New approach to effective field theories

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A systematic discussion of effective field theories, describing a given subset of fields of a quantum field theory, is presented within the context of functional integration. Effective field theories are divided into two classes, natural and unnatural, according to certain independence properties of the counterterms of the theory, defined by minimal subtraction. Natural effective field theories allow independent renormalizations for two distinct mass scales of the theory. A set of constraints, which place restrictions on masses and external momenta, allow the effective field theory to be approximated by a local Lagrangian of dimension four. Predictions of the complete theory are compared with those of the local, effective theory in a domain where both are supposed to be valid. The separate renormalization-group improvement with respect to the two independent mass scales of a natural effective field theory is described. Special problems raised by the presence of massless Goldstone bosons are discussed. The general issues are illustrated by examples from scalar field theories in order to present the discussion simply.

I. INTRODUCTION

The possible grand unification of the weak, electromagnetic, and strong interactions is a fascinating proposal[1] for several reasons. In addition to several conceptual advantages, the program offers the promise of providing links between elementary-particle and cosmological issues. The simplest class of such theories is characterized by two vastly separated mass scales $M \sim 10^{14}-10^{16}$ GeV, say, and $m \sim 300$ GeV, the former referring to the mass at which weak, electromagnetic, and strong interactions become unified, while the latter is the mass at which the electroweak $SU(2)\times U(1)$ gauge theory is broken to its final U(1) invariance. As a consequence, one envisions the $SU(2)\times U(1)\times SU(3)_c$ theory to be an *effective* field theory, obtained as a low-energy approximation to the grand unified gauge theory. In addition to the group-theoretic problem of finding the correct grand unifying gauge theory and representation structure, there is also an unsolved dynamical problem associated with this program.[2] That is, it is not known how to impose a gauge hierarchy with $m/M \sim 10^{-12}$ to 10^{-14} in a *natural* way (in the technical sense). A partial solution to this question has been given by Weinberg.[3] However, in his work certain masses must be set (approximately) to zero by hand. Therefore, a gauge hierarchy is possible, but as yet there is no known natural reason why it must occur.

In spite of the importance of this problem, there has been little systematic discussion of the sequence of steps involved in constructing an arbitrary effective field theory. Further, in practice some approximations are required in order to perform actual calculations, and a detailed discussion of the domain of validity of these approximations and how they may be *systematically* improved, is a necessity. Heretofore, one began such an analysis with a postulated effective field theory, appealing to the discussion of Appelquist and Carazzone[4] for justification. This may involve a number of complications if there is spontaneous symmetry breaking,[5] and if there are symmetry relations linking the counterterms of the complete theory to those of the low-energy residual field theory.

It appears far more logical to us to begin with the complete field theory, and describe the sequence of steps necessary for the extraction of a useful effective field theory. In this paper we describe a systematic discussion of the construction of effective field theories by means of the formalism of functional integration. Our presentation has some advantages over earlier discussions in that (1) the systematic methods presented in this paper paper provide a technical simplification in the actual construction of effective field theories, (2) our analysis allows us to separate several distinct issues which emerge in the construction, and (3) the calculation of corrections to a given approximation to an effective field theory appears to be rather straightforward.

In Sec. II of this paper we present a general construction of effective field theories from a larger field theory. In so doing, we subdivide effective field theories as natural or unnatural, depending on whether the parameters of the effective field theory can be made finite by the counterterms available in the original Lagrangian or not. It is emphasized that effective field theories are in general nonlocal, infinite polynomials. However, for tree-level processes, a useful local approximation to the effective Lagrangian, which ignores terms of dimension greater than four, can always be found by putting constraints on masses and momenta. In processes involving one or more loops, it would appear that the full nonlocal Lagrangian with terms of dimension greater than four must be employed. However, we will show in a subsequent publication that the effects of nonlocality and of terms of dimension greater than four can often be removed by finite, mass-independent wave-function renormalization and by redefinition of the effective mass and coupling parameters [or are suppressed by positive powers of (heavy mass)$^{-1}$]. It follows that the local effective Lagrangian described above can often be used to compute one- (and higher-) loop, as well as tree-level, processes.

In practical calculations an approximate effective field theory is obtained by a perturbation expansion in a subset of fields $\{\chi_\alpha\}$ of the original Lagrangian. Such a perturbative calculation of the effective field theory may be renormalization-group improved with respect to the χ fields' renormalization mass scale. We define an effective field theory without approximations, so that it can be used to reconstruct the predictions of the original field theory. Once one or more approximations are made, it is no longer obvious that the effective field theory will give the same predictions as the original field theory in any kinematical domain. This aspect of the construction must therefore be verified.

