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Model of U(1) chiral-symmetry breaking

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An SU(2) gauge-theory model of spontaneous chiral-symmetry breaking through the nonvanishing vacuum expectation value of spin-zero fields is examined. For the case of two fermion flavors, the model explicitly demonstrates the 't Hooft resolution of the U(1) problem through instanton effects. The absence of any conflict with the current-algebra relations of chiral perturbation theory point to a clearer understanding of the gauge-theory vacuum θ periodicity in the case of broken chiral symmetries.

I. INTRODUCTION

A long-standing difficulty of any fermion constituent model of the strong interactions such as quantum chromodynamics (QCD) is that the requirement of an SU(N) chiral-flavor-symmetry group naturally entails a full U(N) chiral symmetry. Thus, if, as is normally assumed, the SU(2) chiral symmetry is broken to yield the (nearly) massless Goldstone pion, there should also be an isosinglet state with about the same mass. The methods of current algebra which are so successful[1] (to the order of 7–10%) for the pion should then be valid for the isosinglet pseudoscalar meson as well. However, the only natural candidate for this state in the meson spectrum is the η(550 MeV) whose mass exceeds the upper bound placed on it by the techniques of current algebra.[2] The absence of any light pseudoscalar meson corresponding to the U(1) chiral current is the U(1) problem of QCD.

The discovery of the instanton configurations[3] in non-Abelian gauge field theories, together with the Adler-Bell-Jackiw anomalous divergence[4] of the chiral singlet current,

$$\frac{\partial}{\partial x^\mu}\left(\sum_{f=1}^{N} \tfrac{1}{2}\bar{\psi}_f\gamma^\mu\gamma^5\psi_f\right)=\frac{Ng^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}\cdot F_{\alpha\beta}\,, \qquad (1)$$

provide an apparent resolution of the dilemma. For if there really is no conserved chiral charge operator, Goldstone's theorem does not apply to the singlet state and there is no reason to expect that the η will be light as the pion. This loophole was pointed out by 't Hooft[5] who employed the semiclassical methods devised for computing instanton effects—independently of any spontaneous breaking of the full U(N) chiral group. Indeed, the dynamics of this symmetry breaking and subsequent appearance of the pion may not be accessible to present approaches at all.

On the other hand, several authors have suggested that the SU(N)-symmetry breaking is intimately connected to the resolution of the U(1) problem. Models have been proposed in which the instanton effects are taken into account by means of the induced $2N$-point fermion interaction treated in a self-consistent manner similar to the Nambu–Jona-Lasinio model.[6] A natural consequence of this approach is that the same interaction that generates the fermion masses and binds them into massless Goldstone bosons also automatically solves the U(1) problem. Crewther[7] has argued along different lines that the 't Hooft resolution of the U(1) problem is actually inconsistent with current-algebra estimates of SU(2)$^{\rm ch}$ breaking arising from small explicit quark mass terms added to the QCD Lagrangian.

The same author has also pointed out an apparent difficulty of the 't Hooft resolution of the U(1) problem with respect to the periodicity of the gauge-theory vacuum. According to the now standard prescription,[8] the pseudoparticle configurations necessitate an additional parameter in the theory θ which enters the Lagrangian multiplying the (CP-violating) topological charge density $(g^2/32\pi^2)F_{\mu\nu}\cdot\tilde{F}_{\mu\nu}$. If the integral of this density is an integer ν, all physical amplitudes manifestly have a θ periodicity of 2π. This leads to a selection rule[7] which would seem to *prevent* the removal of the U(1) Goldstone boson by the 't Hooft loophole for $N>1$. When SU(N)$^{\rm ch}$ is broken, the θ periodicity would have to change in order to avoid this conclusion, even though θ enters the theory in exactly the same way as in the unbroken case.

With the aim of clarifying this situation by means of a concrete example, a model of chiral-symmetry breaking for $N=2$, containing scalar and pseudoscalar fields Yukawa coupled to the fermions is considered in this paper.[9] When the scalar isosinglet field develops a vacuum expectation value, four pseudoscalar bosons appear corresponding to the four generators of the broken U(2)$^{\rm ch}$ group. However, the inclusion of instanton effects shifts the mass of the isosinglet state away from zero.

The model is illustrative in several respects. It demonstrates that the $SU(2)^{ch}$ symmetry breaking and concomitant fermion mass generation *must* occur in order to solve the U(1) problem in a way consistent with the current-algebra estimates of Crewther. Of course, it cannot be claimed that the scalar field model realistically describes the dynamics of chiral-symmetry breaking in QCD. This dynamics may be quite complicated without necessarily being due to instanton effects. Indeed, the $\eta(550)$ may not be interpretable as a would-be Goldstone particle at all. Nevertheless, the necessity of the $SU(2)^{ch}$ breaking for the 't Hooft resolution of the U(1) problem is demonstrated by the model and computed in a systematic perturbative way.

The model also provides an explicit example of the avoidance of the selection rule cited by Crewther as an obstacle to the 't Hooft solution for integral ν. This leads to a simple and direct resolution of the paradox of the θ periodicity of physical amplitudes in the case of spontaneous symmetry breakdown, which is both interesting and instructive.

The details of the calculational scheme (for $\theta=0$) are presented in Sec. II. This is supplemented by the one-loop corrections to the lowest-order result which are presented in Appendix A. In Sec. III the question of the θ dependence is taken up. It is shown that although a chiral rotation corresponding to $\theta\to\theta+2\pi N$ (for $N=2$ here) is necessary to return to the same vacuum state, a rotation of 2π brings us to another vacuum state which is in all respects physically equivalent to the first, thus removing the formal periodicity problem without altering the physical content of the θ parameter. Since the case of exact $SU(2)\times SU(2)^{ch}$ symmetry is somewhat singular (the vacuum phases being fixed only by arbitrary choice), Appendix B considers the case in which this symmetry is explicitly broken. In this case $\theta\to\theta+2\pi$ does indeed return us to the same vacuum state but the previous avoidance of the selection rule of Crewther still holds—now because of the terms explicitly breaking the symmetry at some values of θ. Finally, Sec. IV contains a discussion of the results, expecially as they relate to the current-algebra estimates of Crewther and shed light on the relationship between $SU(2)^{ch}$-symmetry breaking and the solution of the U(1) problem.

II. THE MODEL

The Lagrangian of the model is taken to be

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}+i\sum_{f=1}^{2}\bar\psi_f\gamma^\mu(\partial_\mu-igA_\mu)\psi_f+i\lambda\sum_{f,f'=1}^{2}\bar\psi_f\Phi_{ff'}\psi_{f'}+\mathcal{L}_\Phi\,,$$

where

$$8\mathcal{L}_\Phi=-\mathrm{Tr}(\partial^\mu\Phi\partial_\mu\Phi^\dagger)+\frac{a}{2}\mathrm{Tr}(\Phi\Phi^\dagger)-\frac{b}{4}(\mathrm{Tr}\Phi\Phi^\dagger)^2-\frac{c}{4}\mathrm{Tr}(\Phi\Phi^\dagger\Phi^\dagger\Phi)\,, \tag{2}$$

which is the Lagrangian of an SU(2) (color) gauge theory with two species (flavors) of fermions that are Yukawa coupled to a set of spinless fields Φ described by a 2×2 complex matrix in flavor space. If Φ is written in terms of four real scalar and pseudoscalar fields, we have

$$\Phi=\sigma-i\eta\gamma^5+i\pi_i\tau_i\gamma^5+\phi_i\tau_i \tag{3}$$

in a suggestive notation. The traces in (2) are over both Dirac and flavor indices with τ_i, $i=1,2,3$, the 2×2 Pauli matrices. This accounts for the factor of 8 in the definition of $\mathcal{L}_\Phi$.

