{n'}_{m_2m'} =\frac{1}{2n+1}\delta_{nn'}\delta_{mm_2}\delta_{m_1m'}\,, \quad (21)$$

which when used in (20) yields the average T matrix given by

$$\langle T_{nm,n'm'}\rangle=\frac{1}{2n+1}\sum_{m_1,m_2}\hat{T}_{nm_1,nm_2}\delta_{mm_2}\delta_{m_1m'}\delta_{nn'}\,. \quad (22)$$

For spheroidal scatterers

$$\hat{T}_{nm_1,nm_2}=\hat{T}_{nm_1,nm_1}\delta_{m_1m_2}\,, \quad (23)$$

so that Eq. (22) becomes

$$\langle T_{nm,n'm'}\rangle=\hat{T}_{nm,nm}\delta_{nn'}\delta_{mm'}\,. \quad (24)$$

IV. PROPAGATION CHARACTERISTICS OF THE COHERENT FIELD

Now that the average over all possible orientations have been performed, the remaining steps are identical to those given in Ref. 1. If Eqs. (9a) and (9b) are multiplied from the left-hand side by T, we obtain a set of coupled equations for the scattered field coefficients B and C alone which are now already averaged over all possible orientations. The average over all possible positions of the scatterers is still to be performed. As in Ref. 1, we assume that the two-scatterer distribution function is given by

$$p(\vec{r}_i,\vec{r}_j)=\begin{cases}1/V, & |\vec{r}_i-\vec{r}_j|\geq 2a\,,\\ 0, & 0\leq|\vec{r}_i-\vec{r}_j|<2a\,,\end{cases} \quad (25)$$

where V is the large but finite volume occupied by the scatterers. Equation (25) implies that there is no interpenetration of the scatterers; otherwise, all correlations are neglected. We may note that from Eq. (25), although the scatterers may be nonspherical, the statistics are spherical. It is not too difficult to generalize to nonspherical statistics by making the exclusion volume conform to the geometry of the scatterers.

In this paper, since we are concerned with random orientation of nonspherical scatterers, it is appropriate to assume spherical statistics. When Eqs. (9a) and (9b) are averaged over all possible positions keeping the ith scatterer fixed, it is seen that we get a hierarchy of equations for the scattered field coefficients with more and more scatterers fixed. As in Ref. 1, we truncate the hierarchy by making Lax's quasicrystalline approximation which states that the neighborhood of every scatterer is the same, so that

$$\langle B^j_{nm}\rangle_{ij}=\langle B^j_{nm}\rangle_j\,. \quad (26)$$

Thus, from Eq. (9), by omitting all details which may be found in Ref. 1 we obtain the average scattered field coefficients as follows:

$$\begin{bmatrix}\langle B^i_{nm}\rangle_i\\ \langle C^i_{nm}\rangle_i\end{bmatrix}=\begin{bmatrix}\langle T^{11}\rangle & \langle T^{12}\rangle\\ \langle T^{21}\rangle & \langle T^{22}\rangle\end{bmatrix}\begin{bmatrix}\langle\psi^i_{n_1m_1}\rangle\\ \langle\chi^i_{n_1m_1}\rangle\end{bmatrix}, \quad (27)$$

where

$$\langle\psi^i_{n_1m_1}\rangle=\frac{2n_1+1}{n_1(n_1+1)}i^{n_1}\frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i}[\delta_{m_1,1}+n_1(n_1+1)\delta_{m_1,-1}] +\frac{1}{V}\sum_{j=1}^{N}{}'\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\int_{V'}[\langle B^j_{n_2m_2}\rangle B^{n_2m_2}_{n_1m_1}(\vec{r}_i-\vec{r}_j) +\langle C^j_{n_2m_2}\rangle C^{n_2m_2}_{n_1m_1}(\vec{r}_i-\vec{r}_j)]d\vec{r}_j \quad (28a)$$

and

$$\langle\chi^i_{n_1m_1}\rangle=\frac{2n_1+1}{n_1(n_1+1)}i^{n_1}\frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i}[\delta_{m_1,1}-n_1(n_1+1)\delta_{m_1,-1}] +\frac{1}{V}\sum_{j=1}^{N}{}'\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\int_{V'}[\langle B^j_{n_2m_2}\rangle C^{n_2m_2}_{n_1m_1}(\vec{r}_i-\vec{r}_j) +\langle C^j_{n_2m_2}\rangle B^{n_2m_2}_{n_1m_1}(\vec{r}_i-\vec{r}_j)]d\vec{r}_j\,. \quad (28b)$$

In Eqs. (28a) and (28b), V' denotes the volume of the medium V excluding a sphere of radius $2a$ centered at O_i. For identical scatterers $\sum_{j=1}^{N'}=N-1$ and $4\pi(N-1)a^3/3V\simeq c$, the volume concentration of scatterers provided N is large enough.