The discussion of Sec. II and subsequent sections is mostly heuristic, with no formal proofs of the assertions presented, although the detailed discussion should give sufficient motivation for the claims. In the remaining sections of the paper we present examples from scalar field theories to illustrate the issues. Sections III and IV give examples of natural and unnatural effective field theories. In Sec. V we show how, in our examples, restricted kinematics lead to an approximately local tree-level effective field theory. No assumption of a mass hierarchy is required. We examine in Sec. VI the circumstances under which the effective field theory in our example gives a reasonable approximation of the predictions of the complete theory. We emphasize that in a *natural* effective field theory, there are two *independent* renormalization schemes, with mass scales μ_1 and μ_2 associated with M and m, the large and small mass scales, respectively. Therefore, a *natural* effective field theory allows for *independent* renormalization-group improvement with respect to μ_1 and μ_2. Renormalization-group improvements of the effective Lagrangian with respect to mass scale μ_1 is discussed in Sec. VII. If the effective field theory is *unnatural*, then the situation is more complicated since the counterterms of the unnatural effective field theory link the large and small masses M and m, and the large and small mass scales μ_1 and μ_2. In Sec. VIII we extend the discussion to three scalar fields in order to delineate special problems related to the Goldstone boson. This example demonstrates that a Goldstone boson must always be included in the $\{\chi_\alpha\}$ partition in order for the effective field theory to be natural.

II. THE EFFECTIVE FIELD THEORY

A. Natural and unnatural effective field theories

Consider a multicomponent scalar field Φ_A described by a renormalizable Lagrangian density $\mathcal{L}$. The couplings of $\mathcal{L}$ may or may not be restricted by a group of internal symmetries G, perhaps softly broken. (Since Symanzik has shown[6] that an arbitrary quantum field theory can be represented by a multicomponent scalar field, in principle, our discussion is sufficiently general to include both gauge and fermion fields.) Now partition the components of Φ_A into two distinct sets $\{\phi_a\}$ and $\{\chi_\alpha\}$. That is,

$$\Phi_A = \phi_a \oplus \chi_\alpha , \tag{2.1}$$

where the capital latin subscripts refer to Φ, lower-case latin subscripts label the ϕ, and lower-case Greek subscripts enumerate the χ. No particular symmetry or mass scale need be attributed to the interactions of the ϕ_a or χ_α.

The generating functional for the Green's functions of this field theory is

$$Z[J] = N \int [d\Phi_A] \exp\left(i \int d^4x[\mathcal{L}(x) + J_A(x)\Phi_A(x)]\right), \tag{2.2}$$

where N is an appropriate normalization. Equivalently, the generating functional for this same field theory can be written as

$$Z[j_a, j_\alpha] = N \int [d\phi_a][d\chi_\alpha] \times \exp\left(i \int d^4y[\mathcal{L}(y) + j_a(y)\phi_a(y) + j_\alpha(y)\chi_\alpha(y)]\right), \tag{2.3}$$

where $j_a(y)$ is a source for $\phi_a(y)$ and $j_\alpha(y)$ is a source for $\chi_\alpha(y)$. When expressed in terms of ϕ_a and χ_α, the Lagrangian can be written as

$$\mathcal{L}(x) = \mathcal{L}_\phi(x) + \mathcal{L}_\chi(x) + \mathcal{L}_{\phi\chi}(x) + \Delta\mathcal{L}_\phi(x) + \Delta\mathcal{L}_\chi(x) + \Delta\mathcal{L}_{\phi\chi}(x), \tag{2.4}$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\chi$ refer to terms which are self-interacting in ϕ_a and χ_α, respectively; $\mathcal{L}_{\phi\chi}$ is the interaction which couples these two fields, while the three $\Delta\mathcal{L}$ are the counterterms of the theory, partitioned in the same way. If we were to restrict ourselves to a renormalizable field theory with a softly broken symmetry G, the divergent part of

$\Delta\mathfrak{L}_{\Phi}$ would be invariant under G by Symanzik's theorem,[6] where

$$\Delta\mathfrak{L}_{\Phi}=\Delta\mathfrak{L}_{\phi}+\Delta\mathfrak{L}_{\chi}+\Delta\mathfrak{L}_{\phi\chi}\,. \tag{2.5}$$

In this particular case, even though the divergent part of $\Delta\mathfrak{L}_{\Phi}$ is invariant under G, in general Symanzik's theory does not give information on the symmetries of the separate counterterms which appear on the right side of (2.5). In our work we will define the counterterms by means of dimensional regularization, with minimal subtractions, which means that

$$\Delta\mathfrak{L}=(\Delta\mathfrak{L})_{\text{div}} \tag{2.6}$$

for each of the counterterms, where the $(\Delta\mathfrak{L})_{\text{div}}$ are the pole terms of the dimensionally regulated counterterms.