The Lagrangian is invariant under the $U(2)\times U(2)^{ch}$ transformation

$$\delta\psi=\frac{i}{2}(\alpha_0+\vec\alpha\cdot\vec\tau)\psi+\frac{i}{2}(\beta_0+\vec\beta\cdot\vec\tau)\gamma^5\psi \tag{4}$$

if we require

$$\begin{aligned}&\delta\sigma=-\beta_0\eta+\vec\beta\cdot\vec\pi,\quad \delta\eta=\beta_0\sigma+\vec\beta\cdot\vec\phi\,,\\&\delta\vec\phi=\vec\alpha\times\vec\phi+\beta_0\vec\pi-\vec\beta\eta,\quad \delta\vec\pi=\vec\alpha\times\vec\pi-\beta_0\vec\phi-\vec\beta\sigma\,.\end{aligned} \tag{5}$$

Thus, the eight spin-zero fields are in the $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\times SU(2)^{ch}$, which may be recognized also as the vector representation of O(4). In fact, if the fields are rearranged into two four-vectors, $v=(\vec\phi,\eta)$ and $w=(\vec\pi,\sigma)$, then v and w rotate jointly under the O(4) transformation given by (5) (with $\alpha_0=\beta_0=0$). Hence, v^2, w^2, and $v\cdot w$ all remain invariant under this transformation. Under the chiral $U(1)^{ch}$ transformation, the two four-vectors mix, leaving only the two combinations $v^2+w^2=\frac{1}{8}\mathrm{Tr}(\Phi\Phi^\dagger)$ and $(v^2-w^2)^2+4(v\cdot w)^2=\frac{1}{8}\mathrm{Tr}(\Phi\Phi^\dagger\Phi^\dagger\Phi)$ invariant under the full symmetry group. Therefore, the only quartic invariants are those included in $\mathcal{L}_\Phi$ of Eq. (2).

If $c<0$, the appearance of the $\mathrm{Tr}(\Phi\Phi^\dagger\Phi^\dagger\Phi)$ term in $\mathcal{L}_\Phi$ implies that the minimum of the potential occurs when v and w are parallel, and we limit our attention to this case. As long as $b>-c>0$, this leads to no instability since $\mathrm{Tr}(\Phi\Phi^\dagger)\geq\mathrm{Tr}(\Phi\Phi^\dagger\Phi^\dagger\Phi)$. An O(4) rotation brings the vacuum expectation values of v and w to zero in all but their fourth component. A $U(1)^{ch}$ transformation may then be performed so that we are free to choose $\langle\sigma\rangle\neq0$, with all other fields having zero vacuum expectation values, consistent with ordinary U(2) and *CP* symmetry being preserved (in the $\theta=0$ theory). We find $\langle\sigma\rangle=[a/(b+c)]^{1/2}$, the chiral $U(2)^{ch}$ is broken, the fermions receive a mass $\lambda\langle\sigma\rangle$ from their coupling to Φ, and there are four massless pseudoscalar Goldstone bosons,

η and π_i, $i=1,2,3$.

Now consider the effects of the Yang-Mills (color) pseudoparticle configurations (in Euclidean space) on these Goldstone states. Since Φ couples to the gauge field only through its Yukawa coupling to the fermions, we treat λ as the small perturbative expansion parameter. Although this may not be a reasonable expansion for making contact with low-energy meson phenomenology, it has the advantage of being systematic and illustrative of our more general conclusions.

We also restrict the calculation to instantons of a fixed scale size ρ. The problem of integrating over ρ and the resulting infrared difficulties must be faced in any complete treatment of instanton effects, but the infrared cutoff and confinement mechanism is not at all relevant to the questions being addressed here and so may be ignored for present purposes.

With $\langle\sigma\rangle$ and ρ fixed, we can always satisfy $\lambda\langle\sigma\rangle\rho \ll 1$ for λ sufficiently small. This means that the fermion mass is small compared to the energy scale set by the instantion size ρ. In the limit of zero mass, the effects of an instanton of definite size ρ on fermions are described by the effective Lagrangian[10]

$$\mathcal{L}_{\text{eff}}(z)=\kappa D(\rho)\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \epsilon_{ff'}\bar{\psi}_f(x_1)V(x_1,z,y_1)\times\psi_1(y_1)\bar{\psi}_{f'}(x_2)V(x_2,z,y_2)\psi_2(y_2)\,, \quad (6)$$

where $\kappa=e^{-8\pi^2/g^2}$, V is the nonlocal vertex function composed of normalized zero modes $u_0(x,z)$ in the presence of one pseudoparticle at z, and $D(\rho)$ is the result of the calculation of quantum fluctuations[5] about the classical solution, given by

$$V(x,z,y)=\overrightarrow{\not\partial}_x u_0(x,z)u_0^\dagger(y,z)\overleftarrow{\not\partial}_y\,,$$
$$D(\rho)=\frac{1}{\rho^2g^8}(4\pi)^8(0.6397)(\mu_0\rho)^6\,. \quad (7)$$

Here μ_0 is the renormalization point of the gauge coupling constant g.

In order for the corrections to the effective-Lagrangian approximation to be small, the dilute-gas limit for multi-instanton configurations must be considered. This means that ρ must be small enough so that g is small and $\kappa \ll 1$, which is certainly consistent with our previous requirement, $\lambda\langle\sigma\rangle\rho\ll 1$.

The contributions to the η self-energy to lowest order in $\mathcal{L}_{\text{eff}}$ (i.e., κ), λ, and the boson couplings b, c are illustrated in Fig. 1. Note that of the two possible four-point vertices generated by (6), only one appears in Fig. 1. This is because the exchange terms, obtained by interchanging two of the fermion lines emerging from the pseudo-

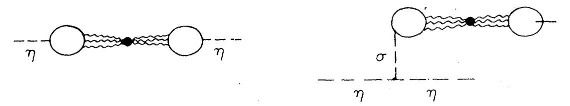

FIG. 1. Lowest-order pseudoparticle-induced η self-energy graphs. The dot represents the pseudoparticle (having $\nu=\pm 1$). A horizontal bar on a (solid) fermion line denotes a single mass insertion $\lambda\langle\sigma\rangle$ on otherwise massless fermion propagators.

particle interaction vanish identically owing to the flavor structure of the four-point vertex function. If the momentum flowing through the external legs is p, then the self-energy contributions from pseudoparticles of topological charge ± 1 add to yield

$$\Sigma_\eta(p^2)=4\lambda^2\kappa D(\rho)[F(p^2)+F(0)]\,, \quad (8)$$

where

$$F(p^2)=\left|\int d^4x\, u_0^\dagger(x,0)u_0(x,0)e^{ip\cdot x}\right|^2,$$
$$F(0)=1.$$

If multi-instanton configurations are considered, the self-energy diagrams of Fig. 1 are simply iterated in the dilute-gas limit. The resulting, geometric series, when summed, exhibits a massive pole in the η propagator. Evidently, in the approximation being considered,

$$m_\eta^{\,2}\cong\Sigma_\eta(p^2=0)=8\lambda^2\kappa D(\rho) \quad (9)$$

explicitly demonstrating the avoidance of Goldstone's theorem for the isosinglet state.

