PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Parity, charge-conjugation, and time-reversal violation in models of relativistic quantum field theory

M. Flato
Physique Mathématique, Faculté des Sciences Mirande, 21000 Dijon, France

R. Rączka
Institute for Nuclear Research, 00-681 Warsaw, Hoza 69, Poland

(Received 29 January 1979)

An analysis of dynamical, S-operator, and spontaneous violations of space parity P, charge conjugation C, time reversal T, and PCT operations in models of quantum field theory is presented. In particular, it is shown that P and T can be dynamically violated in $P(\Phi)_2$ and sine-Gordon models and that P is dynamically violated in the Federbush model. An interesting example of spontaneous P violation in $\lambda\Phi^4{}_2$ theory is discussed. The connection between θ and PCT symmetries in quantum field theory is elucidated and it is shown that in some $P(\Phi)_2$ and Federbush models, though θ remains a good symmetry, PCT symmetry is dynamically violated.

I. INTRODUCTION

In this work we consider several quantum-field-theoretical models in which various discrete symmetries are violated. The main purpose of this analysis is to clarify the meaning of discrete symmetries operations in cases when these symmetries are not conserved. The need for such an analysis follows from the fact that some authors who investigated this problem both theoretically and experimentally[1,2] reached the conclusion that in the case of parity violation the corresponding parity operators are not defined on the carrier Hilbert space of physical states. In particular, Lee (cf. Ref. 1, Sec. 1.4) observes that since P and $e^{it\mathcal{H}}$ are representatives of a geometrical space inversion and a time translation, if defined, they must necessarily commute. Thus if one checks out that for a given dynamical system $[P,\mathcal{H}]\neq 0$, this will mean that P is not defined. This point of view, however, brings up many difficult questions. In particular, if the parity operator cannot be defined, then how can we determine its eigenvalues and eigenfunctions and in what sense is the parity violated?

We think that a definition of nonconserved physical quantities should reflect the conditions in which these quantities are measured. Thus in our opinion it is not worthwhile to consider the commutator $[P,\mathcal{H}]$ because we have no possibilities to measure this quantity during the time evolution of the dynamical system. What is actually known or measured are the parities, energies, momenta, spins, etc., of incoming and outgoing particles. Thus one should introduce the concept of observed quantities for incoming particles such as parity P_{in}, charge conjugation C_{in}, and even time reversal T_{in}, etc., and verify, if possible, that during the time evolution these quantities are or are not conserved. Since in every quantum field theory satisfying the Haag-Ruelle assumption[3] the space H_{in} of incoming particles is a subspace of the Hilbert space H of the physical state vectors, the "in" observables are always well defined. In addition, if asymptotic completeness is satisfied then $H_{\text{in}}=H=H_{\text{out}}$. Thus defining physical observables in terms of in or out quantities furnishes a good framework for analysis of conservation or violation of a given physical quantity.

We now give a precise definition of conservation (or violation) of P, C, T, and PCT operations. We shall always assume that we have a quantum field theory satsifying the Haag-Ruelle assumptions so that the corresponding interacting fields have then asymptotic in and out fields and $H_{\text{in}}\subseteq H$, $H_{\text{out}}\subseteq H$. We begin with the space parity operator.

The analysis of the concept of parity violation suggests the following three distinct *definitions:*

(i) We say that the *parity operator* P_{in} defined in the carrier Hilbert space $H_{\text{in}}\subseteq H$ is *dynamically violated* if

$$P_{\text{in}}\Phi_{\text{in}}(t,\vec{x})P_{\text{in}}^{-1}=\eta_P\Phi_{\text{in}}(t,-\vec{x})$$

but (1.1)

$$P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1}\neq\eta_P\Phi(t,-\vec{x}),$$

where η_P equals $+1$ or -1.

(ii) We say that *parity* P_{in} is *not S-conserved* if

$$[P_{\text{in}},S]\neq 0. \tag{1.2}$$

(iii) Let $\mathcal{L}_T(\Phi,\psi)$ be a total Lagrangian for, say, a spinless boson Φ and a fermion ψ field, which is invariant under space reflections:

$$\Phi(t,\vec{x})\to\eta_P\Phi(t,-\vec{x}),\quad \psi(t,\vec{x})\to\eta_P'\gamma^0\psi(t,-\vec{x}), \tag{1.3}$$

$\eta_P=\pm 1$, $\eta_P'=\pm 1,\pm i$. Let the carrier space H be

$$H = \bigoplus_{\omega} H_\omega , \tag{1.4}$$

where ω are pure Wightman states which define a given superselection sector (~phase). We say that *space parity is spontaneously broken* if there exists a Wightman state ω which is not invariant under space reflection.

Remarks.

(1) It may occur that parity is dynamically violated but still S-conserved. This indeed holds in several models, e.g., in the Federbush model (for certain values of the coupling constant) considered in Sec. III. On the other hand, if parity is dynamically conserved, then it is also S-conserved.

(2) The case (iii) holds, e.g., even in the case of $\lambda\Phi^4_2$ theory. In this case, the Wightman states which are not invariant under space reflection are simply vacuums or Wightman states defining soliton and antisoliton superselection sectors (cf. Sec. II C).

(3) We have stated the above definitions for parity. But, of course, the same definitions hold for any *other discrete symmetry*, e.g., *charge conjugation* C_{in}, $C_{\text{in}}P_{\text{in}}$, G parity, etc. Only in the case of T and PCT symmetries must some additional care be taken in defining them because of the antilinearity of these operations, as follows.

(i) The *time-reflection* symmetry T_{in} is *dynamically violated* if

$$T_{\text{in}}\Phi_{\text{in}}(t,\vec{x})T_{\text{in}}^{-1} = \eta_T \Phi_{\text{out}}(-t,\vec{x}) ,$$

but (1.5)

$$T_{\text{in}}\Phi(t,\vec{x})T_{\text{in}}^{-1} \neq \eta_T \Phi(-t,\vec{x}) ,$$

where $\eta_T = \pm 1$ for a Poincaré scalar field. For tensor and fermion fields appropriate matrix factors on the right-hand side will appear, e.g., $i\gamma^1\gamma^3$ for the Dirac field in R^4.

(ii) We say that *time reflection* is *S-conserved* if

$$T_{\text{in}} S T_{\text{in}}^{-1} = S^* . \tag{1.6}$$

It seems that there is some confusion with respect to the concept of PCT symmetry. Namely, the analysis of this notion in the axiomatic quantum field theory shows that PCT symmetry is identified with the operation θ^{-1} defined by the formula [c.f., e.g., Ref. 4, Eq. (3.67)]

$$\theta^{-1}\psi_{\alpha\dot\beta}(x)\theta = (-1)^{j} i^{F(\psi)} \psi^*_{\alpha\dot\beta}(-x) , \tag{1.7}$$

where (j,k) is the index characterizing irreducible representations $D^{(j,k)}$ of SL$(2,C)$ and

$$F(\psi) = \begin{cases} 0 \text{ for } j+k \text{ even} \\ 1 \text{ for } j+k \text{ odd} . \end{cases}$$

This operation should be contrasted with the operation PCT, which is the product of the individual P, C, and T. Obviously, since we are in the Wightman framework, θ is always a symmetry of the theory for any dynamics; on the other hand, the operation PCT might or might not be a symmetry and, in general, may be chosen to be different from θ. We show this explicitly in Sec. V for $P(\Phi)_2$ models and for the Federbush model.