It is convenient to rewrite (2.3) and (2.4) as follows:

$$Z[j_a,j_\alpha]=N\int[d\phi_a]\exp\left(i\int d^4x[\mathfrak{L}_\phi+(\Delta\mathfrak{L}_\phi)_1+j_a\phi_a]\right)\times\int[d\chi_\alpha]\exp\left(i\int d^4y[\mathfrak{L}_\chi+\mathfrak{L}_{\phi\chi}+\Delta\mathfrak{L}_\chi+\Delta\mathfrak{L}_{\phi\chi}+(\Delta\mathfrak{L}_\phi)_2+j_\alpha\chi_\alpha]\right), \tag{2.7}$$

where the counterterm $\Delta\mathfrak{L}_\phi$ has been divided into two parts

$$\Delta\mathfrak{L}_\phi=(\Delta\mathfrak{L}_\phi)_1+(\Delta\mathfrak{L}_\phi)_2\,. \tag{2.8}$$

The precise separation in (2.8) will be defined constructively in what follows. Equation (2.7) can also be written as

$$Z[j_a,j_\alpha]=N\int[d\phi_a]\exp\left(i\int d^4x[\mathfrak{L}_\phi+(\Delta\mathfrak{L}_\phi)_1+j_1\phi_a]\right)Z[j_\alpha;\phi]\,, \tag{2.9}$$

where

$$Z[j_\alpha;\phi]\equiv e^{iW[j_\alpha;\phi]}=\int[d\chi_\alpha]\exp\left(i\int d^4y[\mathfrak{L}_\chi+\mathfrak{L}_{\phi\chi}+\Delta\mathfrak{L}_\chi+\Delta\mathfrak{L}_{\phi\chi}+(\Delta\mathfrak{L}_\phi)_2+j_\alpha\chi_\alpha]\right) \tag{2.10}$$

is the generating functional for the Green's function of the quantum field theory describing the interaction of the quantum field χ_α in the presence of a background classical field ϕ_a defined by the Lagrangian $\bar{\mathfrak{L}}_{\phi\chi}$, with

$$\bar{\mathfrak{L}}_{\phi\chi}=\mathfrak{L}_\chi+\mathfrak{L}_{\phi\chi}+\Delta\mathfrak{L}_\chi+\Delta\mathfrak{L}_{\phi\chi}+(\Delta\mathfrak{L}_\phi)_2\,. \tag{2.11}$$

Similarly, $W[j_\chi;\phi]$ generates the *connected* Green's function of the same field theory. (In performing the χ-functional integration one of course takes proper account of the statistics of the χ_α fields.)

At this stage it is necessary to emphasize that (2.11) may not be the most general field theory describing the interaction of the quantum field χ_α with the classical field ϕ_a if the original Lagrangian in (2.4) is restricted by a symmetry group G. As a consequence of this restriction, there are symmetry transformations relating $\bar{\mathfrak{L}}$ of (2.11) to $\mathfrak{L}_\phi+(\Delta\mathfrak{L}_\phi)$ of Eq. (2.9), as well as restricting the form of $\bar{\mathfrak{L}}$ itself. This issue will be relevant in the classification of effective field theories.

If one is interested in processes with external ϕ_a lines only, then (2.9) and (2.10) may be simplified by setting the source $j_\chi=0$. For this special case

$$Z[j_a,0]=N\int[d\phi_a]\exp\left(i\int d^4x[\mathfrak{L}_\phi+(\Delta\mathfrak{L}_\phi)_1+j_a\phi_a]e^{iW[0;\phi]}\right) \tag{2.12a}$$

$$=N\int[d\phi_a]\exp\left(i\int d^4x[(\mathfrak{L}_\phi)_{\text{eff}}+j_a\phi_a]\right), \tag{2.12b}$$

where Eqs. (2.12) *define* the effective Lagrangian density $(\mathfrak{L}_\phi)_{\text{eff}}$. Since $W[0;\phi]$ describes the connected Green's functions of the quantum field χ_α in the background field ϕ_a to all orders, $(\mathfrak{L}_\phi)_{\text{eff}}$ is clearly *nonlocal*, as all possible closed loops of the χ_α fields are included in $W[0;\phi]$. The generator of one-particle-irreducible (1PI) Green's functions, $\Gamma(\bar\phi)$, is defined by the Legendre transform

$$\Gamma(\bar\phi)=-i\ln\left[\int[d\phi_a]\exp\left(i\int d^4x[(\mathfrak{L}_\phi)_{\text{eff}}+j_a\phi_a]\right)\right]-\int d^4x\,j_a\bar\phi_a\,. \tag{2.13}$$

It is obvious that

$$\Gamma(\bar\phi)\neq\int(\mathfrak{L}_\phi)_{\text{eff}}\,d^4x$$

except in the tree approximation for the ϕ_a field. Since no approximations have been made in arriving at (2.12) and (2.13), the exact 1PI Green's functions with external ϕ lines are obtained from $(\mathcal{L}_\phi)_{\text{eff}}$ by means of (2.13) which, to reiterate, is nonlocal. There are particular circumstances for which $(\mathcal{L}_\phi)_{\text{eff}}$ is well approximated by a *local* Lagrangian density, a topic which we will discuss shortly. In summary, we *define* the effective field theory of the field ϕ_a relative to the field theory $\mathcal{L}_\Phi$ by the Lagrangian $(\mathcal{L}_\phi)_{\text{eff}}$, as obtained by the construction (2.1)–(2.12).