It is instructive to check this calculation of the η mass shift by means of a chiral Ward identity. The chiral current

$$J^{\mu 5}=\tfrac{1}{2}\sum_{f=1}^{2}\bar{\psi}_f\gamma^\mu\gamma^5\psi_f+\eta\partial^\mu\sigma - \sigma\partial^\mu\eta+\phi_i\partial^\mu\pi_i-\pi_i\partial^\mu\phi_i \quad (10)$$

has an anomalous divergence as in (1), giving the relation

$$\int d^4x\, iq_\mu\langle T\eta(x)J^{\mu5}(0)\rangle e^{iq\cdot x} = -i\langle\sigma\rangle+\int d^4x\, e^{iq\cdot x}\left\langle T\eta(x)\frac{g^2}{16\pi^2}F_{\mu\nu}\cdot\tilde{F}^{\mu\nu}(0)\right\rangle. \quad (11)$$

Without the anomaly *and* the pseudoparticle effects (zeroth order in κ), the right side of (11) is just $-i\langle\sigma\rangle$, indicating the presence of the massless Goldstone pole q^μ/q^2 in the amplitude at left before contraction with q_μ (Fig. 2): $m_\eta^{\,2}=0$ to this order.

To first order in κ, the η self-energy terms

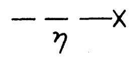

FIG. 2. Lowest-order graph in $\langle T\eta J^{\mu 5}\rangle$. The cross denotes the $J^{\mu 5}$ vertex with $\langle\sigma\rangle\,\partial^{\mu}\eta$ being the relevant term here.

appear on the left side of (11) (illustrated in Fig. 3) and the identity remains valid with the inclusion of the term contributing to the right side with the same singular $1/q^2$ behavior as $q^2 \to 0$ shown in Fig. 4.

Now, when the sum over multi-instanton configurations is performed, the η becomes massive and the left side of (11) behaves like $q^2/(q^2+m_\eta{}^2)$ as $q^2 \to 0$. The massless pole no longer exists and the right side of (11) now features a term illustrated as in Fig. 4 with the η *massive* propagator inserted. This term appears to be proportional to κ from the explicit appearance of the pseudoparticle interaction; however, $m_\eta{}^2 \sim \kappa$ so that as $q^2 \to 0$, the massive propagator contributes a factor of $1/\kappa$ which causes the result to be independent of κ. Thus, the Ward identity may be reconsidered to zeroth order in κ. The left side of (11) vanishes: There is no Goldstone pole. Instead we obtain a consistency check on the η mass as given by (9). The Euclidean version of (11) gives

$$0 = -\langle\sigma\rangle - \frac{i}{m_\eta{}^2}\left[2\lambda^2\kappa D(\rho)\langle\sigma\rangle\right] \times \sum_{\nu=\pm 1} 2\nu\left[-i\int d^4x\, u_0^\dagger(x)\gamma^5 u_0(x)\right] \times\left[\int d^4x\, u_0^\dagger(x) u_0(x)\right]. \tag{12}$$

Using $\gamma^5 u_0 = \mp u_0$ for $\nu = \pm 1$ and solving for $m_\eta{}^2$ then recovers (9), thereby confirming the previous direct computation and explicitly demonstrating the removal of the 0^- Goldstone boson through the chiral Ward identity.

Because the isotriplet chiral current does not have an anomalous divergence, the pseudoparticle-induced $\vec{\pi}$ self-energy graphs do not give the $\vec{\pi}$ field a mass, as may be verified by direct calculation. Goldstone's theorem is valid in all orders of perturbation theory and the $\vec{\pi}$ remains massless. These elementary lowest-order results already illustrate the essential features of

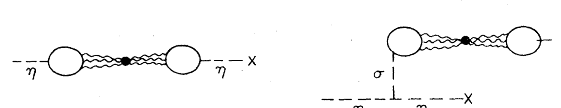

FIG. 3. First-order self-energy insertions of pseudoparticle interaction contributing to $\langle T\eta J^{\mu 5}\rangle$.

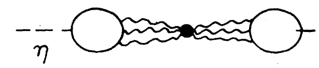

FIG. 4. Lowest-order contribution to $\langle T\eta F\tilde{F}\rangle$.

the model; however, the one-boson-loop corrections further point out the subtleties of the κ expansion and for that reason are treated in Appendix A.

III. θ DEPENDENCE

The existence of the instanton solutions in non-Abelian gauge theories has been shown to require a more complicated vacuum state[8] than the naive vacuum (for which $\nu = 0$). In fact, the stable vacuum state is specified by the phase angle θ which enters the functional integral prescription for transition amplitudes through a factor $e^{i\nu\theta}$ multiplying the amplitude in the topological sector where

$$\nu = \frac{g^2}{32\pi^2}\int d^4x\, F_{\mu\nu}\cdot\tilde{F}_{\mu\nu}$$

is fixed (in Euclidean space). An equivalent description is to consider this term as part of the action, so that θ appears in the Lagrangian multiplying the topological charge density $(g^2/32\pi^2)$ $\tilde{F}_{\mu\nu}\cdot F_{\mu\nu}$. The class of theories with $\theta \neq 0$ (or $2\pi n$) would thus seem to violate P and CP invariance strongly.

However, when $\mathcal{L}$ possesses a $U(1)^{ch}$ symmetry as in pure (massless) QCD or the present model, it is possible to show that the θ parameter by itself is not physically meaningful. For, if a chiral U(1) rotation of (finite) magnitude β is performed on the fields,

$$\begin{aligned}\psi &\to \psi' = e^{i\beta\gamma^5/2}\psi\,,\\ \sigma &\to \sigma' = \sigma\cos\beta - \eta\sin\beta\,,\\ \eta &\to \eta' = \eta\cos\beta + \sigma\sin\beta\end{aligned} \tag{13}$$

in the functional integral, then because of the chiral current anomaly (1), we find $\theta \to \theta' = \theta - 2\beta$. Thus, a chiral rotation of $\beta = \theta/2$ and redefinition of the fields through (13) returns us to a theory in all respects physically equivalent to the original one with $\theta = 0$ and the old fields. This is the realization of Peccei and Quinn that a conserved CP operation can be defined for all θ in QCD or any theory with fermions and a $U(1)^{ch}$ symmetry.[11]

When the $U(1)^{ch}$ symmetry is broken, either spontaneously through scalar fields or by the addition of an explicit mass matrix to the Lagrangian

$$\bar{\psi}M\frac{(1-\gamma_5)}{2}\psi + \bar{\psi}M^\dagger\frac{(1+\gamma_5)}{2}\psi\,, \tag{14}$$

we are still free to make the transformation (13). However, β is already determined by the condition that M be Hermitian so that the fermion mass(es) are real. Thus $\alpha = \theta - \arg(\det M)$ is fixed and is not in general equal to 0, unless the dynamics of the instanton effects require $\arg(\det M) = \theta$. In fact, Peccei and Quinn[11] showed that this is indeed the case in the theory of a single-flavor fermion Yukawa coupled to a spinless field that has its $U(1)\times U(1)^{ch}$ symmetry spontaneously broken down to U(1). Following their methods it can be shown that the same phenomenon occurs in the present model, which is really just the model of Ref. 11 for the case of two flavors ($N=2$).

