Systematics of higher-spin gauge fields

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Free-field theories for symmetric tensor and tensor-spinor gauge fields have recently been obtained which describe massless particles of arbitrary integer or half-integer spin. An independent discussion of these field theories is given here, based on a hierarchy of generalized Christoffel symbols with simple gauge transformation properties. The necessity of certain constraints on gauge fields and parameters is easily seen. Wave equations and Lagrangians are expressed in terms of the Christoffel symbols, and the independent modes of the system are counted in covariant gauges. Minimal-coupling inconsistency and a combined system of higher-spin boson gauge fields interacting with relativistic particles is discussed.

I. INTRODUCTION

Gauge invariance has proved crucial in constructing field theories for particles of spins 1, $\frac{3}{2}$, and 2. The integer-spin cases are known to be associated with fundamental forces in Nature, while spin $\frac{3}{2}$ will be required in the future if Bose-Fermi symmetry is relevant. It is interesting to consider the extension of gauge principles to fields that describe higher-spin particles. Recently, this extension has been obtained for free fields of arbitrary integer[1] and half-integer[2] spins. Bose particles of spin s and Fermi particles of spin $s+\frac{1}{2}$ are described, respectively, by totally symmetric rank-s tensors $\varphi_{\mu_1\mu_2\cdots\mu_s}$ and tensor-spinors $\psi_{\mu_1\mu_2\cdots\mu_s\alpha}$ where α is a spinor index. The fields are subject to certain gauge invariances and constraints (to be discussed below) which are crucial to prove that the theories are free of ghosts and describe precisely two propagating modes of helicity $\pm s$ for bosons or $\pm(s+\frac{1}{2})$ for fermions. The theories were obtained in Refs. 1 and 2 by considering the massless limit of massive higher-spin theories.[3] The same results were rederived[4] using gauge invariance and supersymmetry transformations which relate higher-spin massless fields to known lower-spin cases. The wave equations for spin $\frac{5}{2}$ and 3 were actually obtained long ago,[5] while the spin-$\frac{5}{2}$ system with its gauge invariance was reobtained directly by requiring the absence of ghosts.[6]

In this paper we give a simple self-contained discussion of arbitrary-spin gauge fields based on the gauge transformation laws

$$\delta\varphi_{\mu_1\mu_2\cdots\mu_s}=\sum_{\mu_1}\partial_{\mu_1}\xi_{\mu_2\mu_3\cdots\mu_s}\,, \tag{1.1}$$

$$\delta\psi_{\mu_1\mu_2\cdots\mu_s}=\sum_{\mu_1}\partial_{\mu_1}\epsilon_{\mu_2\mu_3\cdots\mu_s\alpha}\,, \tag{1.2}$$

where $\sum_{\mu_1}$ indicates a symmetrized sum of s terms, and where the transformation parameters $\xi_{\mu_2\cdots\mu_s}$ and $\epsilon_{\mu_2\cdots\mu_s\alpha}$ are totally symmetric rank-$(s-1)$ tensors and tensor-spinors. We define and study the properties of a set of natural quantities, called generalized Christoffel symbols and curvature tensors, with simple gauge transformation properties. The structure of the Christoffel symbols shows that in order to have gauge-invariant second-order (boson) and first-order (fermion) wave equations, one must require that the gauge parameters be traceless, i.e.,

$$\xi_\rho{}^\rho{}_{\mu_4\cdots\mu_s}=0\,, \tag{1.3}$$

$$\gamma^\rho\epsilon_{\rho\mu_3\cdots\mu_s\alpha}=0\,, \tag{1.4}$$

as was found previously. The wave equations are expressed simply in terms of the generalized Christoffel symbols. The independent degrees of freedom of the fields are counted in covariant gauges. For bosons this gives a very natural generalization of the Lorentz and de Donder gauges for spins 1 and 2. For fermions the method involves novel use of "regauge" transformations. Gauge-invariant actions are also expressed in terms of the Christoffel symbols, and the Bianchi identities which express the gauge invariance are obtained.

We find that the free arbitrary-spin gauge fields have transparent and even tantalizing regularities, suggestive of an underlying mathematical (geometric?) structure. Clarification of this structure is related to the question of interactions. We have not been successful in formulating interacting field theories for high-spin gauge fields, and some of the difficulties encountered are summarized here. The systematic structure found for free fields does suggest a general form of the force law and action for the interaction of a relativistic *particle* with a higher-spin Bose gauge field. This gives a combined particle-gauge-field system which is gauge invariant in lowest order and is described below.

II. HIERARCHY OF CHRISTOFFEL SYMBOLS

The purpose of this section is to construct a set of quantities linear in derivatives of the fields $\varphi_{\mu_1\mu_2\cdots\mu_s}$ with simple properties under the gauge transformation law (1.1). They generalize the Christoffel symbols of ordinary gravity. The first-order spin-s Christoffel symbol is

$$\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s}=\partial_\rho\varphi_{\mu_1\cdots\mu_s}-\sum_{\mu}{}^{1}\partial_{\mu_1}\varphi_{\rho\mu_2\cdots\mu_s}\,. \quad (2.1)$$

The variations of the terms on the right-hand side of (2.1) contain two different forms, viz.,

$$\partial_\rho\sum_{\mu}{}^{1}\partial_{\mu_1}\xi_{\mu_2\cdots\mu_s}\text{ and }\sum_{\mu}{}^{2}\partial_{\mu_1}\partial_{\mu_2}\xi_{\rho\mu_3\cdots\mu_s}\,, \quad (2.2)$$

where the notation $\sum_{\mu^2}$ indicates a sum of $\frac{1}{2}s(s-1)$ independent index permutations of the μ_i. Obviously it is not possible to define a gauge-invariant quantity of first order in derivatives, and the relative coefficient in (2.1) was chosen to eliminate the first term in (2.2), so that

$$\delta\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s}=-2\sum_{\mu}{}^{2}\partial_{\mu_1}\partial_{\mu_2}\xi_{\rho\mu_3\cdots\mu_s}\,.$$

For spin 2 this coincides with the linearized Christoffel symbol, and we will see that this choice leads to higher-rank gauge-invariant quantities for general spin.