The simplest illustration of nonuniqueness of discrete symmetry operations may be given already in the case of the free Poincaré spinless neutral field Φ_{in}. In this case, one may introduce in the Fock space H_{in} associated with Φ_{in} two parity operators ${}^{+}P_{\text{in}}$ and ${}^{-}P_{\text{in}}$ given by the formula [cf. Ref. 5 (i), Sec. V and Ref. 5 (ii)]

$${}^{\eta_P}P_{\text{in}} = \exp\left\{-\frac{i\pi}{2}\left[N_{\text{in}} - \eta_p \int \frac{d^3\vec{p}}{p_0}\, a^*_{\text{in}}(\vec{p}) a_{\text{in}}(-\vec{p})\right]\right\}$$

$$\eta_P = \pm 1 \tag{1.8}$$

(N_{in} = particle-number operator) which satisfies in H_{in} the following conditions:

$${}^{\eta_P}P_{\text{in}}\Phi_{\text{in}}(t,\vec{x})\,{}^{\eta_P}P_{\text{in}}^{-1} = \eta_P \Phi_{\text{in}}(t,-\vec{x}) , \quad \eta_P = \pm 1 . \tag{1.9}$$

Since in this case $\Phi_{\text{in}} = \Phi_{\text{out}}$, we have in H_{in} two time-reflection operators ${}^{\eta}{}^{T}T, \eta_T = \pm 1$, with the property

$${}^{\eta_T}T_{\text{in}}\Phi_{\text{in}}(t,\vec{x})\,{}^{\eta_T}T_{\text{in}}^{-1} = \eta_T \Phi_{\text{in}}(-t,\vec{x}) , \quad \eta_T = \pm 1 . \tag{1.10}$$

These time-reflection operators can be realized explicitly in H_{in} by the formula

$${}^{\eta_T}T_{\text{in}} = \exp\left\{-\frac{i\pi}{2}\left[N_{\text{in}} - \eta_T \int \frac{d^3\vec{p}}{p_0}\, a^*_{\text{in}}(\vec{p}) a_{\text{in}}(-\vec{p})\right]\right\}K , \tag{1.11}$$

where K is the operator of complex conjugation. We recall that according to Watanabe's classification[6] there can be four kinds of scalar (or tensor) fields depending on their properties with respect to space and time reflections, namely the scalars of the 0-kind (space scalar, time scalar), 1-kind (space pseudoscalar, time scalar), 2-kind (space pseudoscalar and time scalar), and 3-kind (space pseudoscalar and time pseudoscalar). The formulas (1.9) and (1.10) demonstrate that there are no intrinsic space and time parities associated with a spinless neutral free quantum field. A given free field Φ_{in} in Fock space can be converted into a scalar field of arbitrary Watanabe kind at our will by a proper choice of ${}^{\eta_P}P_{\text{in}}$ and ${}^{\eta_T}T_{\text{in}}$ inversion operators.

Since C_{in} in the considered case is trivial, we have in H_{in} the following four PCT operators: ${}^{+}P_{\text{in}}C_{\text{in}}{}^{+}T_{\text{in}}$, ${}^{+}P_{\text{in}}C_{\text{in}}{}^{-}T_{\text{in}}$, ${}^{-}P_{\text{in}}C_{\text{in}}{}^{+}T_{\text{in}}$, and ${}^{-}P_{\text{in}}C_{\text{in}}{}^{-}T_{\text{in}}$. Clearly, the first and the last will coincide with the θ^{-1} operation when acting on the field Φ_{in}; however, the second and third will be distinct from it. It seems that, physically, nothing forces us to correlate a space parity η_P with a time parity η_T for a given field. Thus there is no reason to restrict ourselves in concrete cases to θ^{-1} symmetry and disregard the PCT operators. If we considered a charged Poincaré scalar field, the situation would be even more complex and we would have in H_{in} eight PCT operators with four of them distinct from θ^{-1}. In the case of the interacting fields, the situation is even more drastic since θ is always dynamically conserved, whereas PCT might be dynamically violated (See Sec. V).

The problem of the choice of phases for discrete operations is an old one.[7-9] Evidently, it was recognized early that the choice of phases of P, C, or T is to a large extent arbitrary. In particular, Wick, Wightman, and Wigner[9] have shown that superselection rules such as charge, angular momentum, or baryon-number conservation forbid us to determine uniquely the phases of P, C, or T operations.

However, implicitly or explicitly, most of these works supposed definite transformation properties of the interacting fields under discrete operations. In this work it will be shown that in cases in which discrete symmetries are violated, all *a priori* imagined transformation properties of the interaction fields break down.

It should also be evident that though in most cases the choice of the phases is arbitrary and does not have an *a priori* physical justification, such a choice usually has physical consequences.

If we admit arbitrariness of phases, say, for the T operation as, for instance, is implied by superselection rules, then it is as good to have a theory (supposing PC is violated) in which T is violated and $PCT=\theta^{-1}$ is conserved as it is to have a theory in which T is conserved and PCT (different from θ^{-1}) is violated. Under such conditions no experimental verification of PCT conservation can exist. Or else, if, say, in refined experiments in the K_0 system one is able to get some information on the phase of T, it might still be that experiment can distinguish between these last two possibilities.

The most important conclusion following from our work consists in the fact that the Lagrangian and the field equation can dynamically violate parity, charge conjugation, time reversal, or PCT symmetry, but still these operations might be S-conserved. Thus the standard analysis of discrete symmetries violation based on glancing at the interacting Lagrangian is not satisfactory and might lead to incorrect conclusions. Similar remarks concern in fact also continuous symmetries. One has really to solve the problem and determine the asymptotic fields Φ_{in} and Φ_{out} or scattering operator S in order to verify symmetry violation or conservation. However, in general, it is quite difficult to achieve this last point because of technical difficulties associated with rigorous and nonperturbative construction of the S matrix itself for a given model.

In this work in Sec. II we analyze discrete symmetries violation in $P(\Phi)_2$ and sine-Gordon models. We show that for some $P(\Phi)_2$ interactions, parity is well defined and dynamically violated. We also show an interesting fact that although these models provide examples of local quantum field theory, the product PCT might be dynamically violated. In Sec. II we also give a striking example of spontaneous parity violation in a $\lambda\Phi^4{}_2$ theory.

In Sec. III, we analyze the Federbush model. We show that, for certain values of the coupling constant, the parity is dynamically violated but is, however, S-conserved. We also show that charge conjugation is dynamically and S-conserved. Consequently, $P_{in}C_{in}$ is dynamically violated but is S-conserved.

Finally, in Sec. IV, we show that time-reversal symmetry may be dynamically violated in some $P(\Phi)_2$ models, and in Sec. V we show that the product PCT may be dynamically violated in some $P(\Phi)_2$ models and is dynamically violated in the Federbush model. We also show explicitly that the θ operation in these models is always a symmetry, in agreement with a general theorem (cf. Ref. 4, Sec. 4.3). Several critical remarks and suggestions conclude our article.

This work is a continuation of a previous one where the same problems were analyzed in the case of classical relativistic nonlinear field theory of boson and fermion fields in R^4 (Ref. 10).

II. PARITY VIOLATION IN BOSON FIELD THEORIES

A. $P(\Phi)_2$ models

Consider the weakly coupled $:P(\Phi)_2:$ quantum field theory with $P(\xi)=\sum_{n=3}^{2N}\lambda_n\xi^n$, for $\lambda_{2n}>0$. It was shown that in these models all Wightman axioms are satisfied and there exists an isolated one-particle mass hyperboloid.[11] Hence the Haag-Ruelle scattering theory assures that there exist well-defined asymptotic fields Φ_{in} and Φ_{out} as well as a well-defined scattering operator S:

$$S^{-1}\Phi_{in}S=\Phi_{out}\,. \tag{2.1}$$

We wish to show in this model, similarly as in classical nonlinear relativistic field theory, that the parity operator is well defined but, in general, dynamically violated. We define the in and out parity operators in a conventional manner:

$$\begin{aligned}{}^{\eta P}P_{in}\Phi_{in}(t,\vec{x}){}^{\eta P}P_{in}{}^{-1}&=\eta_P\Phi_{in}(t,-\vec{x})\,,\\ {}^{\eta P}P_{out}\Phi_{out}(t,\vec{x}){}^{\eta P}P_{out}{}^{-1}&=\eta_P\Phi_{out}(t,-\vec{x})\,,\end{aligned} \tag{2.2}$$

where $P_{\text{in,out}}$ is given by formula (1.8) and $\eta_P = \pm 1$.