The relevant quantity for this demonstration is the boson effective potential, corrected for instanton effects. It is possible to argue from general properties of the functional integration over the fermion fields that $\arg(\det M) = \arg(\langle\sigma\rangle + i\langle\eta\rangle)^2$ will be fixed by the value of θ if the vacuum is a minimum of the effective potential, just as in the one-flavor case. However, it will be sufficient to calculate V_{eff} to the lowest nontrivial order ($\sim \kappa\lambda^2$) as in Sec. II.

The θ vacuum-to-vacuum transition amplitude in the one-pseudoparticle sector receives a contribution first order in $\mathcal{L}_{eff}$ and second order in λ from the term, in Euclidean space,

$$\left\langle T\frac{1}{2!}\left[\int d^4x\, \lambda\bar{\psi}_i \Phi_{ij}\psi_j\right]\left[\int d^4y\, \lambda\bar{\psi}_k \Phi_{kl}\psi_l\right]\left[\int d^4z\, \mathcal{L}_{eff}(z)\right]\right\rangle e^{i\theta}. \tag{15}$$

Considering only the flavor structure of $\mathcal{L}_{eff}$, it is clear that the various possible fermion contractions yield

$$\frac{1}{2!}\,2(\delta_{i1}\delta_{k2}-\delta_{i2}\delta_{k1})(\delta_{j1}\delta_{l2}-\delta_{j2}\delta_{l1})\Phi_{ij}^{(+)}(x)\,\Phi_{kl}^{(+)}(y)$$

or (16)

$$\Phi_{11}^{(+)}(x)\Phi_{22}^{(+)}(y)-\Phi_{12}^{(+)}(x)\Phi_{21}^{(+)}(y)+(x \leftrightarrow y)$$

showing the familiar determinant structure. The superscript (+) denotes the topological sector $\nu=+1$. Since $\gamma^5 u_0 = \mp u_0$ for the zero modes in the cases $\nu=\pm 1$, respectively, we have

$$\Phi^{(+)} = \sigma \pm i\eta + (\vec{\Phi} \mp i\vec{\pi})\cdot\vec{\tau}. \tag{17}$$

Thus, the amplitude (15) is

$$\lambda^2\kappa D(\rho)\int d^4z\, d^4x\, d^4y\{[\sigma(x)+i\eta(x)][\sigma(y)+i\eta(y)]-[\vec{\phi}(x)-i\vec{\pi}(x)]\cdot[\vec{\phi}(y)-i\vec{\pi}(y)]\}$$
$$\times e^{i\theta}u_0^\dagger(x,z)u_0(x,z)u_0^\dagger(y,z)u_0(y,z). \tag{18}$$

For $\nu=-1$, we have $e^{i\theta}\to e^{-i\theta}$ and $\Phi^{(+)}\to\Phi^{(-)}=\Phi^{(+)\dagger}$. Thus, the sum of $\nu=\pm1$ contributions is twice the real part of (18).

Now, in the dilute-gas limit, multipseudoparticle configurations simply result in the exponentiation of the single-pseudoparticle amplitude. Hence, we may view the result of the integration over fermion degrees of freedom as generating a $U(1)^{ch}$-breaking term in the boson potential[9]:

$$\Delta V(z) = -2\lambda^2\kappa D(\rho)\int d^4x\, d^4y\, \mathrm{Re}\, e^{i\theta}\{[\sigma(x)+i\eta(x)][\sigma(y)+i\eta(y)]$$
$$-[\vec{\phi}(x)-i\vec{\pi}(x)]\cdot[\vec{\phi}(y)-i\vec{\pi}(y)]\}u_0^\dagger(x,z)u_0(x,z)u_0^\dagger(y,z)u_0(y,z). \tag{19}$$

If we now allow the σ and η fields to develop expectation values (independent of x)

$$\langle\sigma\rangle + i\langle\eta\rangle = ve^{-i\beta}, \tag{20}$$

we find the following value of the potential in the vacuum state:

$$V_{eff} = -\frac{a}{2}v^2 + \frac{(b+c)}{4}v^4 - 2\lambda^2\kappa D(\rho)\cos(\theta-2\beta)v^2, \tag{21}$$

so that $\theta - 2\beta$ must be 0 (or $2\pi n$) at the minimum of V_{eff} implying

$$\beta = \theta/2,\ \theta/2+\pi,\qquad v^2 = \frac{a+4\lambda^2\kappa D(\rho)}{b+c}. \tag{22}$$

The latter equation indicates that v is shifted from its tree value by instanton effects; it can also be calculated from the graph of Fig. 5 (in the $\theta=0$ case). The first equation is more in-

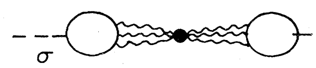

FIG. 5. Graph contributing to shift in $v=\langle\sigma\rangle$.

teresting. It shows that there are two vacuum states which minimize V_{eff} which are therefore in all respects physically equivalent—except in their values of β. As expected, V_{eff} is 2π periodic in θ; however, since a chiral rotation of 2β is equivalent to changing θ, a chiral rotation of only π is required to return to the same physical theory. Since the formal specification of the vacuum requires β (and not θ), and a β chiral rotation of 2π is necessary to return to the same formally specified vacuum state, it is not at all surprising to find that quantities which distinguish the two equivalent vacuums (such as $\langle\sigma\rangle$) have a θ periodicity of *twice* the normal result, i.e., 4π instead of 2π. Indeed, Eqs. (22) show that $\langle\sigma\rangle=+v\cos(\theta/2)$ (for the case $\beta=\theta/2$) has a θ periodicity of 4π in the spontaneous broken case.[12]

IV. DISCUSSION

The fact that some quantities in the theory can show a θ periodicity different from 2π indicates that the θ dependence is quite subtle in the spontaneously broken case. For example, consider the selection rule cited by Crewther[7] which derives from the chiral Ward identity

$$\sum_\nu \int d^4x\, \partial_\mu \left\langle T J^{\mu 5}(x) \prod_n O_n \right\rangle_\nu e^{i\nu\theta} = \sum_\nu \left(2\nu - \sum_m \chi_m\right) \left\langle T \prod_n O_n \right\rangle_\nu e^{i\nu\theta}, \quad (23)$$

where the operators O_n have chirality χ_n. This implies

$$\int d^4x\, \partial_\mu \left\langle T J^{\mu 5}(x) \prod_n O_n \right\rangle_\nu = \left[2\nu - \sum_m \chi_m\right] \left\langle T \prod_n O_n \right\rangle_\nu = 0 \quad (24)$$

if we assume that the left side of (23) vanishes for all θ, as implied by the supposed lack of a Goldstone pole in any physical amplitude *and* if the θ dependence is completely accounted for by the $e^{i\nu\theta}$ factor in Eq. (23). If for the $\Pi_n O_n$ we substitute an operator having the quantum numbers of $\bar\psi(1\pm\gamma_5)\psi$ (for which $\chi=\pm1$), we find that $2\nu-\chi\neq0$ for all integers ν, implying that the matrix element $\langle\bar\psi(1\pm\gamma_5)\psi\rangle$ or $\langle\sigma\pm i\eta\rangle$ must vanish identically. In pure QCD we would then conclude that if (24) were correct, the spontaneous breaking of $U(2)^{ch}$ and generation of quark masses is inconsistent with the 't Hooft resolution of the U(1) problem for integral ν.