It is now straightforward to define generalized higher-rank Christoffel symbols $\Gamma^{(m)}_{\rho_1\rho_2\cdots\rho_m;\mu_1\mu_2\cdots\mu_s}$ recursively by

$$\Gamma^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}\equiv\partial_{\rho_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m;\mu_1\cdots\mu_s}-\frac{1}{m}\sum_{\mu}{}^{1}\partial_{\mu_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m;\rho_1\mu_2\cdots\mu_s}\,, \quad (2.3)$$

where the summation includes s independent permutations of the μ_i. The relative coefficient was chosen to give the gauge transformation law

$$\delta\Gamma^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}=(-)^m(m+1)\sum_{\mu}{}^{m+1}\partial_{\mu_1}\cdots\partial_{\mu_{m+1}}\xi_{\rho_1\cdots\rho_m\mu_{m+2}\cdots\mu_s} \quad (2.4)$$

where the "special indices" ρ_i appear only on the gauge parameter. At this point the motivation for the construction is clear; for $m=s$, $\Gamma^{(m)}$ is *gauge invariant*, and we will call it a generalized curvature tensor

$$R_{\rho_1\cdots\rho_s;\mu_1\cdots\mu_s}\equiv\Gamma^{(s)}_{\rho_1\cdots\rho_s;\mu_1\cdots\mu_s}\,. \quad (2.5)$$

The $\Gamma^{(m)}_{\rho_1\rho_2\cdots\rho_m;\mu_1\cdots\mu_s}$ are manifestly symmetric in the "spin" indices μ_i. They are also symmetric in the special indices ρ_i as can be proved by induction on m using the recursion relations to express $\Gamma^{(m)}$ in terms of $\Gamma^{(m-2)}$. Using these symmetry properties and requiring the transformation rules (2.4), one can directly construct the formula

$$\Gamma^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}=\partial_{\rho_1}\cdots\partial_{\rho_m}\varphi_{\mu_1\cdots\mu_s}+\sum_{j=1}^{m-1}(-)^j\binom{m}{j}^{-1}\sum_{\rho}{}^{j}\sum_{\mu}{}^{j}\partial_{\mu_1}\cdots\partial_{\mu_j}\partial_{\rho_{j+1}}\cdots\partial_{\rho_m}\varphi_{\rho_1\cdots\rho_j\mu_{j+1}\cdots\mu_s}$$
$$+(-)^m\sum_{\mu}{}^{m}\partial_{\mu_1}\cdots\partial_{\mu_m}\varphi_{\rho_1\cdots\rho_m\mu_{m+1}\cdots\mu_s}\,, \quad (2.6)$$

where $\binom{m}{j}$ are binomial coefficients and $\sum_{\mu_j}$ indicates a sum over $\binom{s}{j}$ independent permutations of the index set $\{\mu_i\}$ with a similar definition of $\sum_{\rho_j}$. The expression for $\Gamma^{(m)}$ is the unique quantity involving φ with the indicated symmetries and gauge transformation law.

We now consider the generalized curvatures. The pair exchange property

$$R_{\rho_1\cdots\rho_s;\mu_1\cdots\mu_s}=(-)^s R_{\mu_1\cdots\mu_s;\rho_1\cdots\rho_s} \quad (2.7)$$

follows immediately from (2.6), and one may also prove the relation

$$\sum_{\mu}{}^{1}R_{\mu_1\rho_2\cdots\rho_s;\rho_1\mu_2\cdots\mu_s}=-R_{\rho_1\rho_2\cdots\rho_s;\mu_1\mu_2\cdots\mu_s} \quad (2.8)$$

which is a generalization of the well-known cyclicity property for $s=2$.

To gain familiarity with this hierarchy of curvatures we compare with low-spin cases. For $s=1$,

$$R_{\rho;\mu}=\partial_\rho\varphi_\mu-\partial_\mu\varphi_\rho\,, \quad (2.9)$$

which coincides with the well-known Maxwell field strength. For $s=2$, our generalized curvature with symmetric structure is equal to a linear combination of the standard Riemann curvatures, viz.,

$$R_{\rho_1\rho_2;\,\mu_1\mu_2}=\tfrac{1}{2}(R'_{\rho_1\mu_1;\,\rho_2\mu_2}+R'_{\rho_1\mu_2;\,\rho_2\mu_1})\ , \qquad (2.10)$$

where the prime indicates the linearized Riemann tensor. Thus our approach includes the possibility of interactions for spin 2 which would be equivalent to the standard interactions. It is interesting that a linear combination of antisymmetric Riemann tensors leads to the symmetric tensor found by our construction which generalizes directly to higher spin.

One should also expect identities involving derivatives of R which generalize the known Bianchi identities for $s=1$ and $s=2$. Indeed the general identity

$$\begin{aligned}&\partial_\lambda(\Gamma^{(m)}_{\rho\rho_2\cdots\rho_m;\,\mu\mu_2\cdots\mu_s}-\Gamma^{(m)}_{\mu\rho_2\cdots\rho_m;\,\rho\mu_2\cdots\mu_s})\\&\quad+\partial_\rho(\Gamma^{(m)}_{\mu\rho_2\cdots\rho_m;\,\lambda\mu_2\cdots\mu_s}-\Gamma^{(m)}_{\lambda\rho_2\cdots\rho_m;\,\mu\mu_2\cdots\mu_s})\\&\quad+\partial_\mu(\Gamma^{(m)}_{\lambda\rho_2\cdots\rho_m;\,\rho\mu_2\cdots\mu_s}-\Gamma^{(m)}_{\rho\rho_2\cdots\rho_m;\,\lambda\mu_2\cdots\mu_s})\\&=0 \qquad (2.11)\end{aligned}$$

follows easily from (2.6) for $m=s$ it can be written in terms of curvatures and reproduces the standard Bianchi identities for $s=1$ and $s=2$. We should, however, emphasize that identities such as (2.11) have no significant meaning for higher spin when $m<s$, since the quantities involved are not gauge covariant in that case. In fact, one can construct a large variety of such identities, and several of them reproduce the Bianchi identities for the case of low spins.

The reader should note in connection with Sec. IV that all formulas in this section can be understood as being valid for fermion tensor-spinor gauge fields by direct replacement of $\varphi_{\mu_1\cdots\mu_s}(x)$ by $\psi_{\mu_1\cdots\mu_s\alpha}(x)$, and $\xi_{\mu_1\cdots\mu_s}(x)$ by $\epsilon_{\mu_1\cdots\mu_s\alpha}(x)$.

III. ARBITRARY-SPIN BOSON GAUGE FIELDS

Standard free quantum field theory requires linear second-order wave equations for bosons, but the results of the previous section indicate that fully gauge-invariant quantities for rank-s tensor fields necessarily have at least s derivatives. The simple solution to this problem is to require that gauge parameters be traceless, according to (1.3). Then the once-contracted Christoffel symbols (contracted with the Minkowski metric $\eta_{\mu\nu}$)

$$W^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}\equiv\Gamma^{(m+2)}{}_{\rho_1\cdots\rho_m\sigma}{}^{\sigma}{}_{,\,\mu_1\cdots\mu_s} \qquad (3.1)$$

are invariant under the gauge transformation (1.1) with traceless parameters. For the rest of this section, gauge invariance will have this restricted meaning.