Proposition 2.1. Consider $:P(\Phi)_2:$ theory with $P'(-\xi) \neq -P'(\xi)$. If for the Φ_{in} field $\eta_P = 1$, then P_{in} is dynamically and S-conserved. If $\eta_P = -1$, then P_{in} is dynamically violated.

Proof. For a weakly coupled $:P(\Phi)_2:$ theory all Wightman axioms are satisfied and there exists an isolated one-particle mass hyperboloid. Hence, following Hepp, one may write a Yang-Feldman (YF) equation in the form[3]

$$\Phi(t,\vec{x}) = \Phi_{\text{in}}(t,\vec{x}) + \Delta_R * :P'(\Phi):(t,\vec{x}) . \tag{2.3}$$

The explicit form of $:P'(\Phi):$ in Minkowski space was derived by Schrader[12] (cf. also Ref. 13):

$$:P'(\Phi): = \sum_{n=3}^{2N} n\lambda_n :\Phi^{n-1}: . \tag{2.4}$$

The ordering : : may be taken with respect to the free or interacting measure. The YF equation is defined on the Hepp domain D_{in} of the so-called nonoverlapping vectors.[3] This space consists of all vectors in H of the form

$$|\psi_{\text{in}}(f^n)\rangle = \prod_{k=1}^{n} \Phi_{\text{in}}(f_k)|0\rangle , \quad f^n = (f_1, \ldots, f_n) , \tag{2.5}$$

where the Fourier transform $\tilde{f}_i$ of f_i is an element of the Schwartz space $\mathcal{S}(G_m)$ with

$$G_m = \{p \in R^2,\ 0 < p_0 < (\vec{p}^2 + 4m^2)^{1/2}\} .$$

The supports of $\tilde{f}_i$ are mutually nonoverlapping in velocity space, i.e., for $p_i \in \operatorname{supp}\tilde{f}_i$ we have

$$\frac{1}{\omega_i}\vec{p}_i \neq \frac{1}{\omega_j}\vec{p}_j \quad \text{for } i \neq j, \quad \omega_i = (\vec{p}_i^{\,2} + m_i^{\,2})^{1/2} .$$

It follows from (2.5) that D_{in} is P_{in} invariant.

Let B be a bounded invertible operator in H_{in} such that $BD_{\text{in}} \subset D_{\text{in}}$ and $B\Phi(f)B^{-1} = \Phi(f_{\tilde{B}})$, with $f_{\tilde{B}}$ defined by a coordinate transformation $\tilde{B}$ in R^2 inducing B. Then it follows from the definition of $:\Phi^n:$ that

$$B:\Phi^n:(x)B^{-1} = :(B\Phi B^{-1})^n:(x) .$$

Consider first the case $\eta_P = 1$. Then acting on (2.3) by the P_{in} operator from the left and using P_{in} invariance of D_{in}, we obtain on D_{in} the following:

$$P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1}(P_{\text{in}}|\psi_{\text{in}}(f^n)\rangle) = [\Phi_{\text{in}}(t,-\vec{x}) + \Delta_R * :P'(P_{\text{in}}\Phi P_{\text{in}}^{-1}):(t,\vec{x})]P_{\text{in}}|\psi_{\text{in}}(f^n)\rangle .$$

This equation in the variable $\tilde{\Phi}(t,\vec{x}) = P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1}$ takes the form

$$\tilde{\Phi}(t,\vec{x}) = \Phi_{\text{in}}(t,-\vec{x}) + \Delta_R * :P'(\tilde{\Phi}):(t,\vec{x}) ,$$

which coincides with (2.3) taken for $\Phi(t,-\vec{x})$. Thus $\tilde{\Phi}(t,\vec{x})$ and $\Phi(t,-\vec{x})$ satisfy the YF equation with the same initial condition and the same form of the current $P'(.)$. Consequently, $\tilde{\Phi}$ coincides with Φ, i.e.,

$$P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1} = \Phi(t,-\vec{x}) .$$

Thus P_{in} is dynamically and S-conserved.

Let now $\eta_P = -1$. Then on the domain D_{in} we have

$$P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1} = -\Phi_{\text{in}}(t,-\vec{x}) + \Delta_R * :P'(P_{\text{in}}\Phi P_{\text{in}}^{-1}):(t,\vec{x}) . \tag{2.6}$$

If we suppose that P_{in} is dynamically conserved, i.e.,

$$P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1} = -\Phi(t,-\vec{x}) ,$$

then (2.6) becomes

$$\Phi(t,\vec{x}) = \Phi_{\text{in}}(t,\vec{x}) + \Delta_R * [- :P'(-\Phi):](t,\vec{x}) . \tag{2.6$'$}$$

Taking the difference of (2.3) and (2.6′) we obtain

$$\Delta_R * \sum_{n=1}^{N-1} (2n+1)\lambda_{2n+1} :\phi^{2n}:(x) = 0 .$$

Passing with this equality to momentum space and using the fact that the Fourier transform $\tilde{\Delta}_R(p) \neq 0$, we obtain the following operator equality

$$\sum_{n=1}^{N-1} (2n+1)\lambda_{2n+1} :\Phi^{2n}:(p) = 0 ,$$

which evidently does not hold true in H by Schrader's analysis[12] if $P'(-\xi) \neq -P'(\xi)$, hence $P_{\text{in}}\Phi(t,\vec{x})P_{\text{in}}^{-1} \neq -\Phi(t,-\vec{x})$. Consequently, P_{in} is dynamically violated.

Remarks. (1) One may give an alternative proof of proposition 2.1 using the result of Fröhlich (Ref. 14, corollary 4.4) stating that the dynamical Euclidean measures $d\mu_{:P(\Phi):}$ and $d\mu_{:P(-\Phi):}$ for $P'(-\xi) \neq -P'(\xi)$ are mutually singular. (2) At first sight the pseudoscalar $P(\Phi)_2$ models might seem quite arbitrary. However, one can keep in mind an oversimplified two-dimensional picture of a weak decay of a pion.

B. Sine-Gordon model

The equation of motion has the form

$$(\Box + m_0^2)\Phi(x) = -\lambda :\sin[\epsilon\Phi(x) + \theta]: , \tag{2.7}$$

where $\lambda > 0$, $\epsilon < 0$, $\theta \in [0, 2\pi]$, and : : means the Wick normal-ordering operation. In Ref. 15 the following interesting properties for this model were proven:

(1) For a given $\epsilon^2/4\pi<1$ and for $|\lambda/m_0^2|$ sufficiently small (depending on ϵ) the energy-momentum spectrum for (2.7) has an isolated one-particle mass shell of mass $m>0$.

(2) The field Φ satisfies all Wightman axioms.

(3) For $m_0^2>0$ and $\epsilon^2/4\pi<1$ the scattering operator S is nontrivial.

(4) For $\theta\neq\theta'$ the dynamical Euclidean measures μ_θ and $\mu_{\theta'}$ are mutually singular.

Proposition 2.2. Consider the sine-Gordon model with $\epsilon^2/4\pi<1$ and $|\lambda m_0^{-2}|$ sufficiently small, $\theta\neq0$, π, and let Φ_{in} be the asymptotic field for Φ. Then if $\eta_P=1$, the P_{in} parity is dynamically and S-conserved; if $\eta_P=-1$, the parity is dynamically violated.

Proof. By virtue of properties (1) and (2) and Haag-Ruelle construction, there exist in H_θ quantum fields Φ, Φ_{in}, and Φ_{out} and a Fock space $H_{\text{in}}\subseteq H_\theta$. The parity operator P_{in} is given in H_{in} by the formula (1.8). If $\eta_P=1$ for Φ_{in}, then the proof that P_{in} is dynamically and S-conserved proceeds as in proposition 2.1.