Of course, in the present model (24) is *not* correct and spontaneous symmetry breaking of $U(2)^{ch}$ does occur. The formal loophole is that the θ dependence in the spontaneously broken theory is not as simple as the transition from (23) to (24) would imply. The amplitude $\langle TJ^{\mu5}(x)\Pi_n O_n\rangle_\nu$ is not θ independent since the functional integral requires a choice between the two physically equivalent boson vacuum states indicated in (22), i.e., a specification of β, and the value of β is fixed by θ at the minima of V_{eff}. In fact, β is fixed so that for either choice in (22), the periodicity in θ of the amplitude is *twice* 2π, even though only integral ν has been considered. This causes no contradiction with the obviously true assertion that $\theta\to\theta+2\pi$ must result in the same physical theory; it simply informs us that quantities which distinguish the two equivalent vacuums (such as $\arg\langle\sigma\rangle$) are not physically measurable quantities.

Thus, $\langle TJ^{\mu5}\Pi_n O_n\rangle_\nu$ should not in general be physically measurable. That this is indeed the case in our model is exemplified by the term

$$\int d^4x\, e^{iq\cdot x}\langle TJ^{\mu5}(x)\eta(0)\rangle_{\nu=1}$$

illustrated in Fig. 3. This term has a singular behavior $q^\mu/(q^2)$ as $q^2\to0$ which does not correspond to any physical massless particle in the theory. In fact, it is just the sum over more and more singular terms in the iteration of the self-energy graphs (in higher ν sectors) that results in the *massive* propagator $1/(q^2+m_\eta^2)$ which is the physically meaningful object. The removal of the Goldstone pole in the θ amplitude (summed over ν) does not imply the removal of (unphysical) $q^2\to0$ singularities in the individual ν amplitudes—a result which *would* be implied if the θ dependence were completely given by the $e^{i\nu\theta}$ factor alone in Eq. (23) so that the individual ν sector amplitudes were measurable in the spontaneously broken theory with no Goldstone poles. The 4π θ periodicity of $\langle\sigma\rangle$ is a signal that this is not the case.

The question of the θ periodicity is thus quite clear in the scalar field model presently being considered. However, the only role of the scalar field multiplet in this argument is to provide an explicit spontaneously broken vacuum state with the appropriate quantum numbers. In pure QCD we expect $\langle\sigma\rangle$ and $\langle\eta\rangle$ to be replaced by $\langle\sum_f\bar\psi_f\psi_f\rangle$ $\langle\sum_f\psi_f\gamma^5\psi_f\rangle$, respectively, and that these vacuum expectation values will have a phase angle β that enters the effective potential in the combination $\theta-2\beta$. The same 4π periodicity in θ would then be recovered for all amplitudes which distinguish between the two physically equivalent vacuums and (24) would be invalidated just as in the present scalar model.

However, there is an additional complication. In massless QCD, the selection rule (24) is a statement of the Atiyah-Singer index theorem,[13] valid for all gauge theories definable on a compactifiable space. This theorem relates the numbers of positive-and negative-chirality zero modes (n_+, n_-) of the operator $\not{\partial}-ig\not{A}$ to the topological charge of the gauge field: $\nu = n_- - n_+$. This means that the functional-integral expression for the amplitude $\langle T\,\Pi_n O_n\rangle_\nu$ vanishes unless the chiralities of the O_n satisfy $\sum_n \chi_n = 2\nu$. Thus, this theorem provides a derivation of the selection rule (24) *independently* of the θ periodicity considerations above; and as has already been emphasized, the selection rule is *inconsistent* with the spontaneous breaking of the full $U(2)^{ch}$ symmetry and resultant 4π periodicity of the vacuum.

Notice, however, that for the index theorem to be applicable at all, the fermion functional integral must be considered after the decomposition of the gauge field integral into ν sectors, but *before* any gauge field integrations have been performed. On the other hand, spontaneous symmetry breaking in pure QCD will not occur unless there is some term in the effective potential for the fermions which minimizes V_{eff} at $\langle \sum_f \bar\psi_f \psi_f\rangle \neq 0$; and such a term is possible *only* if the gauge field functional integration is performed first. Thus, the difficulty with the index theorem may be resolved by some noncommutativity of the gauge-field and fermion-field functional integrals in QCD. Alternatively, the compactification assumption required for the index theorem to be applicable may be incorrect in the spontaneously broken case, so that the usual semiclassical WKB methods are inadequate, as suggested by Crewther.[7] There is no need, however, for considering the possibility of nonintegral ν in the scalar model. The sum over ν is simply not a Fourier series expansion of the θ amplitude in this case.

The intimate connection between the solution of the U(1) problem, the index theorem, and $U(2)^{ch}$-symmetry breaking may be seen in another way. Starting with (1) and using the standard current-algebra method of chiral perturbation theory in the small (explicit) fermion mass M, it follows that[7]

$$m_\pi^2 F_\pi^2 = -M\frac{g^2}{16\pi^2}\int d^4x \left\langle T F_{\mu\nu}\cdot \tilde F_{\mu\nu}(x)\left[\sum_f \bar\psi_f \gamma_5 \psi_f\right]\right\rangle + O(m_\pi^4), \tag{25}$$

where the absence of any pseudoscalar massless pole in the physical amplitudes has also been assumed. The relation shows quite clearly that the nontrivial topological charge configurations in QCD give rise to the pion-decay constant F_π demonstrating their bearing on the $U(2)^{ch}$ breaking. Now, in the chiral-symmetry limit $M \sim m_\pi^2 \to 0$, F_π is taken to be virtually constant, i.e., independent of M. If the quantity at right in (25) is estimated in the same limit, then the index theorem informs us that in the ν sector, this amplitude varies with M like $MM^{2|\nu|-1} = M^{2|\nu|}$ as $M \to 0$ for the case $N=2$. Since ν is an integer, we find the leading terms in (25) come from $\nu = \pm 1$ which are of order $M^2 \sim m_\pi^4$, thus arriving at a contradiction unless F_π also varies with M in an unconventional way or $F_\pi = 0$ identically.

This impasse is overcome in the scalar model in a very simple way. The index theorem no longer applies and the fermions obtain a contribution to their mass $\lambda\langle\sigma\rangle$ separately from the explicit mass M. However, after spontaneous symmetry breaking has occurred this term enters the fermion determinant appearing in the functional-integral expression for (25) in exactly the same way as M. Thus, for $\nu = \pm 1$ we obtain the estimate $M(M + \lambda\langle\sigma\rangle) \sim M \sim m_\pi^2$ in the chiral limit $M \to 0$. The contradiction is removed by the explicit appearance of the spontaneously generated fermion mass. If the quark masses in QCD are generated (at least in part) by some dynamical mechanism, this loophole may be operative in QCD also. Thus, we see from another point of view how the conditions for the index theorem to be applicable in the case of spontaneous breaking of $U(2)^{ch}$ symmetry *must not* be satisfied, in order to solve the U(1) problem in a manner consistent with the techniques of chiral perturbation theory. Since the index theorem does not apply to a model with scalar fields as we have been considering, we can explicitly show the absence of any difficulty reconciling the pseudoparticle resolution of the U(1) problem and the techniques of current algebra. The θ-periodicity discussion leads one to believe that this will also be true in QCD. It is for this reason that the self-consistent models of $U(2)^{ch}$ breaking[6] that go beyond the semiclassical approximation and attempt to circumvent the index theorem do not run into any difficulties with current-algebra relations. The effective-potential minimization of Sec. III is an explicit example of how the semiclassical results can be circumvented by the introduction of spontaneous symmetry breaking into the model in a way consistent with the first-order quantum corrections to V_{eff}. It is this avoidance of the most naive semiclassical approximations that Crewther has repeatedly emphasized as necessary to a fuller understanding of the U(1) problem in the (real world) case of broken chiral symmetries.