It is then immediately suggested that the gauge-invariant second-order differential equation

$$\begin{aligned}W_{\mu_1\cdots\mu_s}&\equiv\Gamma^{(2)\sigma}_{\sigma}{}_{,\,\mu_1\cdots\mu_s}\\&\equiv\Box\varphi_{\mu_1\cdots\mu_s}-\sum_{\mu}{}_1\,\partial_{\mu_1}\partial_\rho\varphi^\rho{}_{\mu_2\cdots\mu_s}\\&\quad+\sum_{\mu}{}_2\,\partial_{\mu_1\mu_2}\varphi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}\\&=0 \qquad (3.2)\end{aligned}$$

be adopted as the basic wave equation for spin s. The consistency of this choice will be studied immediately below. For the moment we note that this equation includes the d'Alembertian, Maxwell, and linearized Ricci equations for $s=0$, 1, and 2, respectively, and can thus be considered a natural generalization of these well-known results. It is also equivalent to the equation found by Fronsdal.[1]

In electromagnetism and linearized gravity, gauge fields which satisfy the basic wave equation are characterized by the field strength $F_{\mu\nu}$ and (conformal) curvature tensor $R'_{\mu\nu\rho\sigma}$ which are gauge invariant. For $s>2$ there are no local gauge-invariant quantities of first or second order in derivatives. Instead a gauge field satisfying (3.2) can be characterized by the third-order quantities which are the components of $W^{(1)}_{\rho,\,\mu_1\cdots\mu_s}$ in (3.1).

The algebraic constraint that fields (for $s\geq 4$) are required to be double traceless, i.e.,

$$\varphi_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s}=0 \qquad (3.3)$$

was found necessary in previous treatments. Such a constraint might be expected because the double trace is gauge invariant (indeed the strongest gauge-invariant constraint possible), and describes lower-spin components of the field. If not eliminated, the field theory would not contain pure spin s, and would very likely have negative-metric ghosts. With this physical motivation in mind we will assume the constraint (3.3) and discuss the key properties of (3.2). Later we will come back to the constraint (3.3) which can actually be motivated strongly within the present formalism; specifically, the wave equation (3.2) can only be derived from a Lagrangian if (3.3) is imposed.

The compatibility of (3.2) must still be checked. To do this we compute the first and second traces of the wave operator, viz.,

$$W_{\rho}{}^{\rho}{}_{\mu_3\cdots\mu_s} = 2\Box\varphi_{\rho}{}^{\rho}{}_{\mu_3\cdots\mu_s} - 2\partial_\rho\partial_\sigma\varphi^{\rho\sigma}{}_{\mu_3\cdots\mu_s} + \sum_{\mu}{}_1 \partial_{\mu_3}\partial_\rho\varphi_\sigma{}^{\sigma\rho}{}_{\mu_4\cdots\mu_s} + \sum_{\mu}{}_2 \partial_{\mu_3}\partial_{\mu_4}\varphi_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s}, \quad (3.4)$$

$$W_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s} = 3\Box\varphi_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s} + 3\sum_{\mu}{}_1 \partial_{\mu_5}\partial_\tau\varphi_{\rho\sigma}{}^{\rho\sigma\tau}{}_{\mu_6\cdots\mu_s} + \sum_{\mu}{}_2 \partial_{\mu_5}\partial_{\mu_6}\varphi_{\rho\sigma\tau}{}^{\rho\sigma\tau}{}_{\mu_7\cdots\mu_s}. \quad (3.5)$$

Since the double trace (3.5) vanishes [but (3.4) does not] if (3.3) is satisfied, the wave equation contains precisely the correct number of conditions to determine the components of the symmetric double-traceless field $\varphi_{\mu_1\cdots\mu_s}$.

We now discuss gauge fixing and the counting of the independent dynamical modes of the field. The covariant gauge condition

$$F_{\mu_2\cdots\mu_s} \equiv \partial_\rho\varphi^{\rho}{}_{\mu_2\cdots\mu_s} - \tfrac{1}{2}\sum_{\mu}{}_1 \partial_{\mu_2}\varphi_{\rho}{}^{\rho}{}_{\mu_3\cdots\mu_s} = 0 \quad (3.6)$$

is appropriate, since it contains precisely as many conditions as the number of independent components of the gauge parameter $\xi_{\mu_2\cdots\mu_s}$. This is so because

$$F_{\rho}{}^{\rho}{}_{\mu_4\cdots\mu_s} = \sum_{\mu}{}_1 \partial_{\mu_4}\varphi_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s} \quad (3.7)$$

vanishes because of (3.3). Use of the gauge condition (3.6) is extremely natural for the wave equation (3.2). If (3.6) is imposed, then (3.2) simply implies the d'Alembertian equation

$$\Box\varphi_{\mu_1\cdots\mu_s} = 0 \quad (3.8)$$

for the constrained fields. Thus the particles described by $\varphi_{\mu_1\cdots\mu_s}$ are massless. Further, the gauge variation of the left side of (3.6) is

$$\delta F_{\mu_2\cdots\mu_s} = \Box\xi_{\mu_2\cdots\mu_s}, \quad (3.9)$$

so that one can still regauge certain components of $\varphi_{\mu_1\cdots\mu_s}$ without affecting the gauge condition (3.6), if the gauge parameters satisfy the d'Alembertian equation. This is sufficient for our purpose, because all the field components already satisfy (3.8) in the gauge (3.6). All of this is familiar for $s=1$ and $s=2$; indeed (3.6) reduces to the well-known Lorentz and de Donder gauge conditions in those cases.