To find the average propagation constant K for the bulk medium, we assume plane-wave solutions for $\langle B\rangle$ and $\langle C\rangle$ with unknown amplitudes Y and Z:

$$\langle B^i_{nm}\rangle_i = Y_{nm} e^{i\vec{K}\cdot\vec{r}_i}, \tag{29a}$$

$$\langle C^i_{nm}\rangle_i = Z_{nm} e^{i\vec{K}\cdot\vec{r}_i}. \tag{29b}$$

For the coherent field $\vec{K}$ is parallel to $\vec{k}$ and hence along the z axis. Substituting Eq. (29) into Eq. (27) and performing the required integrations and invoking the extinction theorem as in Ref. 1, we finally obtain

$$Y_{nm} = \frac{6c}{(k^2 - K^2)} a^2 \sum_{q=|n_1-n_2|}^{|n_1+n_2|} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=-n_1}^{n_1} \sum_{m_2=-n_2}^{n_2} (-1)^{m_2} i^{n_2-n_1} \times \delta_{m_1 m_2} (JH)_q \{ Y_{n_2 m_2} [\langle T^{11}\rangle_{nm,n_1m_1} a(n_2,n_1,q)\, a(m_2,n_2|-m_1,n_1|q) - \langle T^{12}\rangle_{nm,n_1m_1} b(n_2,n_1,q)\, a(m_2,n_2|-m_1,n_1|q,q-1)] + Z_{n_2m_2}[\langle T^{12}\rangle_{nm,n_1m_1} a(n_2,n_1,q)\, a(m_2,n_2|-m_1,n_1|q) - \langle T^{11}\rangle_{nm,n_1m_1} b(n_2,n_1,q)\, a(m_2,n_2|-m_1,n_1|q,q-1)]\} \tag{30a}$$

and

$$Z_{nm} = \cdots, \tag{30b}$$

where Eq. (30b) can be obtained from Eq. (30a) by replacing $\langle T^{11}\rangle$ and $\langle T^{12}\rangle$ by $\langle T^{21}\rangle$ and $\langle T^{22}\rangle$, respectively. The term $(JH)_q$ is given by

$$(JH)_q = 2kaj_q(2Ka)h_q'(2ka) - 2Kah_q(2ka)j_q'(2Ka). \tag{31}$$

The only difference between the equations obtained here for Y and Z and those in Ref. 1 is that T is replaced by $\langle T\rangle$. This is one of the basic advantages of formulating the multiple scattering in terms of a T matrix. The expressions for "a" and "b" occurring in Eq. (30) are given by Cruzan[6] in his paper on translation theorems for vector spherical functions. From Eq. (24), we note that for spheroidal scatterers

$$\begin{bmatrix} \langle T^{11}\rangle & \langle T^{12}\rangle \\ \langle T^{21}\rangle & \langle T^{22}\rangle \end{bmatrix}_{nm,n_1m_1} = \begin{bmatrix} \hat{T}^{11} & \hat{T}^{12} \\ \hat{T}^{21} & \hat{T}^{22} \end{bmatrix}_{nm,nm} \delta_{nn_1}\delta_{mm_1}, \tag{32}$$

so that the summations on n_1, m_1, and m_2 disappear on the right-hand side of Eqs. (30a) and (30b). For a random distribution of spherical scatterers, $\langle T\rangle$ and $\hat{T}$ are both identical and diagonal.

Equations (30a) and (30b) are a set of homogeneous linear coupled equations for the coefficients Y and Z of the coherent electric field in the inhomogeneous medium. For a nontrivial solution, we require that the determinant of the truncated matrix should vanish yielding the dispersion equation for the bulk medium. This may be numerically solved to obtain K as a function of $k=\omega/c$. It is important to note that the scatterer property and geometry are, as yet, unspecified. This information is all contained in the T matrix. Thus, the formalism may be ideal for studying coherent wave propagation through randomly distributed and oriented scatterers of arbitrary shape. In particular, we can use the T-matrix of perfect conductors, dielectric, or two-layered scatterers, all of which have been formulated in the literature. The effective wave number K is now complex $(K_1 + iK_2)$ and frequency dependent. The real part K_1 relates to the coherent phase velocity while the imaginary part K_2 relates to coherent wave attenuation.

V. COHERENT WAVE PHASE VELOCITY AND ATTENUATION

The procedure for computing phase velocity and attenuation is similar to the one presented in Refs. 1 and 7. For a given value of ka, the T matrix of the scatterer is computed. Next, the coefficient matrix C corresponding to Y and Z of Eqs. (30a) and (30b) is formed. We have retained as many as 13 simultaneous equations for Y and Z to obtain proper convergence and the complex determinant of the coefficient matrix is computed using standard Gaussian elimination techniques. A proper T-matrix size is chosen for a given ka to satisfy the unitary and symmetry properties. For example, for $a/b=2$ and $ka=2$, a T-matrix size of 20×20 has been employed. The roots $(K=K_1+iK_2)$ of the resulting transcendental equation are obtained by Müller's complex root searching algorithm. For quick convergence, we start from a low value of ka $(=0.05)$ for which the values obtained from Eqs. (36) and (37) of Ref. 1 served as good initial guesses in the root searching algorithm. For $ka<0.05$, the average wave speed given by Eqs. (36) and (37) is an excellent approximation. The values of ka are increased by small increments of the order