In summary, the scalar model of chiral-symmetry breaking shows that there is no formal objection to the solution of the U(1) problem proposed by 't Hooft. Instead, the problem is to understand the detailed dynamical mechanism of chiral-symmetry breaking in QCD which realizes the θ dependence and index theorem inapplicability conjectures presented here in connection with the scalar field model.

ACKNOWLEDGMENTS

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APPENDIX A

Although the lowest-order results of Sec.II are sufficient to illustrate the essential features of the η mass generation and resolution of the U(1) problem through instanton effects and the current anomaly (1), it is instructive to carry out the calculation to the one-boson-loop level (first order in b or c). The relevant self-energy graphs are shown in Figs. 6 and 7. Note that when the isotriplet $\vec{\pi}$ or $\vec{\phi}$ fields couple to the instanton through the fermion loop that the exchange interaction of $\mathcal{L}_{\text{eff}}$ gives the only nonvanishing terms as may be seen from (6) and (16). These terms differ from those of Fig. 6 by an overall relative minus sign and a factor of 3 due to the three possible choices for $i = 1, 2, 3$ and are treated separately below.

The contribution of the terms represented in Fig. 6 to the η self-energy is

$$(b+c)\lambda^2\kappa D(\rho)\int d^4k\,F(k^2)\left\{-\frac{6}{(k^2+m_\sigma^{\,2})^2}-\frac{2}{(k^2)^2}+\frac{2}{(k^2+m_\sigma^{\,2})^2}+\frac{6}{(k^2)^2}-\frac{8(b+c)\langle\sigma\rangle^2}{(k^2+m_\sigma^{\,2})^2(k+p)^2}-\frac{8(b+c)\langle\sigma\rangle^2}{(k^2)^2[(k+p)^2+m_\sigma^{\,2}]}\right\}. \tag{A1}$$

If the lowest-order relation $2(b+c)\langle\sigma\rangle^2 = m_\sigma^{\,2}$ is used, then (A1) vanishes at $p=0$ and there would seem to be no corrections to $\Sigma_\eta(0)$ first order in b or c. However, we know from (22) and Fig. 5 that $\langle\sigma\rangle$ is shifted from its tree level of $[a/(b+c)]^{1/2}$; also, the σ self-energy diagrams corresponding to Fig. 1 for the η show that there is also a σ self-energy due to pseudoparticle effects:

$$\Sigma_\sigma(p^2) = 4\lambda^2\kappa D(\rho)[-F(p^2)+3F(0)]\,, \tag{A2}$$

so that $\Sigma_\sigma(p^2=0)=\Sigma_\eta(p^2=0)=8\lambda^2\kappa D(\rho)$. These results could also have been deduced from (19) and the minimization of V_{eff}. Thus, $2(b+c)\langle\sigma\rangle^2 \neq m_\sigma^{\,2}$ to order κ. However, these corrections contribute to (A1) only in order κ^2 and there must be a contribution to the η self-energy first order in κ, as is evident from the Ward identity (11) and the nonvanishing of the term first order in κ in $\langle T\eta(x)F_{\mu\nu}\cdot\tilde{F}_{\mu\nu}(0)\rangle$. This term, illustrated in Fig. 8, has the value (as $q^2\to 0$)

$$16(b+c)\frac{\lambda^2\kappa}{m_\eta^{\,2}}D(\rho)\int d^4k\,F(k^2)\frac{1}{k^2}\,\frac{1}{k^2+m_\sigma^{\,2}}\,. \tag{A3}$$

The problem is that the graphs of Fig. 6 do not give the only contributions to Σ_η to this order. The correct result is obtained by considering the (ordinary) self-energy graphs of Fig. 9 with the propagator functions for the internal η and σ lines *corrected* by the η and σ self-energies of (8) and (A2). The sum of these terms (at $p=0$) is

$$(b+c)\int d^4k\left\{\frac{3}{k^2+2a+\Sigma_\sigma(k^2)}+\frac{1}{k^2+\Sigma_\eta(k^2)}+\frac{1}{k^2+2a+\Sigma_\sigma(k^2)}-\frac{3}{k^2+\Sigma_\eta(k^2)}+\frac{4(b+c)\langle\sigma\rangle^2}{[k^2+\Sigma_\eta(k^2)][k^2+2a+\Sigma_\sigma(k^2)]}\right\}. \tag{A4}$$

Of course, to lowest order in κ, this vanishes, which is just a statement of Goldstone's theorem at the one-loop level. Expanding to first order in κ recovers the terms in (A1) [corresponding to the $F(k^2)$ part of Σ_σ and Σ_η] and, in addition, gives those one-loop diagrams featuring the $F(0)$ part of the σ and η self-energies, inserted onto the internal σ

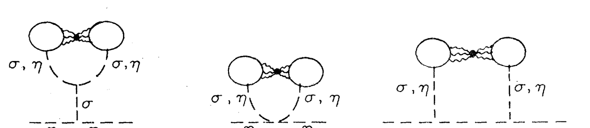

FIG. 6. Pseudoparticle-induced η self-energy graphs first order in the boson couplings b or c, deriving from the first (direct) term in $\mathcal{L}_{\text{eff}}$.

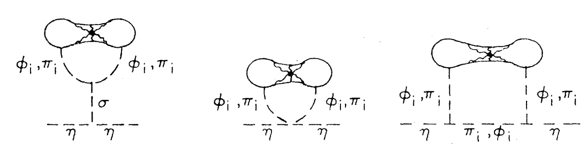

FIG. 7. Pseudoparticle-induced η self-energy graphs first order in b or c deriving from the second (exchange) term in $\mathcal{L}_{\text{eff}}$.

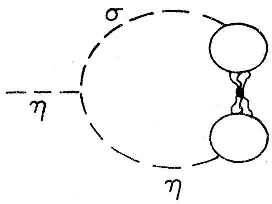

FIG. 8. Direct contribution to $\langle T\eta F\tilde{F}\rangle$ first order in b, c, and κ.

and η loops of Fig. 9. The result of expanding (A4) to first order in κ is

$$(b+c)\int d^4k\left[-\frac{2\Sigma_\sigma(k^2)}{(k^2+2a)^2}+\frac{2\Sigma_\eta(k^2)}{(k^2)^2}-\frac{4a\Sigma_\eta(k^2)}{(k^2)^2(k^2+2a)}-\frac{4a\Sigma_\sigma(k^2)}{(k^2)(k^2+2a)^2}+\frac{16\lambda^2\kappa D(\rho)}{(k^2)(k^2+2a)}\right], \qquad \text{(A5)}$$

where the last term is the correction of $\langle\sigma\rangle^2$ to this same order, as given by (22). Combining terms and using (8) and (A2) gives for (A5)

$$16(b+c)\lambda^2\kappa D(\rho)\int d^4k\,F(k^2)\frac{1}{k^2}\frac{1}{k^2+2a}. \qquad \text{(A6)}$$

Comparison with (A3) and (11) shows that this is just the correction to $m_\eta^{\,2}$ required to confirm the Ward identity to the one-boson-loop level.