These observations permit us to count the independent dynamical degrees of freedom of the field. Using the fact that a general symmetric s-rank tensor in d dimensions has $\binom{s+d-1}{d-1}$ independent components, we find that the field $\varphi_{\mu_1\cdots\mu_s}$ subject to the constraint (3.3) has $2s^2+2$ independent components, whereas we have s^2 independent gauge conditions. Imposing the gauge conditions, and regauging s^2 components by an appropriate choice of the s^2 regauge parameters leaves exactly 2 degrees of freedom, as is required for the description of a massless particle. Let us denote the transverse components of the field in momentum space by $\varphi_{i_1\cdots i_s}(k)$, where $i_1,\ldots,i_s$ are two-valued indices. It is clear that these components are (re)gauge independent and that (3.6) implies the two-dimensional condition[8]

$$\varphi_{jji_3i_4\cdots i_s}(k) = 0 . \quad (3.10)$$

The number of such transverse, traceless components is exactly 2 for all s. Of course, it is entirely expected that these are the physical components of the field. Since the transverse traceless components are s-fold tensor products of transverse polarization vectors, specifically $\epsilon_{\mu_1}(k,+)\epsilon_{\mu_2}(k,+)\cdots\epsilon_{\mu_s}(k,+)e^{-ik\cdot x}$ and $\epsilon_{\mu_1}(k,-)\epsilon_{\mu_2}(k,-)\cdots\epsilon_{\mu_s}(k,-)e^{-ik\cdot x}$, it is clear that the particles of the gauge field carry helicity $\pm s$. This agrees with existing results.[1,4]

The present discussion of the degrees of freedom of the symmetric tensor gauge field has not involved constraint equations on the initial data. One expects that the gauge field equations include some constraints (with at most first-order time derivatives) as well as evolution equations (with second-order time derivatives). Indeed the following components of the original gauge-invariant field equation (3.2),

$$W_{0j_2\cdots j_s} = 0, \qquad W_{00j_3\cdots j_s} - W_{iij_3\cdots j_s} = 0 \quad (3.11)$$

(where i and j are space components, $i,j=1,2,3$), constitute $\frac{1}{2}s(s-1)$ constraints, respectively. Thus we do have the expected s^2 constraints equal to the number of independent gauge parameters. However, when the gauge condition (3.6) is used the constraints become evolution equations, indeed the simple d'Alembertian equation (3.8). Thus the counting can proceed very simply without paying attention to constraints. In a noncovariant gauge the constraints would be relevant, but then there would be no regauge freedom. Thus one would find the same number of independent components.

We now study Lagrangians which lead, upon variation, to the wave equation (3.2). In our approach it is suggested that Lagrangians should be expressed directly in terms of $\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s}$ and $W_{\mu_1\cdots\mu_s}$ which have simple gauge transformation properties. The results are equivalent to the known form[1] which is bilinear in $\partial_\rho\varphi_{\mu_1\cdots\mu_s}$.

We first look for a Lagrangian of the form[9]

$$\mathcal{L}=\tfrac{1}{2}\varphi_{\mu_1\cdots\mu_s}W^{\mu_1\cdots\mu_s}-a\varphi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}W_\sigma{}^{\sigma\mu_3\cdots\mu_s}, \tag{3.12}$$

which generalizes the Ricci scalar action for $s=2$. Possible additional terms in the ansatz (3.12) involving higher traces can be ignored because of the constraint (3.3). By direct computation using the expression (3.2) for $W_{\mu_1\cdots\mu_s}$ in terms of $\varphi_{\mu_1\cdots\mu_s}$, one finds that gauge invariance uniquely selects the value $a=\frac{1}{8}s(s-1)$. This is also the unique value which leads to the wave equation (3.2). Variation of (3.12) gives[1]

$$W_{\mu_1\mu_2\cdots\mu_s}-\tfrac{1}{2}\sum_\mu{}_2\,\eta_{\mu_1\mu_2}W_\rho{}^\rho{}_{\mu_3\cdots\mu_s}=0\,. \tag{3.13}$$

Contraction with $\eta^{\mu_1\mu_2}$ tells us that $W_\rho{}^\rho{}_{\mu_3\cdots\mu_s}=0$, and upon resubstitution in (3.13), one finds (3.2). There is no Lagrangian involving $\varphi_{\mu_1\cdots\mu_s}$ and derivatives which yields (3.2) directly. The intermediate trace procedure is unavoidable.

One subtlety of higher-spin gauge field Lagrangians is that free variation with respect to $\varphi_{\mu_1\cdots\mu_s}$ is incorrect in principle because of the double-traceless condition (3.3). One should actually use projection operators to ensure that the field variation $\delta\varphi_{\mu_1\cdots\mu_s}$ also satisfies the same double-traceless condition. The effect of the projection operators would then lead to additional terms in (3.13) involving second- and higher-order traces of $W_{\mu_1\cdots\mu_s}$. Since such traces vanish self-consistently [see (3.5)] when (3.3) is satisfied, these complications can be ignored, and the naive variation of (3.12) leads to the correct wave equation.

The gauge variation of the action (3.12) implies a Bianchi identity which generalizes the contracted Bianchi identity of gravitation. Specifically, the traceless projection of the divergence of (3.13) vanishes identically

$$\partial_\rho W^\rho{}_{\mu_2\cdots\mu_s}-\tfrac{1}{2}\sum_\mu{}_1\,\partial_{\mu_2}W_\rho{}^\rho{}_{\mu_3\cdots\mu_s}=0\,, \tag{3.14}$$

as can be verified by direct computation.

Let us briefly discuss what happens if one tries to derive (3.2) from a Lagrangian for general tensor fields $h_{\mu_1\cdots\mu_s}$, which are not subject to the double-traceless constraint (3.3). In this case the most general gauge invariant Lagrangian is

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}h_{\mu_1\cdots\mu_s}W^{\mu_1\cdots\mu_s}\\&-\tfrac{1}{8}s(s-1)h_\rho{}^\rho{}_{\mu_3\cdots\mu_s}\\&\times\Big(W_\sigma{}^{\sigma\mu_3\cdots\mu_s}-\sum_\mu{}_2\,\partial^{\mu_3}\partial_{\mu_4}h_{\sigma\tau}{}^{\sigma\tau\mu_5\cdots\mu_s}\Big)+\cdots\,,\end{aligned} \tag{3.15}$$

where omitted terms involve second- and higher-order traces on the fields (which are gauge invariant). The associated wave equation is

$$\begin{aligned}&W_{\mu_1\cdots\mu_s}-\tfrac{1}{2}\sum_\mu{}_2\,\eta_{\mu_1\mu_2}\\&\times\Big(W_\rho{}^\rho{}_{\mu_3\cdots\mu_s}-\sum_\mu{}_2\,\partial_{\mu_3}\partial_{\mu_4}h_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s}\Big)+\cdots=0\,.\end{aligned} \tag{3.16}$$

We then note that the term $\eta_{\mu_1\mu_2}\partial_{\mu_3}\partial_{\mu_4}h_{\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_6}$ cannot be absorbed in a higher-order trace of W, neither can it be canceled by the omitted terms which are proportional to at least two metric tensors $\eta_{\mu\nu}$. Therefore (3.16) would imply (3.2) as well as a set of extra independent equations on the second- and higher-order traces of the fields. However, we would then obtain more independent equations than contained in (3.16), which shows that in order to derive Eq. (3.2) as the universal wave equation from an action principle requires the constraint (3.3) on the fields.

