## Is the effective Lagrangian for quantum chromodynamics a $\sigma$ model?

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A fully interacting effective chiral Lagrangian obeying the anomalous axial-baryon-current conservation law is constructed. This Lagrangian is a generalization of one implied by the 1/N approximation. In a certain sense, the old  $\sigma$  model is recovered. Our Lagrangian displays the dependence of amplitudes on the quantum-chromodynamic vacuum angle  $\theta$ , gives soft  $\eta'$  theorems, and hints at a possible complementarity between the instanton and 1/N approaches. We can rewrite our model in terms of a gauge-invariant gluon field.

Since the fundamental dynamical variables (quarks and gluons) of the theory of quantum chromodynamics are presumably unobservable, it is desirable to construct a low-energy effective Lagrangian in terms of the fields of observable particles. In fact, "chiral" Lagrangians of this type have been fairly successful in fitting the experimental data. There is, however, a formal problem,<sup>1</sup> commonly referred to as the U(1) problem, since the anomalous Ward identities are not manifestly satisfied. Recently, in an interesting paper, Witten<sup>2</sup> has emphasized that the " $1/N_c$ " approximation to quantum chromodynamics (QCD) provides a motivation for treating strong processes in the tree-diagram approximation (as is done for the chiral Lagrangians) and that the U(1)problem should be solved at this level. He further suggests that gluon fields of certain types ("glueballs") appear in the effective Lagrangian. In the same  $1/N_c$  framework Veneziano<sup>3</sup> has succeeded in saturating the anomalous Ward identities using a "ghost" glueball field. Finally, Di Vecchia<sup>4</sup> has produced the noninteracting (i.e., quadratic) part of the corresponding effective Lagrangian. In this note we present the full low-energy effective Lagrangian for QCD. In a certain sense this Lagrangian is a special case of a general  $\sigma$  model previously treated<sup>5</sup> in detail. Thus, we automatically incorporate all the phenomenological successes of that model and can easily incorporate SU(3) and chiral-symmetry breaking while maintaining consistency with QCD results on anomalous Ward identities. Furthermore, we show that this approach is complementary to the instanton approach to the problem by demonstrating how an interaction term of 't Hooft's<sup>6</sup> effective type may arise.

For simplicity we specialize to a world containing spin-0 mesons of three flavors. The spin-0 fields are contained in a (flavor) matrix  $M_{ab}$ which transforms like the quark-field combination  $\bar{q}_b(1+\gamma_5)q_a$  (see Ref. 5 for more details of notation). We can then write the generalized  $\sigma$  model<sup>5</sup> as

$$\mathcal{L} = -\frac{1}{2} \operatorname{Tr} \left( \partial_{\mu} M \partial_{\mu} M^{\dagger} \right) - V(I_{1}, I_{2}, I_{3}, J)$$
  
+ 
$$\operatorname{Tr} \left[ A(M + M^{\dagger}) \right].$$

Here V is an arbitrary function of the chiral U(3)  $\times$  U(3) invariants  $I_n = \text{Tr}(MM^{\dagger})^n$  and of  $J = (\det M + \det M^{\dagger})$ , and the matrix  $A_{ab} = \delta_{ab}A_b$  is proportional to the matrix of quark masses. In this model the  $\eta'$  has a mass, even when A = 0, which is determined by  $\langle \partial V / \partial J \rangle$ . (The angular brackets indicate that the vacuum expectation value is to be taken.) The anomalous Ward identities are not, however, satisfied.

The task confronting us is to modify the above Lagrangian by incorporating gluon degrees of freedom in such a way that these anomalous Ward identities are satisfied. The Ward identities themselves are derived from the anomalous conservation law for the axial-vector baryon current  $J^5_{\mu}$ , which reads (temporarily neglecting quark masses)

$$\partial_{\mu} J^{5}_{\mu} = \partial_{\mu} K_{\mu} . \tag{1}$$

 $K_{\mu}$  is a well-known<sup>1</sup> combination of gluon fields such that  $\partial_{\mu}K_{\mu} = (\sqrt{N_F} g^2/16\pi^2)F\tilde{F}$ . Hence, if we produce an effective Lagrangian which satisfies (1) by virtue of the equations of motion we are guaranteed that the Ward identities will be satisfied to tree order.