The additional graphs of Fig. 7 are analogous to the previous set. That is, in order to correctly account for the total contribution to η self-energy from the isotriplet $\vec{\phi}$ and $\vec{\pi}$ boson loops, the lowest-order $\vec{\phi}$ and $\vec{\pi}$ self-energies must first be calculated from the graphs of Fig. 10. These give

$$\begin{aligned}\Sigma_\phi(p^2)&=4\lambda^2\kappa D(\rho)\left[F(p^2)+\frac{b-c}{b+c}F(0)\right],\\ \Sigma_\pi(p^2)&=4\lambda^2\kappa D(\rho)\,[-F(p^2)+F(0)]\,.\end{aligned} \qquad \text{(A7)}$$

Then these self-energies are inserted into the $\vec{\phi}$ and $\vec{\pi}$ propagators of Fig. 11 to give the following one-loop correction to the η self-energy:

$$\begin{aligned}3(b-c)&\int d^4k\left[\frac{1}{k^2+m_\phi^{\,2}+\Sigma_\phi(k^2)}-\frac{1}{k^2+\Sigma_\pi(k^2)}\right]\\ &+3(b+c)\int d^4k\left[\frac{1}{k^2+\Sigma_\pi(k^2)}-\frac{1}{k^2+m_\phi^{\,2}+\Sigma_\phi(k^2)}\right]\\ &+12c^2\langle\sigma\rangle^2\int d^4k\left[\frac{1}{(k+p)^2+\Sigma_\pi((k+p)^2)}\right]\\ &\qquad\times\left[\frac{1}{k^2+m_\phi^{\,2}+\Sigma_\phi(k^2)}\right],\end{aligned} \qquad \text{(A8)}$$

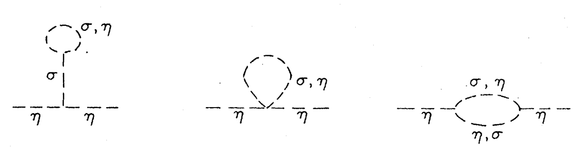

FIG. 9. Ordinary one- (isosinglet-) loop self-energy graphs.

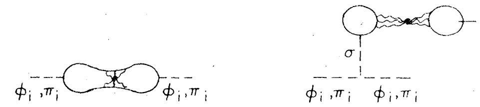

FIG. 10. Lowest-order $\vec{\phi}$ and $\vec{\pi}$ self-energy graphs.

where $m_\phi^{\,2}=-2ac/(b+c)$ is the $\vec{\phi}$ mass squared in tree-level approximation. To zeroth order in κ, (A8) vanishes at $p=0$, again demonstrating Goldstone's theorem to lowest order. Expanding (A8) to first order in κ (at $p=0$) then gives

$$6c\int d^4k\frac{1}{k^2}\frac{1}{k^2+m_\phi^{\,2}}\left[\Sigma_\phi(k^2)-\Sigma_\pi(k^2)+\frac{8c\lambda^2\kappa D(\rho)}{b+c}\right], \qquad \text{(A9)}$$

where the last term again arises from the correction to $\langle\sigma\rangle^2$ from (22). Using (A7), this becomes

$$48c\lambda^2\kappa D(\rho)\int d^4k\,F(k^2)\frac{1}{k^2}\frac{1}{k^2+m_\phi^{\,2}}, \qquad \text{(A10)}$$

which is to be added to (A6) in order to obtain the complete one-loop correction to the η mass of Eq. (9). Comparing these with the appropriate contribution to $\langle T\eta(x)F_{\mu\nu}\tilde{F}_{\mu\nu}(0)\rangle$ illustrated in Fig. 12 and having the value

$$48c\lambda^2\kappa D(\rho)\frac{1}{m_\eta^{\,2}}\int d^4k\,F(k^2)\frac{1}{k^2}\frac{1}{k^2+m_\phi^{\,2}}, \qquad \text{(A11)}$$

we find that the contributions to the η mass of (A6) and (A10) are just those required to satisfy the Ward identity (11) at the one-boson-loop level.

APPENDIX B

In Secs. III and IV it was shown that the scalar field model can exhibit a θ periodicity different from 2π (through the formal specification of the vacuum state) and that this provides a loophole in the derivation of the selection rule, Eq. (24) from Eq. (23)—without the introduction of fractional ν values. However, as the discussion of Sec. IV emphasizes, this θ dependence of formally specified vacuum expectation values is somewhat unphysical, for $\theta\to\theta+2\pi$ does leave physically measurable quantities unchanged. The purpose of this appendix is to show that it is possible to fix the vacuum phases in the model so that they do indeed exhibit a θ periodicity of 2π, while still circumventing the selection rule in (integral) ν sectors.

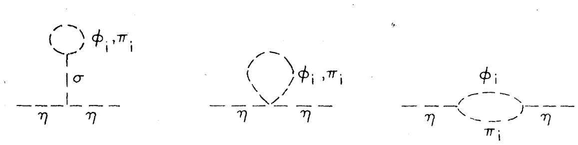

FIG. 11. One- (isotriplet-) loop self-energy graphs.

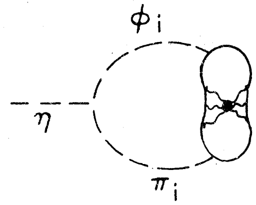

FIG. 12. Exchange contribution to $\langle T\eta F\tilde{F}\rangle$ first order in c and κ.

If, instead of Eq. (20), we let

$$\langle\sigma+i\eta+\phi_3-i\pi_3\rangle=ve^{-i\beta}\,,\qquad \langle\sigma+i\eta-\phi_3+i\pi_3\rangle=ve^{-i\gamma}\,, \tag{B1}$$

which is the most general form for a diagonal fermion mass matrix, then, repeating the effective-potential calculation and using Eq. (19) gives

$$V_{\text{eff}}=-\frac{a}{2}v^2+\frac{(b+c)}{4}v^4-2\lambda^2\kappa D(\rho)\cos(\theta-\beta-\gamma)v^2 \tag{B2}$$

instead of Eq. (21). Now, any choice of β and γ with $\beta+\gamma=\theta$ minimizes V_{eff} and we might just as well choose $\gamma=0$, $\beta=\theta$ instead of the previous specification of $\gamma=\beta=\theta/2$. The expectation values of Eq. (B1) now have a θ periodicity of 2π, not 4π. This is merely another illustration of the artificial, convention-dependent nature of the θ periodicity of physically unmeasurable amplitudes.[14]

With a completely 2π periodic vacuum it may now seem that the problem of the selection rule is again with us. However, this is not the case, because now the assumption that the left side of Eq. (23) vanishes for *all* θ is no longer valid. This can most clearly be seen if we remove all ambiguities in the vacuum phase angles by explicitly breaking the SU(2) × SU(2)$^{\text{ch}}$ symmetry:

$$\mathcal{L}\to\mathcal{L}+\epsilon_0\sigma+\epsilon_3\phi_3\,, \tag{B3}$$

where we may then allow $\epsilon_0,\ \epsilon_3\to 0^+$ at the end of the analysis, if desired.