The final topic of this section is to express the Lagrangian (3.12) in terms of first-order Christoffel symbols. By partial integration of (3.12) and combining terms one finds the equivalent Lagrangian (valid for $s\geq 2$)

$$\mathcal{L}=\frac{1}{2(s-1)}\Gamma^{\rho_1\mu_1\cdots\mu_s}[s\Gamma_{\mu_1,\rho\mu_2\cdots\mu_s}-(s-2)\Gamma_{\rho,\mu_1\cdots\mu_s}]$$
$$+\frac{s}{2(s-1)}\Gamma_\rho{}^{,\rho\mu_2\cdots\mu_s}[(s-2)\Gamma_\sigma{}^{,\sigma}{}_{\mu_2\cdots\mu_s}-(s-1)\Gamma_{\mu_2,\sigma}{}^\sigma{}_{\mu_3\cdots\mu_s}]$$
$$+\tfrac{1}{8}s(s-2)\Gamma^\rho{}_{,\sigma}{}^{\sigma\mu_3\cdots\mu_s}(\Gamma_{\rho,\tau}{}^\tau{}_{\mu_3\cdots\mu_s}-\Gamma_{\mu_3,\tau}{}^\tau{}_{\rho\mu_4\cdots\mu_s})$$
$$+\tfrac{1}{16}s(s-1)(s-2)\Gamma_{\rho,\sigma}{}^{\rho\sigma\mu_4\cdots\mu_s}\Gamma_{\tau,\lambda}{}^{\tau\lambda}{}_{\mu_4\cdots\mu_s}\,. \quad (3.17)$$

For $s=2$ this Lagrangian coincides with a form known for linearized gravitation. For $s=1$ the standard form of the Maxwell Lagrangian is already of this type. Since (3.12) can easily be written in terms of $\varphi_{\mu_1\cdots\mu_s}$ and $\partial_\rho\Gamma_{\sigma,\mu_1\cdots\mu_s}$, it is very likely that (3.12) and (3.17) can be combined to give a first-order formulation of arbitrary-spin boson gauge fields in which $\varphi_{\mu_1\cdots\mu_s}$ and $\Gamma_{\rho,\mu_1\cdots\mu_s}$ are treated as independent fields. One technical problem is to study the effect of the constraints on Γ, such as

$$\Gamma_{\rho,\sigma}{}^{\rho\sigma}{}_{\mu_4\cdots\mu_s}=\tfrac{1}{2}\Gamma_{\mu_4,\rho\sigma}{}^{\rho\sigma}{}_{\mu_5\cdots\mu_s}\,, \quad (3.18)$$

implied by (3.3) in the variation with respect to Γ.

IV. ARBITRARY-SPIN FERMION GAUGE FIELDS

We need gauge-invariant first-order wave equations to describe fermions. This is not possible according to the analysis of Sec. II with general tensor-spinor gauge parameters $\epsilon_{\mu_1\cdots\mu_s\alpha}(x)$, but it does become possible if one simple constraint is imposed. If gauge parameters are required to have vanishing contraction with a Dirac γ matrix as in (1.4), then the γ-contracted Christoffel symbols

$$Q^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}\equiv\gamma^\sigma\Gamma^{(m+1)}_{\sigma\rho_1\cdots\rho_m;\mu_1\cdots\mu_s} \quad (4.1)$$

are invariant.

In this section the term gauge invariance refers to the transformation (1.2) with parameters constrained by (1.4). Further, we use the terms single trace, double trace, etc., to refer to single, double, etc., γ contractions, as, for example, in

$$\gamma^\rho\psi_{\rho\mu_2\cdots\mu_s}\,,$$
$$\gamma^\sigma\gamma^\rho\psi_{\rho\sigma\mu_3\cdots\mu_s}=\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}\,, \quad (4.2)$$
$$\gamma^\tau\gamma^\sigma\gamma^\rho\psi_{\rho\sigma\tau\mu_4\cdots\mu_s}=\gamma^\sigma\psi_\rho{}^\rho{}_{\sigma\mu_4\cdots\mu_s}\,.$$

The discussion in this section will be organized parallel to the previous discussion of boson gauge fields. Topics which are direct generalizations of their boson analogs will be discussed very briefly and more space will be devoted to questions which are more complicated for fermions than for bosons.

Gauge invariance of the contracted Christoffel symbols suggests that the first-order differential equation

$$Q_{\mu_1\cdots\mu_s}\equiv\gamma^\rho\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s}$$
$$=\partial\!\!\!/\psi_{\mu_1\cdots\mu_s}-\sum_{\mu}{}_1\,\partial_{\mu_1}\gamma^\rho\psi_{\rho\mu_2\cdots\mu_s} \quad (4.3)$$

be adopted as the basic wave equation describing a massless fermion of spin $s+\frac{1}{2}$. This equation is equivalent to that found by Fang and Fronsdal[2] and has been noted by Curtright.[4] It is a natural extension of the Dirac equation for spin $\frac{1}{2}$ and an equation equivalent[7] to the Rarita-Schwinger equation for spin $\frac{3}{2}$. For $s\geq 2$ a gauge field satisfying (4.3) can be characterized by the components of $Q^{(1)}_{\rho,\mu_1\cdots\mu_s}$ which involve second derivatives, and are all gauge invariant.

Single and double traces of $\psi_{\mu_1\cdots\mu_s}$ vary under the gauge transformation (1.2) [with (1.4)], but the triple trace is gauge invariant. Since it would otherwise represent lower-spin components of the fields, we impose the triple-traceless constraint (relevant for spin $\geq\frac{7}{2}$)

$$\gamma^\sigma\psi_{\sigma\rho}{}^\rho{}_{\mu_4\cdots\mu_s}=0\,, \quad (4.4)$$

which was also found earlier.[2] Since the wave operator $Q_{\mu_1\cdots\mu_s}$ is triple traceless (but not double traceless) if (4.4) is satisfied, the wave equation contains the appropriate number of independent components.

Precisely as for bosons, some of the components of the wave equation (4.3) can be viewed as constraints on the initial data, rather than evolution equations. Indeed, the analog of (3.11) is

$$Q_{0j_2\cdots j_s} - \gamma_0\gamma^i Q_{ij_2\cdots j_s} = 0 \ . \tag{4.5}$$

These equations contain only spatial derivatives, and indeed comprise $2s(s+1)$ constraints, which is the number of components of the traceless gauge parameters $\epsilon_{\mu_2\cdots\mu_s\alpha}$. These constraints play a role in the discussion of the degrees of freedom in the initial-value problem for $\psi_{\mu_1\cdots\mu_s}$ (in contradistinction to the analogous problem for bosons) because the fermion gauge condition does not involve derivatives and cannot (as it did for bosons) convert constraints into evolution equations.