Now consider the case $\eta_P=-1$. By virtue of consideration given in the proof of proposition 2.1, the field $\tilde\Phi(f)=P_{\text{in}}\Phi(f)P_{\text{in}}^{-1}$ is well defined on Hepp's domain D_{in}. Suppose that $\tilde\Phi(t,\vec{x})=-\Phi(t,-\vec{x})$. Then under P_{in}, the dynamical equation (2.7) will transform into the following one:

$$(\Box+m_0^2)\Phi(t,\vec{x})=-\lambda:\sin[\epsilon\Phi(t,\vec{x})+2\pi-\theta]:.$$

Since $\theta'=2\pi-\theta\neq\theta$ by assumption and the dynamical Euclidean measures μ_θ and $\mu_{\theta'}$ are mutually singular, P_{in} would transform the carrier space H_θ into a carrier space $H_{\theta'}$ orthogonal to it. This would imply that P_{in} could not be defined in H_θ contrarily to the construction (1.8); hence Φ cannot be pseudoscalar with respect to P_{in}. Consequently, for $\eta_P=-1$, P_{in} is dynamically violated.

C. Spontaneous parity violation

The foregoing examples could suggest that parity is violated in a "natural manner," i.e., when the underlying dynamics is not parity invariant. The following examples show that parity might be violated even when the interaction Lagrangian is parity symmetric. The most striking example is provided by the $\lambda\Phi^4_2$ theory.

In order to understand better the physical aspects of this problem, consider first the classical theory. In this case the total Hamiltonian can be taken in the form[17]

$$\mathcal{H}(\varphi,\pi)=\mathcal{H}_0(\varphi,\pi)+\mathcal{H}_I(\varphi,\pi)\,, \tag{2.8}$$

with

$$\mathcal{H}_0=\tfrac{1}{2}[\pi^2+(\partial_{\vec{x}}\varphi)^2]\,,\quad \mathcal{H}_I=\lambda\varphi^4+\tfrac{1}{2}\sigma\varphi^2+\tfrac{1}{16}\sigma^2\,.$$

We have $\mathcal{H}\geq0$. For $\sigma<0$ we have four minima of $\mathcal{H}$:

$$\pi_\pm=0,\quad \varphi_\pm=\pm\tfrac{1}{2}|\sigma|^{1/2}\quad\text{absolute minima}\,,$$

$$\left.\begin{aligned}\pi_S=0,&\quad \varphi_S(\vec{x})=\tfrac{1}{2}|\sigma|^{1/2}\tanh(\tfrac{1}{2}|\sigma|^{1/2}\vec{x})\\ \pi_{\bar S}=0,&\quad \varphi_{\bar S}=-\varphi_S\,.\end{aligned}\right\}\text{local minima}$$

Notice that for this interaction there exists a so-called topological charge given by the formula

$$Q=\int_{-\infty}^{\infty}d\vec{x}\,\partial_{\vec{x}}\varphi\,, \tag{2.9}$$

which is a constant of motion. It is evident from the expressions for φ_S and $\varphi_{\bar S}$ that under space reflection, $\varphi_S\leftrightarrow\varphi_{\bar S}$.

Now, according to the correspondence principle, one would expect that in this model, on the second quantized level, there will exist two vacuum sectors with charge $q=0$ and two other superselection sectors with charges $q_S=1$ and $q_{\bar S}=-1$, respectively. Since space reflection may transform $Q\to-Q$, parity may be violated in some superselection sectors.

It was shown in Refs. 16 and 17 that for $0<\lambda\ll1$ and $\sigma=-\frac{1}{2}$ the carrier Hilbert space has the following structure:

$$\tilde H=H_{\omega_+}\oplus H_{\omega_-}\oplus H_{\omega_S}\oplus H_{\omega_{\bar S}}\,.$$

Here H_{ω_+} is a vacuum sector of $\tilde H$ associated with a vacuum ω_+, H_{ω_-} is another vacuum sector of $\tilde H$, and H_{ω_S} is the soliton and $H_{\omega_{\bar S}}$ the antisoliton sector, respectively. H_{ω_S} and $H_{\omega_{\bar S}}$ are superselection sectors of $\tilde H$ labeled by the eigenvalue of Q equal to 1 and -1, respectively. The problem of dynamical and spontaneous parity violation may be most clearly presented in the subspace $H=H_{\omega_+}\oplus H_{\omega_-}$ of $\tilde H$ connected with vacuum sectors.

Proposition 2.3. Consider the $\lambda\Phi^4_2$ quantum field theory associated with the Hamiltonian (2.8) with $0<\lambda\ll1$ and $\sigma=-\frac{1}{2}$. Then there exists a parity operator P_{in} in the carrier Hilbert space H such that for $\eta_P=1$, parity is dynamically conserved, and for $\eta_P=-1$, parity is dynamically violated. In addition, for $\eta_P=-1$ parity is spontaneously broken.

Proof. The Wightman distributions

$$\tilde W_n^\pm(x_1,\ldots,x_n)=\langle\omega_\pm,\tilde\Phi(x_1)\cdots\tilde\Phi(x_n)\omega_\pm\rangle \tag{2.10}$$

satisfy all Wightman axioms (including the uniqueness of the physical vacuum). In addition, we have

$$W_1^\pm(x_1)=\varphi_\pm\,.$$

Let $\tilde\Phi^\pm$ be the relativistic quantum fields obtained from (2.10) by the Wightman reconstruction theorem. Define $\Phi^\pm(x)=\tilde\Phi^\pm(x)-\varphi_\pm$. The fields $\Phi^\pm$ are Wightman fields in $H_{\omega_\pm}$, respectively and satisfy the condition $W_1^{\pm}(x_1)=0$. Let $\Phi_{in}^{\pm}$ be the asymptotic in fields associated with $\Phi^{\pm}$. Then utilizing Hepp's[3] and Schrader's[12] construction, one may write in $H_{\omega_\pm}$ the Yang-Feldman equation for $\Phi^{\pm}$ in the form

$$\Phi^{\pm}(x)=\Phi_{in}^{\pm}(x)+4\lambda\Delta_R * :(\Phi^{\pm}+\varphi_{\pm})^3:(x)\,. \tag{2.11}$$

Using (1.8) the parity operator ${}^{\eta_P}P_{in}^{\pm}$ can be defined in $H_{\omega_\pm}$, respectively, by the formula

$${}^{\eta_P}P_{in}^{\pm}\Phi_{in}^{\pm}(t,\vec{x})({}^{\eta_P}P_{in}^{\pm})^{-1}=\eta_P\Phi_{in}^{\pm}(t,-\vec{x})\,.$$

For $\eta_P=+1$ we have that the fields ${}^{+}P_{in}^{\pm}\Phi^{\pm}(t,\vec{x})({}^{+}P_{in}^{\pm})^{-1}$ and $\Phi^{\pm}(-t,\vec{x})$ satisfy the same field equation with the same initial condition. Hence ${}^{+}P_{in}^{\pm}\Phi(t,\vec{x})({}^{+}P_{in}^{\pm})^{-1}$ coincides with $\Phi(t,-\vec{x})$; consequently, for $\eta_P=+1$ parity is dynamically conserved. Now for $\eta_P=-1$ the fields ${}^{-}P_{in}^{\pm}\Phi^{\pm}(t,\vec{x})({}^{-}P_{in}^{\pm})^{-1}$ and $-\Phi^{\pm}(t,-\vec{x})$ satisfy different dynamical equations with the same initial conditions. By a simple calculation we therefore show that for $\eta_P=-1$ parity is dynamically violated.

