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Path-dependent formulation of gauge theories and the origin of field copies in the non-Abelian case

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The introduction of path-dependent potentials, as new relevant objects, allows us to relate in a natural way the path-dependent formulation of gauge theories to other point-function formulations. The absence of field copies in a very general class of gauges is proven in a rather obvious geometric way.

I. INTRODUCTION

As is well known in the electromagnetic case, once some field strength $f_{\mu\nu}$ which satisfies the Bianchi identities is given in a simple connected region, the potential A_μ is locally determined up to gauge transformations. In the non-Abelian case, even in a simply connected region, the field strength $f^a_{\mu\nu}$ does not determine the gauge field.[1-4]

It is the purpose of this paper to comment on the source of the ambiguities in the non-Abelian case and to identify the fundamental object that determines the gauge field. The discussion is presented in the framework of a path-dependent formulation of gauge theories which extends the Mandelstam formulation[5] with the introduction of path-dependent potentials as new relevant objects. As a result of the discussion it will become apparent that the well-known Wu-Yang[6] formulation may be considered as a particular case of this extended path-dependent formulation.

Since the relevant elements of the path-dependent formulation are already present in the electromagnetic case, we shall discuss it in some detail. Let $f_{\mu\nu}(x)$ be some field strength satisfying the Bianchi identities

$$f_{\mu\nu,\lambda}(x)+f_{\nu\lambda,\mu}(x)+f_{\lambda\mu,\nu}(x)=g_{\mu\nu\lambda}(x)\,, \tag{1}$$

where $g_{\mu\nu\lambda}(x)$ is the dual of the magnetic current. We may then introduce the path-dependent potential

$$\mathsf{A}_\mu(x,P(x)) \equiv \int_{P(x)}^{x} dx'^\lambda f_{\lambda\mu}(x')\,, \tag{2}$$

where $P(x)$ is a continuous path going from spatial infinity to the point x. Let us also introduce the differential operator, which we shall refer to hereafter as the parallel derivative, given by

$$\partial_{:\mu}F(x,P(x))=\lim_{\epsilon\to 0}\frac{F(x+\epsilon,P_:(x+\epsilon))-F(x,P(x))}{\epsilon}\,, \tag{3}$$

where $F(x,P(x))$ is any path-dependent functional, ϵ some displacement in the "μ" direction, and $P_:(x+\epsilon)$ is obtained from $P(x)$ by parallel transport to the point $x+\epsilon$.

From (2) and (3) it is easy to obtain

$$\partial_{:\nu}\mathsf{A}_\mu(x,P(x))=\int_{P(x)}^{x} dx'^\lambda f_{\lambda\mu,\nu}(x') \tag{4}$$

and using the Bianchi identities it is possible to recover the point function $f_{\mu\nu}(x)$ from the path-dependent potential by the familiar expression

$$f_{\mu\nu}(x)=\mathsf{A}_{\nu:\mu}(x,P(x))-\mathsf{A}_{\mu:\nu}(x,P(x)) + \int_{P(x)}^{x} dx'^\lambda g_{\mu\nu\lambda}(x')\,. \tag{5}$$

This construction shows in a transparent way the connection between the Bianchi identities, the field strength, and the potential. It is, in fact, the coordinate version of a natural geometric construction which represents nonclosed k-forms in a manifold by $(k-1)$-forms in the manifold of paths of the original manifold. The decomposition (5) arises as a very general homotopy associated with the construction. A more detailed discussion of this formulation of electromagnetism will be reported elsewhere.[7]

Since parallel derivatives are obviously commutative, the addition of a parallel four-gradient $\Lambda_{:\mu}(x,P(x))$ to the path-dependent potential leaves invariant the decomposition (5). Nevertheless, the path-dependent potential given in (2) is a gauge-invariant object which contains all the relevant information of the problem.