Our aim can be achieved by introducing a pseudovector glueball field  $K_{\mu}$  and modifying the generalized  $\sigma$  model so that the effective Lagrangian is now

$$\mathcal{L} = -\frac{1}{2} \operatorname{Tr} \left( \partial_{\mu} M \partial_{\mu} M^{\dagger} \right) - V_{0} (I_{1}, I_{2}, I_{3}) + \frac{1}{2} c \left( \partial_{\mu} K_{\mu} \right)^{2} + \frac{i}{4\sqrt{3}} \partial_{\mu} K_{\mu} \left( \ln \det M - \ln \det M^{\dagger} \right) + \operatorname{Tr} \left[ A (M + M^{\dagger}) \right].$$
(2)

Every term but the last in (2) is manifestly invariant under SU(3)×SU(3) transformations  $M \rightarrow U_L M U_R^{\dagger}$ . The U(1) matter current is found by Noether's theorem to be  $J_{\mu}^5 = (-i/\sqrt{3}) \operatorname{Tr}(M^{\dagger} \overline{\partial}_{\mu} M)$ so that  $\partial_{\mu} J_{\mu}^5 = (-i/\sqrt{3}) \operatorname{Tr}(M^{\dagger} \Box M - M \Box M^{\dagger})$ . This

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is easily computed from the equation of motion obtained by varying  $\mathcal{L}$  with respect to  $M^{\dagger}$ :

$$-\frac{1}{2}\Box M + \frac{\partial V_0}{\partial M^{\dagger}} + \frac{i}{4\sqrt{3}} \partial_{\mu} K_{\mu} (M^{\dagger})^{-1} - A = 0.$$
 (3)

Multiplying (3) on the left with  $M^{\dagger}$  and subtracting the complex-conjugate equation yields<sup>7</sup>  $\partial_{\mu} J_{\mu}^{5}$  $= \partial_{\mu} K_{\mu} - (2i/\sqrt{3}) \text{Tr}[A(M - M^{\dagger})]$  which is the anomalous conservation equation (1) including the effects of "quark mass" terms. Thus Eq. (2) is a reasonable candidate for the correct effective QCD chiral Lagrangian. The  $(\partial_{\mu} K_{\mu})^{2}$  term has as yet played no role, but varying  $\mathcal{L}$  with respect to  $K_{\mu}$ yields the very interesting equation

$$\frac{\partial}{\partial x_{\mu}} \left[ \partial_{\nu} K_{\nu} - (4\sqrt{3}ic)^{-1} (\ln \det M - \ln \det M^{\dagger}) \right] = 0.$$
 (4)

This evidently requires the gluon-field combination  $\partial_{\mu}K_{\mu}$  which appears in  $\mathfrak{L}$  to differ from the matter-field term  $(4\sqrt{3}ic)^{-1}(\ln \det M - \ln \det M^{\dagger})$ only by a constant (which we take to be zero). Putting this back into  $\mathfrak{L}$  (which is reasonable in the present effective Lagrangian context) gives

$$\mathcal{L} = -\frac{1}{2} \operatorname{Tr}(\partial_{\mu} M \partial_{\mu} M^{\dagger}) - V_{0} + (96c)^{-1} (\ln \det M - \ln \det M^{\dagger})^{2} + \operatorname{Tr}[A(M + M^{\dagger})].$$
(5)

Equation (5) is expressed completely in terms of matter fields; it is actually a special case of the general linear  $\sigma$  model treated in Ref. 5. If one were to treat (5) as the starting point, he would derive a partial conservation law of the form  $\partial_{\mu} J_{\mu}^{5} =$  function of (det*M* and det*M*<sup>†</sup>). Such an equation leads<sup>5</sup> with usual techniques to a sufficiently large mass for the  $\eta'(960)$  particle to circumvent the U(1) problem. On the other hand, it is apparently not consistent with (1) which requires  $\partial_{\mu} J_{\mu}^{5}$  to equal a combination of gluon fields rather than matter fields. As we have just seen, the constraint (4) performs the task of equating the gluon-field combination to the matter-field one. This would seem to provide a justification for the use of the  $\sigma$  model for practical calculations.

Actually, without modifying our conclusions, the additional terms  $\sum_{m} h_m (\partial_{\mu} K_{\mu})^m$  can be added to the Lagrangian (2). Here the  $h_m$  are arbitrary functions of the invariants  $\operatorname{Tr}(MM^{\dagger})^m$ . This addition will not change the equation for  $\partial_{\mu} J_{\mu}^5$  so all Ward identities will still be satisfied to tree order. However, the "constraint" (4) will be modified so that  $\partial_{\mu} K_{\mu}$  is to be replaced by a more complicated combination of matter fields. Since this leads again to the general linear  $\sigma$  model of Ref. 5 we continue our treatment with the simpler form (2).