Consider now the problem of spontaneous breaking of parity. It follows from Ref. 1 that the cutoff dynamical measures $d\mu_{\pm}^{\Lambda}(\varphi)$ are given by the formula

$$d\mu_{\pm}^{\Lambda}(\varphi)=Z_{\Lambda}^{-1}\exp\left\{-\left[\int_{\Lambda}(\lambda:\varphi^4:(x)-\tfrac{3}{4}:\varphi^2:(x))\,d^2x+\delta S_{\pm}\right]\right\}d\mu_0(\varphi)\,, \tag{2.12}$$

where

$$\Lambda\equiv L\times T=\{x=(t,\vec{x});\ -\tfrac{1}{2}L\leqslant\vec{x}\leqslant\tfrac{1}{2}L\,,\ -\tfrac{1}{2}T\leqslant t\leqslant\tfrac{1}{2}T\}$$

and $\delta S_{\pm}$ is a so-called boundary term given by the formula

$$\delta S_{\pm}=\int_{-T/2}^{T/2}dt\left[\int_{-(1/2)(L+1)}^{-L/2}+\int_{L/2}^{(1/2)(L+1)}\right]d\vec{x}[\varphi_{\pm}\varphi(x)-\tfrac{1}{2}\varphi_{\pm}^2\chi_L(\vec{x})](-\partial_{\vec{x}}^2+1)\chi_L(\vec{x})\,, \tag{2.13}$$

with

$$\chi_L(\vec{x})=\begin{cases}1 \text{ on } [-\tfrac{1}{2}L,\tfrac{1}{2}L]\\ 0 \text{ on } (-\infty,-\tfrac{1}{2}(L+1)]\cup[\tfrac{1}{2}(L+1),\infty)\,.\end{cases}$$

It follows from (2.13) that in the limit $L\to\infty$ the boundary term goes to zero, and we are left with an interaction Lagrangian invariant under the substitution

$$\varphi(t,\vec{x})\to P\varphi(t,\vec{x})=\eta_P\varphi(t,-\vec{x})$$

with $\eta_P=\pm1$. However, if we keep the (T,L) cutoff finite, for $\eta_P=-1$, by virtue of the boundary term, we have $d\mu_+^{\Lambda}(P\varphi)=d\mu_-^{\Lambda}(\varphi)$. It was shown in Ref. 17 that this relation persists also in the limit $\Lambda\to R^2$, i.e., for $\eta_P=-1$,

$$d\mu_+(P\varphi)=d\mu_-(\varphi)\,. \tag{2.14}$$

This relation implies that although the interaction Lagrangian for the $\lambda\varphi^4{}_2$ theory in the noncutoff limit is space-reflection invariant for arbitrary η_P, the Wightman states ω_+ and ω_- are not invariant. Hence for $\eta_P=-1$, parity is spontaneously broken in $H_{\omega_+}\oplus H_{\omega_-}$.

Remark 1. Let T_S be the intertwining operator between the vacuum H_{ω_+} sector and the soliton sector H_{ω_S} and between H_{ω_-} and $H_{\omega_{\bar{S}}}$.[17] Then one can extend the fields $\Phi^{\pm}$ to $H_{\omega_S}\oplus H_{\omega_{\bar{S}}}$ by the formula $\Phi_S^{\pm}=T_S\Phi^{\pm}T_S^{-1}$ and show that parity transforms Q into $-Q$. Hence parity permutes charged superselection sectors and therefore is also spontaneously broken in the soliton-antisoliton sectors. The detailed proof is simple and will be omitted here.

Remark 2. If we consider high-order polynomial $P(\Phi)_2$ models, then one might obviously obtain more than two vacuum superselection sectors. For instance, for[18]

$$P(\varphi)_2=\tfrac{1}{2}\lambda:\varphi^6:-\lambda^{1/2}:\varphi^4:+(\tfrac{1}{2}-\nu):\varphi^2:$$

there exist three vacuum sectors ω_0, ω_+, and ω_-. Using the above analysis, one may show that parity is conserved in the H_{ω_0} sector for arbitrary η_P and is spontaneously broken in $H_{\omega_+}\oplus H_{\omega_-}$ for $\eta_P=-1$.

III. *P*, *C*, AND *PC* SYMMETRIES IN THE FEDERBUSH MODEL

Let ψ^1 and ψ^2 be two fermion massive classical fields satisfying the following dynamical equations in R^2:

$$\begin{aligned}(\partial_\mu\gamma^\mu+m^1)\psi^1(x)&=i\lambda\gamma^\mu\psi^1(x)\epsilon_{\mu\nu}J^{2\nu}(x)\,,\\ (\partial_\mu\gamma^\mu+m^2)\psi^2(x)&=i\lambda J^{1\mu}(x)\epsilon_{\mu\nu}\gamma^\nu\psi^2(x)\,,\end{aligned}\quad x\in R^2 \tag{3.1}$$

where

$$J^{l\mu}(x)=i\bar{\psi}^l(x)\gamma^\mu\psi^l(x)\,,\quad l=1,2\,. \tag{3.2}$$

$\epsilon_{\mu\nu}$ is an invariant skew-symmetric tensor in R^2 and

$$\gamma^0=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma^5=\gamma^0\gamma^1=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{3.3}$$

Equations (3.1) can be derived from the Lagrangian $\mathcal{L}=\mathcal{L}_0(\psi^1)+\mathcal{L}_0(\psi^2)-J^{1\mu}\epsilon_{\mu\nu}J^{2\nu}$. Equations (3.1) admit explicit solutions in terms of the free solutions ψ_0^l, $l=1,2$ of the Dirac equation

$$(\partial_\mu\gamma^\mu+m^l)\psi_0^l(x)=0 \quad l=1,2$$

in the form

$$\psi^1(x)=\exp[-i\lambda\sigma_0^2](x)\psi_0^1(x)\,,\qquad \psi^2(x)=\exp[i\lambda\sigma_0^1](x)\psi_0^2(x)\,, \tag{3.4}$$

where

$$\sigma_0^l(x)=-2im^l\int\bar{\Delta}(x-y:0)\bar{\psi}_0^l(y)\gamma^5\psi_0^l(y)d^2y \tag{3.5}$$

and

$$\bar{\Delta}(x;m)=(2\pi)^{-1}P\int[-k^2+m^2]^{-1}\exp[-ikx]d^2k\,.$$

The second-quantized quantum field $\psi^l(x)$, $l=1,2$ was constructed by Federbush[19] and Wightman[20] by the technique of ⋮⋮ ordering. The resulting quantum fields have the form

$$\psi^1(x)=\vdots\exp[i\lambda\sigma_0^2]\vdots(x)\psi_0^1(x)\,,\qquad \psi^2(x)=\vdots\exp[i\lambda\sigma_0^1]\vdots(x)\psi_0^2(x)\,, \tag{3.6}$$

where

$$\sigma_0^l(x)=-2im^l\int\bar{\Delta}(x-y;0):\bar{\psi}_0^l(y)\gamma^5\psi_0(y):d^2y\,,\quad l=1,2 \tag{3.7}$$

and :: denotes the ordinary Wick ordering. We recall that ⋮⋮ ordering is defined for successive powers of the local field σ recursively and, for instance, for the third power it has the form[20]

$$\begin{aligned}\vdots\sigma^3\vdots(x)=\lim_{\xi\to 0,\,\xi^2<0}\,&[\sigma(x+\xi)\sigma(x)\sigma(x-\xi)\\ &-\langle\sigma(x+\xi)\sigma(x)\rangle_0\sigma(x-\xi)\\ &-\langle\sigma(x+\xi)\sigma(x-\xi)\rangle_0\sigma(x)\\ &-\langle\sigma(x)\sigma(x-\xi)\rangle_0\sigma(x+\xi)]\,.\end{aligned}$$

In the second-quantized theory the currents (3.2) are defined by the following limiting procedure[20]:

$$J^{l\mu}(x)=\lim_{\xi\to 0}\tfrac{1}{2}[\phi^{l\mu}(x+\xi)\psi^l(x)-\bar{\psi}^l(x)\bar{\phi}^{l\mu}(x-\xi)]\,, \tag{3.8}$$

where $\phi^{l\mu}(x)=\bar{\psi}^l(x)\gamma^\mu$.