Ordinary point-function potentials may be obtained from the path-dependent potential by selecting a fixed path for every point x. For instance, let us consider some fixed reference path C going from spatial infinity to the origin of coordinates. We may then select for every point x the path $C(x)$ obtained from C by parallel transport from the origin of coordinates to the point x. The point-function object which arises from this path specialization will be denoted as

$$A_\mu(x)=\mathsf{A}_\mu(x,P(x))|_{P(x)=C} \text{ or } A_\mu(x)=\mathsf{A}_\mu(x,C)\,. \tag{6}$$

It is an immediate consequence of the definition given in (3) that

$$\partial_{;\nu} \mathsf{A}_\mu(x, P(x))|_{P(x)=C} = A_{\mu,\nu}(x) \quad (7)$$

and then it follows from (5) that $A_\mu(x)$ is an ordinary potential with its string singularities related to C in an obvious geometrical way. For instance, in the case of a static monopole at the origin of coordinates, one may obtain from (6) the Dirac potential with a generic string[8]:

$$\vec{A}(\vec{x}) = -\int_{C'} d\vec{a} \times \vec{B}(\vec{x} - \vec{a}), \quad (8)$$

where the string C' is symmetric to the reference path C with respect to the origin.

The Wu-Yang[6] formulation of electromagnetism may be considered as a particular case of the path-dependent formulation. To see this, let us consider some finite covering of space-time by open sets $D_1, D_2, \ldots, D_N$ with nonvanishing intersections D_{ij}. Let us assign a reference path C_i to each region D_i and then let us fix parallel paths $C_i(x)$ for each point in this region. Within a region D_i one must then have the usual representation of the field strength in terms of the ordinary point-function potential $\mathsf{A}_\mu(x, C_i)$. For x belonging to some intersection region D_{ij}, the potential is doubly defined but it is immediately seen that the two potentials are related by a gauge transformation. In fact, a simple calculation using the Bianchi identities and the Stokes theorem shows that for x in D_{ij} one has

$$\mathsf{A}_\mu(x, C_i) - \mathsf{A}_\mu(x, C_j) = \partial_\mu\left(\tfrac{1}{2}\int_{\Sigma_{ij}}^{x} d\sigma^{\nu\lambda} f_{\nu\lambda}\right), \quad (9)$$

where Σ_{ij} is some two-dimensional surface drawn from $C_j(x)$ to $C_i(x)$. A variation of this surface will change the integral in (9) by the amount of the magnetic charges enclosed in the variation. Hence, the potentials in adjacent regions are related in the intersection by the gradient of a multiple-valued function, a familiar characteristic of the Wu-Yang formulation.

Since the reference paths C_i may be changed at will, the whole Wu-Yang construction is equivalent to a path-dependent potential which only exhibits the degrees of freedom associated with the path when going from region to region in some covering of space-time. One may then consider more refined coverings and in the limit when the regions reduce to points the Wu-Yang potential reduces to the path-dependent potential given in (2).

II. THE NON-ABELIAN CASE

The generalization of these ideas to the non-Abelian case is rather straightforward. Following Bialynicki-Birula[9] and Mandelstam[10] one may introduce the path-dependent field strength

$$\mathsf{F}^a_{\mu\nu}(x, P(x)) = \left\{\exp_{\text{ordered}}\left[-g\int_{P(x)}^{x} dx'^\lambda A^j_\lambda(x')\,\mathsf{C}_j\right]\right\}^a_b f^b_{\mu\nu}(x)$$
$$\equiv \{\Phi_{\infty P x}\}^a_b f^b_{\mu\nu}(x), \quad (10)$$

where C_j are the structure constants, $f^b_{\mu\nu}(x)$ is the ordinary field strength, and $A^j_\lambda(x)$ is the point-function connection. One may also introduce the well-known Mandelstam[4] covariant derivative D_μ which satisfies

$$D_\lambda \mathsf{F}^a_{\mu\nu}(x, P(x)) = \{\Phi_{\infty P x}\}^a_b(\partial_\lambda \delta^b_c - gA^j_\lambda C_j{}^b{}_c) f^c_{\mu\nu}(x). \quad (11)$$