We have demonstrated (by construction) that the presence of the term proportional to  $\partial_{\mu}K_{\mu}(\ln \det M)$ 

-  $\ln \det M^{\dagger}$ ) in the effective chiral Lagrangian (2) ensures the anomalous conservation law for the axial-vector "U(1) current"  $J_{\mu}^{5}$ . This holds as a result of the Lagrange equations of motion. Thus, consequences of the anomalous conservation law, i.e., the Ward identities, will also hold automatically. Since the loop expansion preserves Ward identities order by order the treatment of the effective chiral Lagrangian at the tree level provides a consistent realization of the consequences of the equation of motion.

The above by itself is not sufficient to solve the U(1) problem which, for present purposes, may be defined as the problem of giving a nonzero mass to the SU(3) singlet  $\eta'$  meson in the limit when the quark masses [the  $A_a$  in Eq. (2)] are absent. The U(1) problem in the present model is solved by the presence of extra terms involving arbitrary chiral-invariant combinations of the guage-invariant glueball field  $\partial_{\mu} K_{\mu}$ . The prototype term of this form is  $\frac{1}{2}c(\partial_{\mu}K_{\mu})^2$  in (2). These extra terms give, as a result of the equation of motion for  $K_{\mu}$ , the very interesting constraint that  $\partial_{\mu}K_{\mu}$ behaves like the quantity  $(\ln \det M - \ln \det M^{\dagger})$ . As we shall show later, the quantity is approximately proportional to the  $\eta'$  field in the  $\sigma$  model. Then it is apparent that the third and fourth terms in (2) become  $\eta'$  mass terms. Our constraint equation is the field-theoretical realization of Witten's postulated cancellation between  $\eta'$  and glueball matrix elements [see Eq. (11) of Ref. 2]. We may note that the constraint equation expresses a mattergluon duality in which the  $\eta'$  meson can be thought of as composed (approximately) of quarks and antiquarks in the usual way or of gluon fields. From our point of view this is very different from considering the  $\eta'$  as a linear combination of a quark and antiquark and a particlelike glueball. As we shall see later, one can do this formally but the glueball behaves then like a peculiar tachyon rather than like a particle.

Since the quantity  $(\ln \det M - \ln \det M^{\dagger})^2$  can be<sup>5</sup> expressed as a function of  $J = (\det M + \det M^{\dagger})$  the general  $\sigma$  model, which emerges when the glueball field  $\partial_{\mu}K_{\mu}$  is eliminated in favor of matter fields by equation (4), is precisely of the form we first wrote. This model has been discussed in detail in the literature (see Ref. 9 and references therein) where successful treatments of SU(3)-symmetry-breaking effects have been given. In particular,  $\Delta I = 1$  mass differences and the puzzling process  $\eta \rightarrow 3\pi$  were explained. Although much work already has been done on this model, now that there appears to be a more solid connection with QCD, it can be pursued still further. Explorations in this direction will be described elsewhere. For the remainder of this paper we shall make the

very simplest approximations to the  $\sigma$  model in order to emphasize the physics of the situation.

We would now like to give a brief sketch of how the effective U(1)-violating determinant term in our model may be related to the one derived by 't Hooft<sup>6</sup> in the instanton approach. This may be relevant since there has been some controversy about whether the instanton approach is correct or compatible with the  $1/N_c$  approach to QCD. At least at the level of our model we shall see that they both give essentially the same result.

The chiral symmetry must break spontaneously, so one has  $^{\scriptscriptstyle 5}$ 

 $\langle M \rangle = \langle M^{\dagger} \rangle = \alpha 1$ ,

with  $\alpha = F_{\pi}/2$ . We then expand in powers of the "fluctuation" det $M - \langle \det M \rangle$  by writing det $M = \langle \det M \rangle + (\det M - \langle \det M \rangle)$ . This leads to

 $\ln \det M - \ln \det M^{\dagger} \approx \alpha^{-3} (\det M - \det M^{\dagger}).$ 

The square of this can be rewritten<sup>8</sup> as  $\alpha^{-6}(\det M + \det M^{\dagger})^2$  plus U(3)×U(3)-invariant terms which can be absorbed in  $V_0$ . Expanding in powers of the fluctuation once more we have