It was shown recently by Challifour and Wightman[21] that the solutions (3.6) have the asymptotic in and out fields in the form

$$\psi^1_{\text{in, out}}=\exp(\mp i\pi\lambda Q^2)\psi_0^1\,,\qquad \psi^2_{\text{in, out}}=\exp(\pm i\pi\lambda Q^1)\psi_0^2\,, \tag{3.9}$$

where Q^l are the fermion-number operators

$$Q^l=i\int d\sigma_\mu(x):\bar{\psi}_0(x)\gamma^\mu\psi_0(x):=\int p_0^{-1}d\vec{p}[a^{l*}(\vec{p})a^l(\vec{p})-b^{l*}(\vec{p})b^l(\vec{p})]\,. \tag{3.10}$$

Consequently, there exists a nontrivial unitary scattering operator in the form

$$S=\exp(2\pi i\lambda Q^1Q^2)\,. \tag{3.11a}$$

At this point an important remark should be made. There seems to be a discrepancy in the literature between, e.g., Refs. 21 and 22. The results of Ref. 21 for the scattering operator based on the Lehmann-Symanzik-Zimmermann formalism of asymptotic fields are different from the results of Ref. 22 based on Bogolyubov-Parasiuk-Hepp-Zimmermann perturbation theory. To be more explicit, the form of the S operator calculated in Ref. 22 is

$$S=\exp\left\{2\pi i\lambda\iint(dk^1/k^{10})(dk^2/k^{20})q^2(k^2)\times\epsilon(k^{2\mu}\epsilon_{\mu\nu}k^{1\nu})q^1(k^1)\right\}, \tag{3.11b}$$

where $q^l(k^l)$ are the densities of Q^l, $l=1,2$.

It is evident from formulas (3.11a) and (3.11b) that, e.g., for positive integer coupling constants, the expressions for the matrix elements of the S operator coincide.

Since we are interested in a framework for testing our ideas on violation of discrete symmetries, the choice of the value of the coupling constant is irrelevent for us. We shall therefore, whenever considering the Federbush model, assume from now on that *the coupling constant is a positive integer*. Such a version of the Federbush model will be called, for brevity, the *"integer Federbush model."*

To go on, it will be convenient for further analysis to express σ_0^l and Q^l in terms of ψ_{in}^l fields, $l=1,2$. Using (3.9) we obtain

$$\sigma_0^l(x)=-2im^l\int\bar{\Delta}(x-y;0):\bar{\psi}_{\text{in}}^l(y)\gamma^5\psi_{\text{in}}^l(y):d^2y\,,\qquad Q^l=i\int d\sigma_\mu(x):\bar{\psi}_{\text{in}}^l(x)\gamma^\mu\psi_{\text{in}}^l(x):d^2x\,. \tag{3.12}$$

We now pass to the definition of a parity operator. The in parity operator will be defined by the formula $P_{\text{in}}=P_{\text{in}}^1\otimes P_{\text{in}}^2$ with

$$P_{\text{in}}^l\psi_{\text{in}}^l(t,\vec{x})P_{\text{in}}^{l-1}=\eta_P\gamma^0\psi^l(t,-\vec{x})\,,\qquad P_{\text{in}}^l\bar{\psi}^l(t,\vec{x})P_{\text{in}}^{l-1}=-\bar{\eta}_P\bar{\psi}^l(t,-\vec{x})\gamma^0\,. \tag{3.13}$$

This operator, as in the case of scalar fields, can be expressed directly in terms of $\psi^l_{\rm in}$ fields. We have the following.

Proposition 3.1. In the integer Federbush model, parity is dynamically violated but S-conserved.

Proof. By virtue of (3.7) and (3.12) we obtain

$$P_{\rm in}\sigma^l_0(t,\vec{x})P_{\rm in}^{-1}=-\sigma^l_0(t,-\vec{x})\,,\quad P_{\rm in}Q^lP_{\rm in}^{-1}=Q^l\,,\quad l=1,2\,. \tag{3.14}$$

Hence

$$P_{\rm in}\psi^1(t,\vec{x})P_{\rm in}^{-1}=:\exp[-i\lambda\sigma^2_0]:(t,-\vec{x})\eta_P\gamma^0\psi^1_0(t,-\vec{x}) \neq\eta_P\gamma^0\psi^1(t,-\vec{x})\,,$$

$$P_{\rm in}\psi^2(t,\vec{x})P_{\rm in}^{-1}\neq\eta_P\gamma^0\psi^2(t,-\vec{x})\,.$$

Thus the operator $P_{\rm in}$ is not the parity operator for the interacting fields (3.6). Consequently, parity is dynamically violated. Since, for positive integer coupling constants (the case considered by us) the scattering operator is trivial, parity is S-conserved.

Consider now the charge conjugation operator $C_{\rm in}$ defined by the formula $C_{\rm in}=C^1_{\rm in}\otimes C^2_{\rm in}$. It follows from the transformation property of the free Dirac equation that $C^l_{\rm in}$, $l=1,2$, must be taken in the form

$$C^l_{\rm in}\psi^l_{\rm in}(x)C_{\rm in}^{l-1}=\eta_C C\psi^{l*}_{\rm in}(x)\,,\quad\text{with } C=\gamma^1,\quad l=1,2\,. \tag{3.15}$$

Proposition 3.2. In the integer Federbush model, charge conjugation is dynamically conserved and S-conserved.

Proof. Using (3.15) we obtain

$$C_{\rm in}\bar{\psi}^l_{\rm in}C_{\rm in}^{-1}=-\bar{\eta}_C\psi^{lT}_{\rm in}C^*\,. \tag{3.16}$$

Hence, by virtue of (3.12), we have

$$C_{\rm in}\sigma^l_0(x)C_{\rm in}^{-1}=-\sigma^l_0\,,\quad C_{\rm in}Q^lC_{\rm in}^{-1}=-Q^l\,. \tag{3.17}$$

Therefore by (3.6) we obtain

$$C_{\rm in}\psi^lC_{\rm in}^{-1}=\eta_C C\psi^{l*}\,, \tag{3.18}$$

i.e., the solutions of the dynamical equation are charge-conjugation invariant. Hence charge conjugation is dynamically as well as S-conserved.

As a consequence of propositions 3.1 and 3.2 we obtain that $P_{\rm in}C_{\rm in}$ is dynamically violated but S-conserved. Thus when we say that CP is a good quantum observable for a given dynamical system we have to determine carefully in what sense the CP conservation is defined.

It should be remarked that for arbitrary (noninteger) λ, according to Wightman and Challifour's version, parity will still be S-conserved, while according to the version of Schroer *et al.*, parity would be S-violated.

IV. TIME-REVERSAL AND *CPT* INVARIANCE

We shall now analyze the problem of time reversal and CPT invariance for boson and fermion models of quantum field theory. We show that for a given interaction, T and CPT invariance may or may not hold depending on the choice of the reflection properties of the fields under consideration. We demonstrate that these reflection properties are not, in general, fixed by the form of interaction and may be chosen arbitrarily, depending on the physical identification of the considered particles.

A. $P(\Phi)_2$ model

As discussed in Sec. II A, for weak coupling there exists in this model asymptotic fields $\Phi_{\rm in}$ and $\Phi_{\rm out}$ given by the Haag-Ruelle construction and $H_{\rm in}\subseteq H$, $H_{\rm out}\subseteq H$. Hence one can define in H the time-reversal operator $T_{\rm in}$ given by formula (1.11):

$${}^{\eta_T}T_{\rm in}\Phi_{\rm in}(t,\vec{x}){}^{\eta_T}T_{\rm in}^{-1}=\eta_T\Phi_{\rm out}(t,-\vec{x})\,. \tag{4.1}$$

Proposition 4.1. If $\eta_T=1$ in $P(\Phi)_2$ theory with $P'(-\xi)\neq-P'(\xi)$, then time reversal is dynamically and S-conserved. If $\eta_T=-1$, time reversal is dynamically violated.