In this formulation the Yang-Mills equations may be written as

$$D^\nu \mathsf{F}^a_{\mu\nu}(x, P(x)) = J^a_\mu(x, P(x)), \quad (12a)$$

$$D_\lambda \mathsf{F}^a_{\mu\nu}(x, P(x)) + D_\nu \mathsf{F}^a_{\lambda\mu}(x, P(x)) + D_\mu \mathsf{F}^a_{\nu\lambda}(x, P(x)) = 0. \quad (12b)$$

The path-dependent field strength is a gauge-invariant object since it may be obtained as the phase factor around a closed loop beginning and ending at spatial infinity and enclosing some infinitesimal surface $\delta\sigma_{\mu\nu}$ spanned by the vectors u_μ and v_μ at the point x:

$$\{\Phi_{\infty P x, x+u, x+u+v, x+v, x P\infty}\}^a_b = \delta^a_b - \tfrac{1}{2} g\, \mathsf{F}^c_{\mu\nu}(x, P(x))\delta\sigma^{\mu\nu} C_{cb}{}^a. \quad (13)$$

Hence, for asymptotically vanishing gauge transformations, the field strength is gauge invariant.

Although the Mandelstam formulation is a complete description of gauge theories in terms of fundamental gauge-invariant objects, it is not clear how to relate it to other formulations given in terms of point-function objects. For this purpose it is more convenient to use the parallel derivative operators introduced in (3) and to define the gauge-invariant path-dependent potential as

$$\mathsf{A}^a_\mu(x, P(x)) \equiv \int_{P(x)}^{x} dx'^\lambda \mathsf{F}^a_{\lambda\mu}(x', \not{P}(x')), \quad (14)$$

where $\not{P}(x')$ is that portion of P leading to x'.

From (3) and (10) it is then possible to write the general relation between the Mandelstam covariant derivative and the parallel derivative:

$$D_\lambda \mathsf{F}^a_{\mu\nu}(x, P(x)) = [\partial_{;\lambda}\delta^a_b - g\mathsf{A}^c_\lambda(x, P(x)) C_c{}^a{}_b]\, \mathsf{F}^b_{\mu\nu}(x, P(x)), \quad (15)$$

which allows us to write the path-dependent Yang-Mills equations in the form

$$[\partial_{:\nu}\delta^a_b - g\mathsf{A}^c_\nu(x,P(x))C_c{}^a{}_b]\mathsf{F}^{b\nu}_\mu(x,P(x)) = J^a_\mu(x,P(x)), \tag{16a}$$

$$[\partial_{:\nu}\delta^a_b - g\mathsf{A}^c_\nu(x,P(x))C_c{}^a{}_b]\bar{\mathsf{F}}^{a\nu}_\mu(x,P(x)) = 0\,. \tag{16b}$$

From the Bianchi identities (12b) and the relation (15) between the Mandelstam and the parallel derivatives, it is possible to obtain the familiar decomposition

$$\mathsf{F}^a_{\mu\nu}(x,P(x)) = \mathsf{A}^a_{\nu:\mu}(x,P(x)) - \mathsf{A}^a_{\mu:\nu}(x,P(x)) + gC^a{}_{bc}\mathsf{A}^b_\mu(x,P(x))\mathsf{A}^c_\nu(x,P(x)), \tag{17}$$

which holds for asymptotically vanishing path-dependent field strengths. These expressions show clearly the connection between the path-dependent formulation and the ordinary formulations in terms of point-function objects. In fact, if one selects parallel paths for every point in some region, it follows from (15)–(17) and (7) that $\mathsf{A}^a_\mu(x,C)$ is an ordinary potential associated with the point-function field strength $\mathsf{F}^a_{\mu\nu}(x,C)$.

The path-dependent potential is a gauge-invariant object since it may be constructed from the gauge-invariant path-dependent field strength. However, different families of parallel paths lead to different point-function potentials which are related by a gauge transformation.

The relation of the path-dependent formulation to the Wu-Yang formulation is now readily obtained following exactly the same ideas discussed in the electromagnetic case.