 $(96C)^{-1}(\ln \det M - \ln \det M^{\dagger})^{2}$ 

 $\approx (24C)^{-1}(\det M + \det M^{\dagger}) + \cdots$ 

If we finally take the liberty of interpreting<sup>9</sup> M as proportional to  $\overline{q}(1+\gamma_5)q$  we are led to expect an effective U(1)-violating Lagrangian of the form  $\det \overline{q}(1+\gamma_5)q$  +H.c. This is precisely the local approximation to 't Hooft's effective term<sup>6</sup> (with further neglect of color indices). The numerical constant c in our Lagrangian is thus indirectly related to 't Hooft's integral over instanton sizes.

Another interesting aspect of QCD physics which can be illuminated by the present model is the dependence of amplitudes on the "vacuum angle"  $\theta$ . This angle is usually introduced into the theory by inclusion of a term

$$rac{- heta g^2}{32\pi^2}\,F ilde{F}$$

in the QCD Lagrangian. In the limit where the quark masses go to zero the  $\theta$  dependence of amplitudes can be eliminated by a chiral U(1) transformation. In what follows we shall verify that the correct term above emerges by making a chiral U(1) transformation in the massless version of (2). We will further show that, in the presence of quark mass terms, the  $\theta$  dependence is simply introduced into the effective Lagrangian by a modification of the constraint Eq. (4). Namely (in the simplified case where reference to the scalar mesons of the theory is suppressed), we eliminate  $\vartheta_{\mu}K_{\mu}$  in terms of

$$\eta' + \frac{\alpha}{\sqrt{3}}\theta$$

rather than just  $\eta'$ . Our modified constraint equation is essentially equivalent to Witten's<sup>2</sup> soft  $\eta'$  theorem. Once the  $\theta$  dependence is introduced into the theory it will appear in terms containing pieces which are functions of

 $e^{i\theta} \det M + \text{H.c.}$ 

The last term of Eq. (2) will continue to be the "mass term." Alternatively, one can make a chiral transformation so that the  $e^{i\theta}$  is eliminated in front of det*M* but it then appears in a parity-violating piece of the mass term which looks like

 $i\sin\frac{1}{3}\theta \operatorname{Tr}[A(M-M^{\dagger})].$ 

Which of the two equivalent presentations of the theory is adopted is a matter of taste or convenience.

It is instructive to ask what happens to our basic effective Lagrangian (2) in the limit of zero quark masses (i.e., A=0) under a chiral U(1) transformation on the matter fields  $M - \exp(i\theta/3)M$ . This angle may then be considered<sup>10</sup> to be the vacuum phase angle. The first three terms in (2) are unchanged while the fourth picks up an additional piece  $-(\theta/2\sqrt{3})\partial_{\mu}K_{\mu} = -(\theta g^2/32\pi^2)F\tilde{F}$ , in terms of the field-strength tensor F. This is just the usual term in the exact QCD Lagrangian so we have a check on the consistency of our procedure.

In order to discuss the schematic effective Lagrangian and the soft  $\eta'$  theorem of Witten,<sup>2</sup> it is helpful to consider an approximation to the linear  $\sigma$  model in which the scalar fields (i.e., the combinations  $M + M^{\dagger}$ ) become very massive. Then our model essentially becomes<sup>11</sup> a nonlinear  $\sigma$  model. All that is required is to set the field M equal to  $\alpha \exp(i\phi/\alpha)$ , where  $\phi$  is the pseudoscalar nonet. We isolate the  $\eta'(960)$  which is of special interest by writing  $\phi = \phi' + (1/\sqrt{3})\eta' 1$  with  $\operatorname{Tr} \phi' = 0$ . Then Eq. (2) becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} (\partial_{\mu} \eta')^{2} - \frac{\alpha^{2}}{2} \operatorname{Tr} \left( \partial_{\mu} e^{i \phi' / \alpha} \partial_{\mu} e^{-i \phi' / \alpha} \right) + \frac{c}{2} (\partial_{\mu} K_{\mu})^{2} \\ &- \frac{1}{2\sqrt{3}} \partial_{\mu} K_{\mu} (\theta + \sqrt{3} \eta' / \alpha) \\ &+ \operatorname{Tr} \left[ A \exp \left( \frac{i \phi}{\alpha} + \frac{i \eta'}{\sqrt{3} \alpha} \right) + \mathrm{H.c.} \right] \,, \end{aligned}$$