Proof. As was said before, in a weakly coupled $P(\Phi)_2$ model all Wightman axioms are satisfied and there exists an isolated one-particle mass hyperboloid.[11] Hence following Hepp, one may write the YF equation in the form[3]

$$\Phi(t,\vec{x})=\Phi_{\rm in}(t,\vec{x})+\Delta_R*:P'(\Phi):(t,\vec{x}) =\Phi_{\rm out}(t,\vec{x})+\Delta_A*:P'(\Phi):(t,\vec{x})\,. \tag{4.2}$$

The explicit form of $:P'(\Phi)_2:$ in Minkowski space was determined by Schrader[12] (cf. also Ref. 13). Equation (4.2) holds true on the Hepp domain $D=D_{\rm in}\oplus D_{\rm out}\subset H$ of nonoverlapping vectors, where $D_{\rm out}$ is the subspace of vectors $|\psi_{\rm out}(f^n)\rangle$ of the form (2.5) with $\Phi_{\rm in}$ replaced by $\Phi_{\rm out}$. It is evident that D is $T_{\rm in}$ invariant. Hence the operators $T_{\rm in}\Phi_{\rm in}T_{\rm in}^{-1}$, $T_{\rm in}\Phi T_{\rm in}^{-1}$, and $\Delta_A*T_{\rm in}P'(\Phi)T_{\rm in}^{-1}$ are well defined on D.

Consider now the case $\eta_T=1$. Then acting on (4.2) by the $T_{\rm in}$ operator we obtain on D the following:

$$T_{\rm in}\Phi(t,\vec{x})T_{\rm in}^{-1}=\Phi_{\rm out}(-t,\vec{x}) +\Delta_R*P'(T_{\rm in}\Phi T_{\rm in}^{-1})(t,\vec{x})\,.$$

This equation in the variable $\check{\Phi}(t,\vec{x})=T_{\rm in}\Phi(t,\vec{x})T_{\rm in}^{-1}$ takes the form

$$\check{\Phi}(t,\vec{x})=\Phi_{\rm out}(-t,\vec{x})+\Delta_A*P'(\check{\Phi})(t,\vec{x})\,,$$

which coincides with (4.2) taken for $\Phi(-t,\vec{x})$. Hence $\check{\Phi}(t,\vec{x})$ and $\Phi(-t,\vec{x})$ satisfy the YF equation with the same initial conditions and the same form of the current. Thus $\check{\Phi}(t,\vec{x})$ and $\Phi(-t,\vec{x})$ must coincide, i.e.,

$$T_{\rm in}\Phi(t,\vec{x})T_{\rm in}^{-1}=\Phi(-t,\vec{x})\,.$$

This implies that T_{in} is dynamically and S-conserved.

Let now $\eta_T=-1$. Then on the domain D we have

$$T_{\text{in}}\Phi(t,\vec{x})T_{\text{in}}^{-1}=-\Phi_{\text{out}}(-t,\vec{x}) + \Delta_R * P'(T_{\text{in}}\Phi T_{\text{in}}^{-1})(t,\vec{x}). \quad (4.3)$$

If we suppose that T_{in} is dynamically conserved, i.e.,

$$T_{\text{in}}\Phi(t,\vec{x})T_{\text{in}}^{-1}=-\Phi(-t,\vec{x}),$$

then (4.3) becomes

$$\Phi(t,\vec{x})=\Phi_{\text{out}}(t,\vec{x})+\Delta_A*[-P'(-\Phi)](t,\vec{x}). \quad (4.4)$$

Taking the difference of (4.1) and (4.4) we obtain

$$\Delta_A * \sum_{n=3}^{N}(2n-1)\lambda_{2n-1}:\Phi^{2n-1}:=0.$$

Repeating now the same arguments as in the proof of proposition 2.1, we conclude that T_{in} is dynamically violated.

Remark 1. As in the case of the proof of proposition 2.1, one can give an alternative proof of proposition 4.1 using the Fröhlich result that the Euclidean dynamical measures $\mu_{P(\Phi)}$ and $\mu_{P(-\Phi)}$ for $P'(-\xi)\neq -P'(\xi)$ are mutually singular. It seems, however, that the present proof is more clear.

Remark 2. The conclusions of proposition 4.1 apply also to weakly coupled sine-Gordon models with $\mathcal{L}_I=\lambda\epsilon^{-1}:\cos(\epsilon\Phi+\theta):$, $0<|\lambda m_0^{-2}|\ll 1$, $\theta\neq 0,\pi$.

B. Integer Federbush model

It follows from the requirement of invariance of the Dirac equation in R^2 that the field ψ^l_{in}, $l=1,2$ must transform in the following manner:

$$T_{\text{in}}\psi^l_{\text{in}}(t,\vec{x})T_{\text{in}}^{-1}=\eta^l_T T\psi^l_{\text{out}}(-t,\vec{x}) \quad (4.5)$$

with $T=\gamma^5$ and $|\eta_T|^2=1$.

Proposition 4.2. For an arbitrary choice of η_T in the integer Federbush model T_{in} is dynamically and S-conserved.

Proof. In order to find out the transformation properties of the interacting fields ψ^l, $l=1,2$ with respect to T_{in} operation we take advantage of the fact that, by virtue of (3.12), $\sigma^l_0(x)$ and $Q^l(t)$ can be expressed in terms of ψ^l_{in} fields, $l=1,2$. These quantities, by virtue of (4.5), have the following transformation properties:

$$T_{\text{in}}\sigma^l_0(t,\vec{x})T_{\text{in}}^{-1}=-S^{-1}\sigma^l_0(-t,\vec{x}),$$
$$T_{\text{in}}Q^l(t)T_{\text{in}}^{-1}=-Q^l(-t). \quad (4.6)$$

Hence, by virtue of (3.6), we get

$$T_{\text{in}}\psi^l(t,\vec{x})T_{\text{in}}^{-1}=\eta_T\gamma^5\psi^l(-t,\vec{x}),$$
$$T_{\text{in}}J^{l\mu}(t,\vec{x})T_{\text{in}}^{-1}=-J^l_\mu(-t,\vec{x}). \quad (4.7)$$

We now verify the transformation properties of field equations. Acting on both sides of Eq. (3.1) by T_{in} we obtain [$\partial^T_\mu\equiv(\partial_{\vec{x}},-\partial_{x_0})$, and we follow the conventions given in Ref. 23, Ch. IV, Sec. 5]:

$$(\partial^T_\mu\bar{\gamma}^\mu+m^1)\gamma^5\psi^1(-t,\vec{x})=-i\bar{\gamma}^\mu\gamma^5(\psi^1\epsilon_{\mu\nu}J^{2\nu})(-t,\vec{x}),$$
$$(\partial^T_\mu\bar{\gamma}^\mu+m^2)\gamma^5\psi^2(-t,\vec{x})=-i(J^{1\mu}\epsilon_{\mu\nu}\bar{\gamma}^\nu\gamma^5\psi^2)(-t,\vec{x}).$$

Multiplying both sides by γ^5 and using the fact that $\bar{\gamma}^\mu=-\gamma^\mu$, $(\gamma^5)^2=I$, we obtain that $\psi^l(-t,\vec{x})$ satisfies the same field equation in the reflected reference frame as did $\psi^l(t,\vec{x})$. Thus the field equations are T_{in} invariant; hence T_{in} is dynamically as well as S-conserved.