The origin of field copies may now be understood in geometrical terms. Let us consider some ordinary point-function field strength $f^a_{\mu\nu}(x) \equiv \mathsf{F}^a_{\mu\nu}(x,C)$. The ordinary potential $\mathsf{A}^a_\mu(x,C)$ cannot be constructed in a unique way since (14) also requires the information contained in $\mathsf{F}^a_{\mu\nu}(x', C\!\!\!/(x'))$, which is in general different from $f^a_{\mu\nu}(x') \equiv \mathsf{F}^a_{\mu\nu}(x', C(x'))$. If any only if the reference path C is a straight line does $C(x')$ coincide with $C\!\!\!/(x')$ and may $\mathsf{A}^a_\mu(x,C)$ be constructed explicitly from $f^a_{\mu\nu}(x)$. But if one selects a family of rectilinear parallel paths along some unitary spatial vector n_μ, the potential will be given by

$$\mathsf{A}^a_\mu(x,n) = \int_n^x dx'^\lambda\, \mathsf{F}^a_{\lambda\mu}(x',n) \tag{18}$$

and since dx'^λ is collinear with n^λ, one must have

$$n^\mu \mathsf{A}^a_\mu(x,n) = 0\,. \tag{19}$$

Hence, selecting parallel rectilinear paths is equivalent to considering ordinary potentials within the gauge where the subsidiary condition (19) holds. Then this geometrical argument suggests that within such rectilinear gauges, field copies should not exist. In fact, this result has been recently proven by Halpern[11,12] in the case of the axial gauge.

It is convenient at this point to present an independent proof of this result based on the following remark. If some ordinary potential $A^a_\mu(x)$ associated with some field strength $f^a_{\mu\nu}(x)$ is given, the path-dependent field strength and the path-dependent potential can be uniquely constructed using (10) and (14), respectively. Selecting then some parallel family of paths C, one may determine the ordinary objects $f'^a_{\mu\nu}(x) \equiv \mathsf{F}^a_{\mu\nu}(x,C)$ and $A'^a_\mu(x) \equiv \mathsf{A}^a_\mu(x,C)$. It is evident from the same construction that $f'^a_{\mu\nu}$ and A'^a_μ are gauge transforms of $f^a_{\mu\nu}$ and A^a_μ, respectively.

Let us now assume that some point function $f^a_{\mu\nu}(x)$ is given with two nonequivalent potentials $A^a_{\mu 1}(x)$ and $A^a_{\mu 2}(x)$, both satisfying the condition (19). It follows then immediately from (10) that the two path-dependent field strengths that may be constructed satisfy $\mathsf{F}^a_{\mu\nu}(x,n) = f^a_{\mu\nu}(x)$ and hence both lead to the ordinary potential $\mathsf{A}^a_\mu(x,n)$. According to the above remark, both $A^a_{\mu 1}$ and $A^a_{\mu 2}$ must be related to $\mathsf{A}^a_\mu(x,n)$ by a gauge transformation and then they must also be gauge transforms of each other. We have then a contradiction.

This geometrical proof leads immediately to a wide generalization of Halpern's result. Let us consider some spatial unitary vector field $n_\mu(x)$ which defines a three-parameter family of infinite nonintersecting tangent lines which have the same asymptotic direction at spatial infinity. Then in the gauge

$$n^\mu(x) A^a_\mu(x) = 0\,, \tag{20}$$

field copies cannot exist.

The proof is straightforward. By assumption, the differential equations

$$\frac{\partial}{\partial y_3} z^\mu(y_0,y_1,y_2,y_3) = n^\mu[z(y)] \tag{21}$$

define a nonsingular change of coordinates $y^\mu = y^\mu(x)$. In the y system the condition (20) reads

$$A^a_3(y) = 0 \tag{22}$$

and, under such a condition, it has been proven that field copies cannot exist. Moreover, it is easy to convince oneself that the proof was in fact independent of the chosen coordinates. Hence, the absence of field copies in the gauge (22) implies also the absence of field copies in the much more general class of gauges given by condition (20).

The asymptotic condition imposed on $n^\mu(x)$ is necessary to guarantee that the paths in space-time have a definite and unique asymptotic direction and are then acceptable in the path-dependent formulation.

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