$$(6)$$

wherein we have also inserted the  $\theta$  dependence discussed in the last paragraph. Note that  $V_0$  in (2) just becomes a number and was thus dropped. Furthermore, the first two terms (which correspond to Witten's matter Lagrangian) are trivially invariant under  $\eta' \to \eta'$  + const. The remaining soft  $\eta'$  theorem which essentially states that differentiation with respect to  $\theta$  is the same as differentiation with respect to  $\sqrt{3\eta'}/\alpha$ . It is revealing to rewrite (6) using the contraint (4) [which, including the  $\theta$  dependence of the states in the effective Lagrangian, now implies  $\partial_{\mu}K_{\mu} = (2\alpha c)^{-1}(\eta' + \alpha\theta/\sqrt{3})$ ]:

$$\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \eta')^{2} - \frac{1}{8\alpha^{2}c} (\eta' + \alpha \theta / \sqrt{3})^{2} - \frac{\alpha^{2}}{2} \operatorname{Tr} (\partial_{\mu} e^{i \phi' / \alpha} \partial_{\mu} e^{-i \phi' / \alpha}) + \operatorname{Tr} \left[ A \exp \left( \frac{i \phi'}{\alpha} + \frac{i \eta'}{\sqrt{3}\alpha} \right) + \mathrm{H.c.} \right].$$
(7)

This (setting  $\theta = 0$ ) is an old-fashioned nonlinear  $\sigma$  model.<sup>12</sup> Note that all interactions of the  $\eta'$  are in the symmetry-breaking term and hence would vanish as the quark masses (here the matrix A) vanish.<sup>13</sup> Further, note that (in the limit of zero quark masses)  $m_{\eta'}^2 = 1/4\alpha^2 c = 1/F_{\pi}^2 c$ . This determines the constant c in the Lagrangian (2) to be positive. This in turn implies<sup>3</sup> that the field  $K_{\mu}$  is a ghost field since its "kinetic term" in (2) has an unusual sign.

Actually the last statement is not very precise since  $K_{\mu}$  is not a gauge-invariant quantity and  $\frac{1}{2}c(\partial_{\mu}K_{\mu})^2$  is not a true kinetic term. We can clarify the situation by introducing a gauge-invariant pseudoscalar glueball field  $G' = \partial_{\mu}K_{\mu}$ . Further, defining  $G = \lambda G'$  we can rewrite our basic Lagrangian (2) as

$$\mathfrak{L} = \lim_{\lambda \to 0} \left( -\frac{1}{2} (\partial_{\mu} G)^2 + \frac{c}{2\lambda^2} G^2 + \frac{iG}{4\sqrt{3}\lambda} (\ln \det M - \ln \det M^{\dagger}) \right.$$
  
+ pure matter terms), (8)

wherein the limit  $\lambda \rightarrow 0$  is to be taken at the very end of the calculation. From (8) it is evident that the ghost field G behaves as a very heavy (infinitely heavy as  $\lambda \rightarrow 0$ ) particle of *imaginary* mass. The presence of the pseudoscalar ghost field G may provide the key to understanding the differences between the pseudoscalar nonet and the conventional, magically mixed nonets (e.g.,  $1^{--}$ ). Witten has argued that for all meson multiplets it is the generalized gluon-annihilation diagrams that provide deviations from magical mixing and violation of the Okubo-Zweig-Iiuzuka (OZI) rule. What makes the pseudoscalars so special is that it is only in this channel that we expect a ghost glueball field to play such a large role. The ghost induces strong mixing while in the other channels, mixing and probably glueball effects are small. The glueball field in Eq. (8) can be eliminated by the constraint (4). This raises the question of whether gauge-invariant glueball fields always end up as ghosts which eventually get eliminated from the theory or whether they may in fact appear as physical particles. Even if the latter is true it seems that the ghost is the most influential of all the glueballs. It is instructive to consider the  $\eta, \eta'$ , glueball (call it G) mixing in our Lagrangian (8). In the  $\eta, \eta'$  G basis the mass matrix has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\alpha\lambda} \\ 0 & -\frac{1}{2\alpha\lambda} & -\frac{c}{\lambda^2} \end{bmatrix}$$

where we have neglected quark masses. Upon diagonalizing we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4\alpha^2 c} & 0 \\ 0 & 0 & -\frac{c}{\lambda^2} \end{pmatrix} + O(\lambda) .$$
 (9)

Equation (9) explicitly shows how the ghost decouples and how the wrong sign of the ghost squared mass is crucial to generate positive  $m_{\eta}$ ,<sup>2</sup> =  $1/4\alpha^{2}c$ . It is this contribution which is unique to the pseudoscalars and distinguishes them from the more conventional nonets.<sup>14</sup> It would still be necessary to invoke an additional mechanism<sup>15</sup> to give a more detailed explanation of the general mixing problem.