A suitable covariant gauge condition for fermion fields is

$$F_{\mu_2\cdots\mu_s} \equiv \gamma^\rho\psi_{\rho\mu_2\cdots\mu_s} - \frac{1}{2s}\sum_{\mu}{}_1\gamma_{\mu_2}\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s} = 0 \ , \tag{4.6}$$

which is traceless and therefore contains $2s(s+1)$ independent conditions. Further, the gauge variations of $F_{\mu_1\cdots\mu_s}$ is

$$\delta F_{\mu_2\cdots\mu_s} = \not\partial\epsilon_{\mu_2\cdots\mu_s} - \frac{1}{s}\sum_{\mu}{}_1\gamma_{\mu_2}\partial_\rho\epsilon^\rho{}_{\mu_3\cdots\mu_3} \tag{4.7}$$

so that the condition (4.6) allows further regauge transformations with parameters $\epsilon_{\mu_2\cdots\mu_s}$ for which (4.7) vanishes. Equation (4.7) is traceless when $\epsilon_{\mu_2\cdots\mu_s}$ is traceless ($\gamma^\rho\epsilon_{\rho\mu_3\cdots\mu_s}=0$) and defines an initial-value problem containing no constraints in which tracelessness is preserved in time. Regauging therefore allows removal of $2s(s+1)$ degrees of freedom.

The number of dynamical components of the field can now be counted. The number of components of the triple traceless $\psi_{\mu_1\cdots\mu_s}$ is $6s^2+6s+4$. From this, one must subtract the number of constraints (4.5), the number of gauge conditions (4.6), and the number of regauge possibilities, which represent $2s(s+1)$ components each. This leaves 4 degrees of freedom for all s; more specifically, the freedom of four complex functions for a "Dirac" fermion field, and four real functions for a "Majorana" field.[8]

At this point the dynamical components of the field $\psi_{\mu_1\cdots\mu_s}$ have been counted accurately, but the situation is not yet optimal because the field equation in the class of gauges defined by (4.6) is

$$\not\partial\psi_{\mu_1\cdots\mu_s} - \frac{1}{2s}\sum_{\mu}{}_2(\gamma_{\mu_1}\partial_{\mu_2}+\gamma_{\mu_2}\partial_{\mu_1})\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s} = 0 \tag{4.8}$$

rather than the Dirac equation. The situation can be remedied by using part of the regauge freedom to make a further gauge transformation to eliminate $\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}$. The gauge variation of $\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}$ is

$$\delta\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s} = 2\partial_\rho\epsilon^\rho{}_{\mu_3\cdots\mu_s} \ . \tag{4.9}$$

Given a field satisfying (4.8), with nonvanishing double trace, we make the regauge transformation with parameters

$$\epsilon_{\mu_2\cdots\mu_s}(x) = \frac{-1}{2s}\int d^4y\, D(x-y)\not\partial_y \times\sum_{\mu}{}_1\gamma_{\mu_2}\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}(y) \ , \tag{4.10}$$

where $D(x-y)$ is a Green's function of the d'Alembertian operator. One can check using (4.8) that the gauge parameters $\epsilon_{\mu_2\cdots\mu_s}(x)$ defined by (4.10) are traceless, and that they do satisfy the regauge condition (4.7) and eliminate $\psi_\rho{}^\rho{}_{\mu_3\cdots\mu_s}$.

Thus, after the regauging above, one finds that the gauge condition (4.6) has become

$$\gamma^\rho\psi_{\rho\mu_2\cdots\mu_s} = 0 \ , \tag{4.11}$$

and the remaining (traceless) field components now satisfy the Dirac equation,

$$\not\partial\psi_{\mu_1\cdots\mu_s} = 0 \ , \tag{4.12}$$

showing that the field describes massless particles. There is still further regauge freedom with traceless gauge parameters which satisfy

$$\not\partial\epsilon_{\mu_2\cdots\mu_s} = 0 \ . \tag{4.13}$$

Note that the original regauge parameters (4.10) were not unique, since any traceless solution of (4.13) could be added without changing the properties needed.

The initial-value problem for the fermion gauge field is now in a form which is naively expected, and more or less analogous to the boson gauge field problem. Since the use of an intermediate regauging is new, it is a useful check to recount the degrees of freedom of the initial-value problem defined by (4.11)–(4.13). The traceless spinor field $\psi_{\mu_1\cdots\mu_s}$ has $2s^2+6s+4$ independent components. The Dirac equation (4.12) for a traceless tensor-spinor field comprises both the evolution equations and constraints. The latter are still expressed by Eq. (4.5) [with (4.11) taken into account]. Hence, we still have $2s(s+1)$ constraint equations on the initial data. By analogous argument one finds that the regauge problem (4.13) contains $2s(s-1)$ constraint equations on the gauge parameters. Thus the number of degrees of freedom in the regauge initial-value problem is the number of components of a tracelsss $\epsilon_{\mu_2 \cdots \mu_s}$, namely $2s(s+1)$, minus the constraints, leaving $4s$ independent functions. Since the original regauge degrees of freedom were $2s(s+1)$, we see that $2s(s-1)$ were used in the first regauging to eliminate $\psi_\rho{}^\rho{}_{\mu_3 \cdots \mu_s}$. As expected from (4.9) this is precisely the number of components contained in the divergence $\partial_\rho \epsilon^\rho{}_{\mu_3 \cdots \mu_s}$ or in $\psi_\rho{}^\rho{}_{\mu_3 \cdots \mu_s}$. The recount of the degrees of freedom is now straightforward. We counted $2s^2+6s+4$ components in the traceless field $\psi_{\mu_1 \cdots \mu_s}$, from which we subtract the $2s(s+1)$ constraints and the $4s$ degrees of freedom eliminated by the second regauge transformation, to find four dynamically independent components.