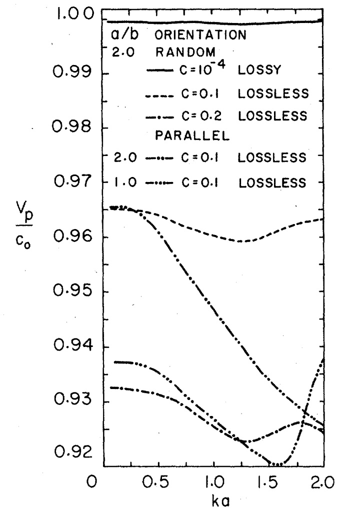

FIG. 3. Normalized phase velocity vs ka for lossy and lossless scatterers for various concentrations c.

of 0.05. The real part K_1 is related to the phase velocity while the imaginary part K_2 is related to attenuation due to geometric dispersion as well as absorption, if any, by the scatterers. We define the normalized phase velocity $V_p/c_0 = k/K_1$, where c_0 is the phase velocity in free space and the coherent attenuation coefficient $S_d = 4\pi K_2/K_1$ so as to make it dimensionless.

In Figs. 3–6, we have plotted phase velocity and attenuation coefficients for spherical and spheroidal dielectric scatterers in free space randomly distributed and randomly oriented. The calculations were performed for both lossless and lossy dielectric scatterers. For lossless scatterers, we assume dielectric constant $\epsilon_r = 3.168$ which

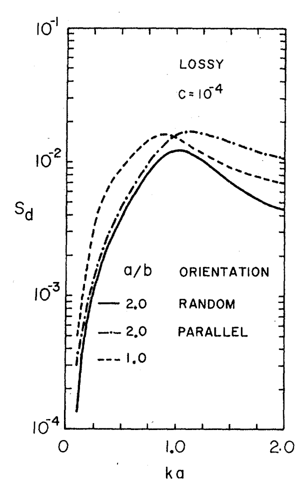

FIG. 4. Attenuation coefficient S_d vs ka for lossy scatterers for $c = 10^{-4}$.

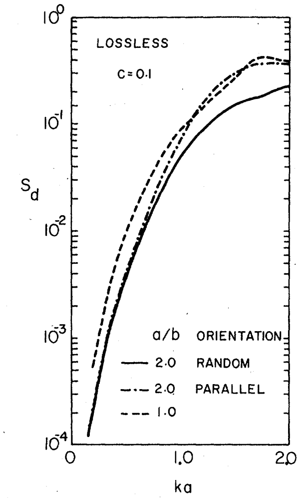

FIG. 5. Attenuation coefficient S_d vs ka for lossless scatters for $c = 0.1$.

corresponds to ice at microwave frequencies. The imaginary part of the dielectric constant for ice is relatively small when compared to the real part; see, for example, Ray (Ref. 8). For the lossy case, we consider complex dielectric constants which correspond to rain particles given by Ray as functions of temperature and frequency. For our calculations, we assume the temperature to be equal to 5 °C.

In Fig. 3, the normalized phase velocity is plotted versus ka for aspect ratios of $a/b = 1.0$ and 2.0 and concentration $c = 10^{-4}$, 0.1, and 0.2. The phase velocity decreases gently as ka increases and tends to increase slightly at higher frequencies. In Fig. 3, we have reproduced the results obtained in Ref. 1 for parallel orientation

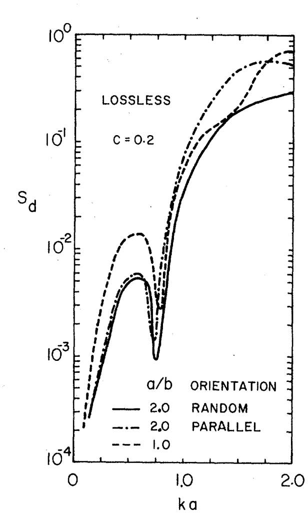

FIG. 6. Attenuation coefficient S_d vs ka for lossless scatterers for $c = 0.2$.

for comparison purposes. The results indicate that there is a significant difference between preferred and random orientations at higher frequencies.

In Fig. 4, we show the attenuation coefficient S_d for lossy dielectric scatterers as a function of ka for concentration $c = 10^{-4}$, where we have plotted the results for spherical and spheroidal $(a/b = 2.0)$ scatterers. In Figs. 5 and 6, we have plotted S_d for lossless spheroidal dielectric scatterers for two different concentrations, $c = 0.1$ and 0.2 and for an aspect ratio $a/b = 2.0$. For comparison purposes we have also reproduced the results of Ref. 1 for parallel orientations. The results indicate that there is a substantial difference in attenuation between preferred and random orientations. It can be seen that for $c = 0.2$, the sharp null is present for both the cases. There is, however, a slight shift in the null towards higher ka for random orientations. The value of the attenuation is lower at the null for random orientation. There is a tendency for the attenuation to increase rapidly again with frequency.

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