Using Eqs. (B1), the effective potential is now

$$V_{\text{eff}}=-\frac{a}{2}v^2+\frac{(b+c)}{4}v^4-2\lambda^2\kappa D(\rho)\cos(\theta-\beta-\gamma)v^2-\frac{\epsilon_0 v}{2}(\cos\beta+\cos\gamma)-\frac{\epsilon_3 v}{2}(\cos\beta-\cos\gamma)\,. \tag{B4}$$

In the limit that ϵ_0, ϵ_3 are small, it is sufficient to minimize V_{eff} with respect to β and γ subject to the constraint $\beta+\gamma=\theta$: i.e., we minimize

$$f(\beta,\gamma)=C(\theta-\beta-\gamma)-\frac{\epsilon_0 v}{2}(\cos\beta+\cos\gamma)-\frac{\epsilon_3 v}{2}(\cos\beta-\cos\gamma) \tag{B5}$$

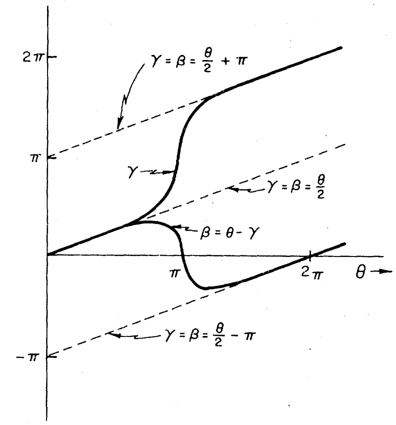

FIG. 13. The dependence of the vacuum phase angles β and γ on θ in the case ϵ_0 and ϵ_3 are both nonzero, with $r=(\epsilon_3/\epsilon_0)\cong 0.05$. The switchover region near $\theta=\pi$ becomes vanishingly small as $\epsilon_3\to 0$.

to obtain

$$C=\sin\beta\left(\frac{\epsilon_0+\epsilon_3}{2}\right)v=\sin\gamma\left(\frac{\epsilon_0-\epsilon_3}{2}\right)v\,,\qquad \beta+\gamma=\theta\,. \tag{B6}$$

If we first consider $\epsilon_3=0$, $\sin\beta=\sin\gamma$ and we have three solutions:

(i) $\beta=\gamma=\theta/2$,

(ii) $\beta=\gamma=\theta/2+\pi$,

(iii) $\beta=\pi-\gamma$ (at $\theta=\pi$).

For a true minimum of the effective potential we

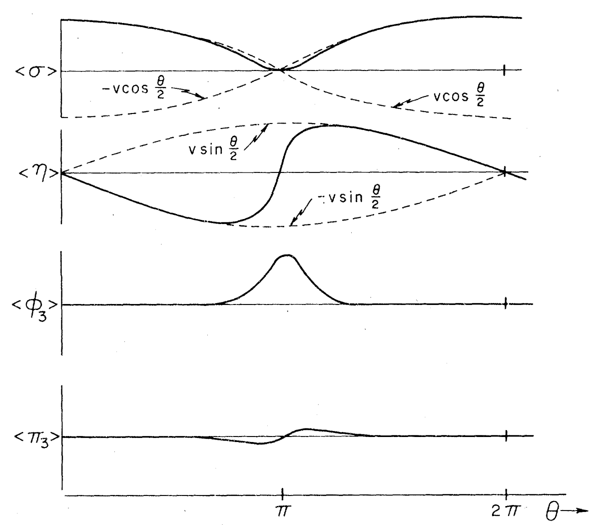

FIG. 14. The corresponding dependence of the vacuum expectation values for $r\cong 0.05$. The $\langle\phi_3\rangle$ field develops an expectation value near $\theta=\pi$ due to the switching from one solution to another in order to minimize V_{eff}.

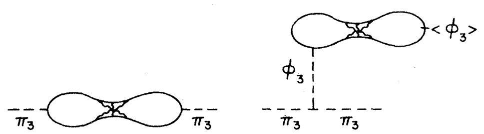

FIG. 15. The π_3 self-energy contributions for the case $\langle\sigma\rangle=\langle\eta\rangle=\langle\pi_3\rangle=0$, $\langle\phi_3\rangle=v$ which obtains at $\theta=\pi$. The two exchange graphs are of the same sign and sum to give a nonvanishing contribution to the π_3 self-energy.

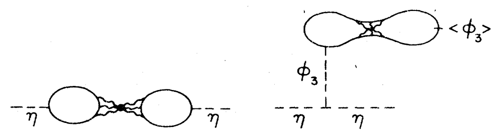

FIG. 16. The η self-energy contributions for $\theta=\pi$. There is a relative minus sign between the direct and exchange contributions which results in a cancellation between the terms as $p^2\to 0$. Hence, the η propagator remains massless.

must have

(i) $\beta=\gamma=\theta/2$ for $0\leq\theta<\pi$,

(ii) $\beta=\gamma=\theta/2+\pi$ for $\pi<\theta\leq 2\pi$,

which is 2π periodic in θ and continuous at $\theta=0$ and 2π. The symmetry breaking ϵ_0 determines the unique minimum of V_{eff} for all θ except $\theta=\pi$. With ϵ_3 strictly zero there is a discontinuous switchover from case (i) to (ii) as θ goes through π. The $\theta\sim\pi$ region may be examined more carefully by allowing $\epsilon_3\neq 0$ and solving Eqs. (B6) for $\beta(\theta)$ and $\gamma(\theta)$. Then, the switchover is continuous[15] and depends only on the dimensionless ratio $r=\epsilon_3/\epsilon_0$. Figures 13 and 14 illustrate the behavior of the phase angles β and γ and the expectation values for the case $r\cong 0.05$.

With the physical vacuum now determined for all θ we indeed find that all the θ dependence is 2π periodic (as it must be, according to the functional-integral expression for amplitudes, properly defined, with no formal ambiguities in the specification of the vacuum). Notice, however, what has happened. At $\theta=\pi$ both the σ and η fields have zero expectation values, but ϕ_3 does not. The σ and ϕ_3 fields have interchanged roles at $\theta=\pi$ relative to $\theta=0$. Correspondingly, it is the π_3 field which acquires a mass from instanton interactions at $\theta=\pi$ while the η remains massless. This is easily verified explicitly from the graphs in Figs. 15 and 16.

Thus, in the case of explicit $SU(2)\times SU(2)^{ch}$ symmetry breaking (as well as in the symmetric case for the choice of phases $\gamma=0,\beta=\theta$ mentioned previously), the η field retains a massless component for some values of θ. The physical vacuum is completely fixed by the symmetry breaking for all θ and is indeed 2π periodic; thus, the Fourier transformation involved in passing from Eq. (23) to Eq. (24) may be correctly performed. However, the left side of Eq. (23) does *not* vanish for all θ, receiving as it does a contribution from the still massless η pole near $\theta=\pi$. Hence, the selection rule $(2\nu-\Sigma_m\chi_m)\langle T\prod_n Q_n\rangle=0$ is still voided (albeit for a somewhat different reason) and the subsequent discussion of Sec. IV applies in this case as well.

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[15]The continuous switchover is special to the $N=2$ case because of the previous ambiguity at $\theta=\pi$. For $N\geqslant 3$ there is expected to be well-defined, *discontinuous* switching from one case to the next as θ passes through $2k\pi/N$, $k=1, \ldots, N-1$. (S. Coleman and N. Christ, private communication.)