In momentum space the positive-frequency forms $\epsilon_{\mu_1}(k,+)\epsilon_{\mu_2}(k,+)\cdots\epsilon_{\mu_s}(k,+)u_\alpha(k,+)e^{-ik\cdot x}$ and $\epsilon_{\mu_1}(k,-)\epsilon_{\mu_2}(k,-)\cdots\epsilon_{\mu_s}(k,-)u_\alpha(k,-)e^{-ik\cdot x}$, constructed from transverse polarization vectors $\epsilon_\mu(k,\pm)$ and positive- or negative-helicity solutions $u_\alpha(k,\pm)$ of the Dirac equation, satisfy (4.11) and (4.12), and are regauge independent. Obviously they carry helicity $\pm(s+\frac{1}{2})$. For Dirac field one adds the corresponding negative-frequency solutions to find the four complex dynamically independent components of the field. For a Majorana field the positive- and negative-frequency parts are complex conjugate to each other, leaving four independent real functions. The results reported here are in agreement with previous work based on a variety of methods.[2,4,6,10]

We now follow the discussion for boson gauge fields, and study Lagrangians which lead to the wave equation (4.3). We look for Lagrangians of the form[11]

$$\mathcal{L} = -\tfrac{1}{2}\bar{\psi}_{\mu_1 \cdots \mu_s} Q^{\mu_1 \cdots \mu_s} + a\bar{\psi}_{\rho\mu_2 \cdots \mu_s}\gamma^\rho\gamma_\sigma Q^{\sigma\mu_2 \cdots \mu_s} + b\bar{\psi}_\rho{}^\rho{}_{\mu_3 \cdots \mu_s} Q_\sigma{}^{\sigma\mu_3 \cdots \mu_s}\ , \tag{4.14}$$

which are directly expressed in terms of the γ-contracted Christoffel symbol and further traces. The values $a=\frac{1}{4}s$ and $b=\frac{1}{8}s(s-1)$ are uniquely determined by gauge invariance. The Lagrangian (4.14) coincides with the Lagrangian of Fang and Fronsdal,[2] and gives the wave equation

$$Q_{\mu_1 \cdots \mu_s} - \tfrac{1}{2}\sum_\mu {}_1 \gamma_{\mu_1}\gamma^\rho Q_{\rho\mu_2 \cdots \mu_s} - \tfrac{1}{2}\sum_\mu {}_2 \eta_{\mu_1\mu_2} Q_\rho{}^\rho{}_{\mu_3 \cdots \mu_s} = 0\ . \tag{4.15}$$

This leads to (4.3) after contraction with γ matrices. There is no Lagrangian leading directly to (4.3) without intermediate trace operations. As for bosons, we have the situation that naive variation of (4.14), ignoring the triple-traceless constraint (4.4), gives the correct wave equation, because triple- and higher-order traces of $Q_{\mu_1 \cdots \mu_s}$ vanish self-consistently for triple-traceless fields. Further, the triple-traceless constraint is required if the wave equation (4.3) is to be obtained from a Lagrangian.

The gauge invariance of the Lagrangian implies again a Bianchi identity. The γ-traceless part of the divergence of the wave operator vanishes identically,[10,12]

$$\partial_\rho Q^\rho{}_{\mu_2 \cdots \mu_s} - \tfrac{1}{2}\not\partial\gamma_\rho Q^\rho{}_{\mu_3 \cdots \mu_s} - \tfrac{1}{2}\sum_\mu {}_1 \partial_{\mu_2} Q_\rho{}^\rho{}_{\mu_3 \cdots \mu_s} = 0\ , \tag{4.16}$$

as can be verified by direct computation.

V. TOWARD INTERACTIONS

The construction of free arbitrary-spin gauge field theories is an important simplification of the previous literature on higher spin. Consistent interacting theories of higher spin remain a difficult problem and one of crucial importance for applications of these theories. It was hoped that the present natural formulation in terms of generalized Christoffel symbols and curvatures would be helpful in constructing interactions. This hope has not yet been realized. In this section we review some of the difficulties concerning higher-spin interactions, and present a novel attempt involving interactions with a *relativistic particle* which is partially successful.

One tries to formulate an interacting field theory of higher spin as a gauge theory of several coupled or self-coupled fields, which is invariant under transformations which generalize the free-field gauge transformations. These transformations would presumably not commute and would generate a non-Abelian symmetry group of the theory. Unfortunately, there is a general theorem[13] that a unitary relativistic S matrix cannot be invariant under transformations whose charges carry spin $\geq \frac{3}{2}$ (which would couple to gauge fields of spin $\geq \frac{5}{2}$). This is confirmed by specific attempts to construct high-spin invariance algebras, which have required that the charges vanish in a positive-metric Hilbert space of particle states.[14,15] Such a requirement is also consistent with the conclusion of Ref. 16 that the coupling of spin $\geq \frac{5}{2}$ particles must vanish in the soft-particle limit. Nevertheless, it may be possible to evade these negative conclusions. For instance, one possibility is that higher-spin gauge-invariant Lagrangians exist, such that spontaneous breakdown of the unacceptable symmetries occurs leaving an S matrix with an admissable symmetry group.

The difficulty of higher-spin interactions may be illustrated by the minimal-coupling inconsistency problem. Here one covariantizes the free-field Lagrangian and transformation rules using electromagnetic or gravitational derivatives. One hopes[4] that the resulting interacting Lagrangian will remain invariant under the combined gauge transformations, or that the validity of the covariantized wave equations [(3.2) or (4.3)] or Bianchi identities [(3.14) or (4.16)] allows nontrivial electromagnetic or gravitational field configurations. For spin $\frac{5}{2}$ it is known that the minimal coupling to electromagnetism[6] or gravity[12,15] fails.

A more direct way to find the inconsistency is to check whether the variation of the covariant wave equation with covariant transformation laws still vanishes in a nontrivial background field. For fermions coupled to gravity, one has the covariant equations

$$Q^{\text{cov}}_{\mu_1\cdots\mu_s} \equiv \gamma^\rho\Big(D_\rho\psi_{\mu_1\cdots\mu_s} - \sum_\mu{}_1 D_{\mu_1}\psi_{\rho\mu_2\cdots\mu_s}\Big) = 0\,, \tag{5.1}$$

$$\delta\psi_{\mu_1\cdots\mu_s} = \sum_\mu{}_1 D_{\mu_1}\epsilon_{\mu_2\cdots\mu_s}\,, \tag{5.2}$$

$$D_\rho\psi_{\mu_1\cdots\mu_s} = (\partial_\rho + \tfrac{1}{2}\omega_\rho{}^{ab}\sigma_{ab})\psi_{\mu_1\cdots\mu_s} - \sum_\mu{}_1 \Gamma_{\rho\mu_1}{}^\sigma\psi_{\sigma\mu_2\cdots\mu_s}\,, \tag{5.3}$$

where $\omega_\rho{}^{ab}$ and $\Gamma_{\mu\nu}{}^\rho$ are the standard gravitational vierbein connection and Christoffel symbol. The variation of (5.1) using (5.2) and (1.4) is (ignoring torsion)

$$\begin{aligned}\delta Q^{\text{cov}}_{\mu_1\cdots\mu_s} &= \gamma^\rho\sum_\mu{}_1 (D_\rho D_{\mu_1} - D_{\mu_1}D_\rho)\epsilon_{\mu_2\cdots\mu_s}\\ &= \tfrac{1}{2}\gamma^\rho\sum_\mu{}_1 R_{\rho\mu_1}{}^{ab}\sigma_{ab}\epsilon_{\mu_2\cdots\mu_s} - \gamma^\rho\sum_\mu{}_2 (R_{\rho\mu_1\mu_2}{}^\sigma + R_{\rho\mu_2\mu_1}{}^\sigma)\epsilon_{\sigma\mu_3\cdots\mu_s}\\ &= \sum_\mu{}_1 R_{\nu\mu_1}\gamma^\nu\epsilon_{\mu_2\cdots\mu_s} - \sum_\mu{}_2 (R_{\rho\mu_1\mu_2}{}^\sigma + R_{\rho\mu_2\mu_1}{}^\sigma)\gamma^\rho\epsilon_{\sigma\mu_3\cdots\mu_s}\,.\end{aligned} \tag{5.4}$$