V. *PCT* AND θ SYMMETRIES

The previous analysis of P, C, and T symmetry violation for several models demonstrates that, with a given dynamical system determined by the interaction Lagrangian, one may associate several *PCT* operations. For instance, in $:P(\Phi)_2:$ models with $P'(-\xi)\neq -P'(\xi)$ we may take Φ as a space pseudoscalar and a time scalar; in this case P_{in} is dynamically violated, whereas T_{in} is dynamically and S-conserved. Since in $:P(\Phi)_2:$ models $C_{\text{in}}=I$ we have that in this case $P_{\text{in}}C_{\text{in}}T_{\text{in}}$ is dynamically violated. Similarly, $P_{\text{in}}C_{\text{in}}T_{\text{in}}$ violation will occur if we take Φ as a space scalar and a time pseudoscalar. In turn, if we choose Φ to be a space scalar and a time scalar or a space pseudoscalar and a time pseudoscalar we obtain $P_{\text{in}}C_{\text{in}}T_{\text{in}}$ dynamical and S-conservation.

Since there is no physical reason for correlating the space and time internal parities for spinless particles we conclude that in $:P(\Phi)_2:$ models we may introduce for $P_{\text{in}}C_{\text{in}}T_{\text{in}}$ the operators

$$^+P_{\text{in}}C_{\text{in}}{}^+T_{\text{in}},\quad {}^-P_{\text{in}}C_{\text{in}}{}^+T_{\text{in}},$$
$$^+P_{\text{in}}C_{\text{in}}{}^-T_{\text{in}},\ \text{and}\ {}^-P_{\text{in}}C_{\text{in}}{}^-T_{\text{in}}. \quad (5.1)$$

The famous θ operation will, however, be uniquely defined by

$$\theta^{-1}\Phi(x)\theta=\Phi(-x)$$

[cf. Ref. 4, Eq. (3.67)]. Hence we see that θ^{-1} may be identified with $^+P_{\text{in}}C_{\text{in}}{}^+T_{\text{in}}$ or $^-P_{\text{in}}C_{\text{in}}{}^-T_{\text{in}}$. However, if we choose Φ to be a space pseudoscalar and a time scalar or a space scalar and a time pseudoscalar field, then the product of operators P_{in}, C_{in}, and T_{in} does not coincide with the θ^{-1} operation.

Notice also that in $:P(\Phi)_2:$ theories, independently of the specific form of the interaction, θ is dynamically and S-conserved. Indeed, acting on the left-hand side of the YF equation by θ^{-1}, we obtain on the Hepp domain D (cf. proof of proposition 4.1)

$$\Phi(-x)=\Phi_{\text{out}}(-x)+\Delta_A*P'(\Phi)(-x),$$

which by virtue of (4.2) equals $\Phi_{in}(-x)+\Delta_R * P'(\Phi)(-x)$. Hence θ^{-1} is dynamically and S-conserved.

An even more interesting situation arises in the integer Federbush model. In this case P_{in} is, independently of a phase η_P, always dynamically violated and S-conserved, whereas C_{in} and T_{in} are dynamically and S-conserved. Therefore the $P_{in}C_{in}T_{in}$ operator is, for all choices of phase η_P, η_C, and η_T, always dynamically violated. Now the θ operation defined by the formula [cf. Ref. 4, Eq. (3.67)]

$$\theta^{-1}\psi(x)\theta=\eta_\theta\psi^*(-x)\,,$$

transforms the currents $J^{l\mu}=i\bar{\psi}^{l}\gamma^\mu\psi^l$, $l=1,2$, in the following manner:

$$\theta^{-1}J^{l\mu}(x)\theta=J^{l\mu}(-x)\,,\quad l=1,2\,.$$

Hence the action integral in the integer Federbush model is θ^{-1} invariant. Consequently, θ^{-1} is dynamically conserved. Hence the product $P_{in}C_{in}T_{in}$ of physical parity, charge-conjugation, and time-reversal operators cannot coincide with the θ^{-1} operation.

We hope that these examples clarify the meaning of θ symmetry in field theory and its connection with the product $P_{in}C_{in}T_{in}$ of space parity, charge conjugation, and time reversal, which, depending on the given dynamics, may or may not be a symmetry of the theory.

This analysis indicates that the so-called *PCT* theorem in axiomatic field theory should (as it actually is by some authors) really be called θ theorem since in fact it concerns the θ symmetry only. In the case where P, C, and T are symmetries of the given dynamics, we necessarily have $\theta^{-1}=P_{in}C_{in}T_{in}$ up to a phase factor.

VI. DISCUSSION

We shall end our work with the following remarks:

(i) It follows from our analysis that S-operator and spontaneous discrete symmetry violations are physically really important. In connection with this, the integer Federbush model is particularly instructive, since in this model P_{in} and $P_{in}C_{in}T_{in}$ are dynamically violated but S-conserved. This illustrates the important fact that being confined to an analysis of the interaction Lagrangian only might give us wrong information on violation or conservation of discrete (as well as continuous) symmetries.

(ii) In $P(\Phi)_2$ and other models, if P_{in} is dynamically violated for $\eta_P=-1$, one could save $P_{in}C_{in}T_{in}$ dynamical conservation by setting $\eta_T=-1$. However, it seems that there are no physical reasons to correlate space and time parities.

(iii) Notice that for polynomial interactions $P(\Phi)_2$ of a single spinless boson field Φ, even if $P'(-\Phi)\neq -P'(\Phi)$, the matrix elements $S_m(x_1,\ldots,x_m)$ of the scattering operator are space-reflection invariant in perturbation theory. This might be most easily seen in the functional formalism for the scattering operator. In this formalism the generating functional $S(\varphi)$ for the scattering operator is given by the formula

$$S(\varphi)=Z^{-1}\exp\left[-\tfrac{1}{2}\frac{\delta}{\delta\varphi}*\Delta_F*\frac{\delta}{\delta\varphi}\right]\times\exp[i:\mathfrak{L}_I(\varphi):]|_{\varphi=\varphi_{in}}$$

and the m-particle S-matrix element at order k of perturbation theory has the form

$$S_m^{(k)}(x_1,\ldots,x_m)=(k!)^{-1}i^{k-m}\frac{\delta^m}{\delta\varphi(x_1)\cdots\delta\varphi(x_m)}\times\exp\left(-\tfrac{1}{2}\frac{\delta}{\delta\varphi}*\Delta_F*\frac{\delta}{\delta\varphi}\right)\times\int\prod_{i=1}^{k}d^mZ_i:\mathfrak{L}_I(Z_i):|_{\varphi=0}\,. \tag{6.1}$$

It follows from (6.1) that $S_m^{(k)}(x_1,\ldots,x_m)$ is a finite sum of space-reflection invariant integrals in R^{km} over finite products of space-reflection invariant Δ_F and Δ_+, which define time and normal ordering, respectively. The same phenomenon holds true for Borel summable expressions for perturbation series, e.g., in $P(\varphi)_2$ models.[24] Thus in order to have a phenomenon of S-matrix parity violation, one must consider a model with several fields. This is precisely the case for which parity violation was observed for physical particles.

(iv) Our analysis clarifies the conceptual problems of discrete symmetries violations in quantum field theory. However, from a realistic point of view, it would be of utmost interest to apply this analysis to quantum field models in four-dimensional space-time. This application is straightforward. Unfortunately, for the time being we have no conclusive nonperturbative results for these models.

(v) Though the technique of constructive field theory (which we mainly adopted here) does not provide us at present with a framework that could be used for our purpose in realistic four-dimensional space-time, much can be at least guessed from formal perturbative calculations.

(vi) It would also be very interesting to extend this analysis for decaying quantum systems, in which naturally several fields appear.

ACKNOWLEDGMENTS

The authors would like to thank D. Sternheimer and B. Nagel for interesting discussions. The authors would also like to thank Arthur Wightman for communicating Ref. 21 and related discussions. This work was partially supported by NSF Grant No. INT-73-20002A01, formerly GF-41958.

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