We would like to remark that the discussion above has explicitly shown how the same result is achieved whether one eliminates  $\partial_{\mu}K_{\mu} = G'$  initially in terms of matter fields or keeps it in the theory and observes that it decouples after performing the diagonalization of fields needed to give a particle interpretation. This seems to be a useful check of the consistency of our procedure.

The question of how to treat QCD at low energy is, at present, a leading theoretical problem. In this note we have shown that the theory is essentially equivalent to a general form of the  $\sigma$  model and therefore incorporates the numerous successful predictions of that model.<sup>5,9</sup> Furthermore, the effective Lagrangian found makes contact with both the  $1/N_c$  and instanton approach to QCD and hints that the two descriptions may in fact be complementary rather than contradictory or incompatible.<sup>16</sup>

One of us (J.S.) would like to thank Per Salomonson for a helpful correspondence.

- <sup>1</sup>For reviews of the subject from somewhat different points of view see R. Crewther, in *Facts and Prospects* of *Gauge Theories*, proceedings of the XVII International Universitatswochen für Kernphysik, Schladming, 1978, edited by P. Urban (Springer, New York, 1978) [Acta. Phys. Austriaca Suppl. <u>19</u> (1978)], p. 47; S. Coleman, "Ettore Majorana lectures, 1977 (unpublished); W. Marciano and H. Pagels, Phys. Rep. <u>36C</u>, 137 (1978).
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  R. Jackiw and C. Rebbi, Phys. Rev. Lett. <u>37</u>, 172 (1976). More papers are listed in Ref. 1.
- <sup>7</sup>Note that  $\operatorname{Tr}(M^{\dagger}\partial V_0/\partial M^{\dagger} M\partial V_0/\partial M) = 0$ . See the second paper of Ref. 5.
- <sup>8</sup>See Eq. (18) in the first paper of Ref. 5.
- <sup>9</sup>A formal correspondence between σ-model parameters and "current-algebra" quark-model parameters is discussed by V. Mirelli and J. Schechter, Phys. Rev. D <u>15</u>, 1361 (1977) and J. Kandaswamy, J. Schechter, and

M. Singer, *ibid.* <u>17</u>, 1430 (1978). In these references the phenomenological consequences of an effective determinant term are treated. The second paper contains many references to other work. The first paper contains a brief survey of work on the  $\eta \rightarrow 3\pi$  decay which can be counted as a success of the linear  $\sigma$  model.

- <sup>10</sup>See Coleman, Ref. 1. Basically the chiral angle and the vacuum phase angle rotate amplitudes in proportional ways for the massless quark theory.
- <sup>11</sup>The transition from the linear to the nonlinear  $\sigma$  model is discussed, for example, by W. Bardeen and B. W. Lee, Phys. Rev. <u>177</u>, 2389 (1969).
- <sup>12</sup>See, for example, J. Cronin, Phys. Rev. <u>161</u>, 1483 (1967).
- <sup>13</sup>Thus the soft  $\eta'$  interaction theorem, which was derived in the A = 0 limit, is trivial for the nonlinear Lagrangian (7). However, it will be nontrivial if the additional terms  $\sum_{m} h_m (\partial_{\mu} K_{\mu})^m$  are added to (2). Note that the analog of the constraint (4) would then introduce  $\theta$  dependence into the  $\eta'$  interaction terms by re-
- quiring  $\partial_{\mu}K_{\mu}$  to be a complicated function of  $\theta + \sqrt{3}\eta'/\alpha$ . <sup>14</sup>Compare this treatment to that of N. Fuchs, Phys. Rev. D 14, 1912 (1976).
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- <sup>16</sup>Recent controversy about this issue can be found in Ref. 2 and in the talk by A. M. Polyakov at the 1979 Photon-Lepton Conference at Fermilab (unpublished).

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