The Ricci identity appears in the first line, and is expressed in terms of the (full nonlinear) Riemann tensor in the second line. Using some γ-matrix algebra and the cyclicity property of the Riemann tensor, one sees that the "spinor terms" contain only the Ricci tensor $R_{\nu\mu_i}$, while the full Riemann tensor remains in the "vector terms." Thus gauge invariance of the higher-spin fermion wave equation fails in the presence of a nontrivial gravitational field. Spin $\frac{3}{2}$ is an exception, since the vector terms are absent. Therefore, as is well known in supergravity, the spin-$\frac{3}{2}$ wave equation is consistent in a background geometry which satisfies $R_{\mu\nu}=0$, the vacuum Einstein field equation. Finally we note that the symmetrized Riemann tensor which appears in the vector terms in (5.4) coincides in linear order with the spin-2 generalized curvature tensor of Sec. II [see (2.10)].

For higher-spin bosons one can perform a similar exercise and find that the variation of the gravitationally covariant wave equation contains the full uncontracted Riemann tensor and its derivatives (in addition to Ricci tensor terms), so that the higher-spin gauge invariance fails in a nontrivial gravitational field.

Our search for higher-spin interactions and for possible applications of the generalized Christoffel symbols led us to consider the interaction of high-spin boson fields with relativistic particles rather than fields. The motivation was the form of the equation of motion of a particle with mass m and electric charge e in an electromagnetic and gravitational field, namely

$$\frac{d^2}{d\tau^2}x^\mu(\tau) = \frac{e}{m}\Gamma^\mu{}_{,\nu}(x(\tau))\frac{dx^\nu(\tau)}{d\tau} + 2\kappa\Gamma^\mu{}_{,\nu_1\nu_2}(x(\tau))\frac{dx^{\nu_1}(\tau)}{d\tau}\frac{dx^{\nu_2}(\tau)}{d\tau}\,, \tag{5.5}$$

where $x^\mu(\tau)$ describes the space-time trajectory as a function of the proper time τ. The Christoffel symbols are defined as in (2.1); the spin-1 symbol coincides with the electromagnetic field tensor, and for spin 2 we recover the standard Christoffel symbol of gravity (modulo a factor -2κ) if we relate the metric tensor $g_{\mu\nu}$, used to raise and lower indices in curved space, to the spin-2 field $\varphi_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\varphi_{\mu\nu} \tag{5.6}$$

(κ is the gravitational coupling constant). Equation (5.5) immediately suggests that we consider generalized laws of force of the form

$$e_s\Gamma^\mu{}_{,\nu_1\cdots\nu_s}(x(\tau))\frac{dx^{\nu_1}(\tau)}{d\tau}\cdots\frac{dx^{\nu_s}(\tau)}{d\tau} \tag{5.7}$$

for the interaction of a particle with a high-spin gauge field. The spin-s coupling constant is e_s.

It is not difficult to find actions that lead to particle equations of motion containing terms like (5.7). For example, the action

$$I=\int_{-\infty}^{\infty} dp\left[\left(g_{\mu\nu}(x(p))\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{1/2} + e_s\left(g_{\rho\sigma}(x(p))\frac{dx^\rho}{dp}\frac{dx^\sigma}{dp}\right)^{(1-s)/2} \times \varphi_{\mu_1\cdots\mu_s}(x(p))\frac{dx^{\mu_1}}{dp}\cdots\frac{dx^{\mu_s}}{dp}\right], \quad (5.8)$$

where $x^\mu(p)$ parametrizes the particle trajectory in terms of a parameter p, yields an equation of motion containing (5.5) and (5.7) and an additional term involving velocities and the acceleration. The action (5.8) is invariant both under reparametrization of the particle trajectory, and under standard general-coordinate transformations. Surprisingly, it is also invariant through first order in e_s under the combined transformations

$$\delta\varphi_{\mu_1\cdots\mu_s}(x)=\sum_{\mu_1} D_{\mu_1}\xi_{\mu_2\cdots\mu_s}\,,$$
$$\delta x^\mu = -e_s\, s(s-1)\left(g_{\rho\sigma}(x(p))\frac{dx^\rho}{dp}\frac{dx^\sigma}{dp}\right)^{1-s/2} \times \xi^\mu{}_{\nu_3\cdots\nu_s}\frac{dx^{\nu_3}}{dp}\cdots\frac{dx^{\nu_s}}{dp}\,, \quad (5.9)$$

which are a covariant extension of the gauge transformation (1.1) and an accompanying transformation of the particle coordinates, invariant under reparametrization of the particle trajectory.

To second order in e_s the variation of (5.8) under (5.9) involves many powers of the velocity, suggestive of the interaction with an even higher-spin gauge field. However, we have not been able to obtain a complete gauge-invariant system which generalizes (5.8), even when many higher spins were considered. One should also note that lowest-order invariance holds for general gauge fields and transformation parameters, without the need for imposing constraints such as (1.3) and/or (3.3). There is, therefore, no natural match with the required constraints on free gauge fields.

Note added: The possibility that deformations of gauge groups provide an underlying mathematical structure for higher-spin interactions has been discussed.[1] The d'Alembertian and Dirac equations for higher-spin fields were also obtained[17] by covariant quantization with suitable auxiliary fields.

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$$N_{\text{boson}} = 2\binom{s+d-5}{s-1}+\binom{s+d-5}{s}\,,$$
$$N_{\text{fermion}} = 2^{d/2}\binom{s+d-4}{s}\,,$$

for the number of degrees of freedom for bosons of spin s and fermions of spin $s+\frac{1}{2}$, respectively. This is precisely the number of components of a transverse, symmetric, traceless rank-s tensor, or tensor-spinor.

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