The concept of an effective field theory becomes increasingly more useful as further restrictions are imposed. We divide effective field theories into two classes by means of a definition.

Definition 1: Natural effective field theory

We define the effective field theory given by (2.12) to be a *natural* effective field theory if the 1PI parts of the quantum field theory of χ in the background field of ϕ, described by the Lagrangian $\overline{\mathcal{L}}_{\phi\chi}$ of Eq. (2.11), is renormalizable to all orders in perturbation theory (i.e., to all orders in χ closed loops), with the restriction that

$$(\Delta\mathcal{L}_\phi)_2 = \Delta\mathcal{L}_\phi \,. \tag{2.14}$$

That is, the effective field theory is natural if $(\mathcal{L}_\phi)_{\text{eff}}$ can be made finite by the full counterterms found in the original Lagrangian. Clearly from (2.8) and (2.14),

$$(\Delta\mathcal{L}_\phi)_1 = 0 \,. \tag{2.15}$$

If an effective field theory is natural in our sense, then the renormalization of the loop expansion of the χ_α fields is independent of the renormalization of the ϕ_a fields. In short,

$$\overline{\mathcal{L}}_{\phi\chi} = \mathcal{L}_\chi + \mathcal{L}_{\phi\chi} + \Delta\mathcal{L}_\chi + \Delta\mathcal{L}_{\phi\chi} + \Delta\mathcal{L}_\phi \tag{2.16}$$

is a renormalizable quantum theory of the χ field in the presence of the background (classical) field ϕ_a. Counterterms for the effective ϕ-field theory are obtained by treating the effective fields and parameters as *bare* quantities which are then multiplicatively renormalized. This definition is not empty, since we will provide explicit examples of natural effective field theories. Further we shall show by example that not all effective field theories are natural.

By contrast we have the following definition:

Definition 2: Unnatural effective field theories

An effective field theory is called an *unnatural* effective field theory if it is not a natural effective field theory. If an effective field theory is unnatural, then either

$$(\Delta\mathcal{L}_\phi)_2 = C\Delta\mathcal{L}_\phi \,, \tag{2.17a}$$

where C is a real number not equal to one or

$$(\Delta\mathcal{L}_\phi)_2 \neq C\Delta\mathcal{L}_\phi \,. \tag{2.17b}$$

In either case, since

$$(\Delta\mathcal{L}_\phi)_1 = (\Delta\mathcal{L}_\phi) - (\Delta\mathcal{L}_\phi)_2 \,, \tag{2.17c}$$

$(\Delta\mathcal{L}_\phi)_1$ does not vanish. It follows that the renormalization of the χ-loop expansion is *not* independent of the renormalization of the ϕ-loop expansion, as a consequence of the unnatural separation of the Φ_A field in (2.1). Since the complete theory is renormalizable, $\overline{\mathcal{L}}_{\phi\chi}$ of (2.11) is of dimension four and all counterterms of (2.11) are of dimension four or less. Even though $(\Delta\mathcal{L}_\phi)_2$ may not appear in the original Lagrangian, it is also at most of dimension four. In other words, if the effective field theory is unnatural, $\Delta\mathcal{L}_\phi$ is linearly *dependent* on $\Delta\mathcal{L}_{\phi\chi} + \Delta\mathcal{L}_\chi$ (see Sec. IV).

We emphasize that our definition of effective field theory, and the separation into natural and unnatural effective theories, is independent of questions of locality and the existence of a mass hierarchy. At this stage in our discussion, we still have an exact description of the complete field theory. Consider the generating function $W[0;\phi]$, defined by (2.10) and (2.12), expressed as an infinite power series in ϕ_a. In momentum space this can be written as

$$W[0;\phi] = \sum_{n=1}^{\infty} \int d^4k_1 \cdots d^4k_n (2\pi)^4 \delta(k_1 + \cdots + k_n) \times W^{(n)}(k_1, \ldots, k_n)\phi_a(k_1) \cdots \phi_b(k_n) \,, \tag{2.18}$$

where $W^{(n)}(k_1, \ldots, k_n)$ is the connected Green's function, with n external ϕ lines carrying momenta $k_1, \ldots, k_n$, respectively, computed to all orders in χ. If (2.18) is inserted into (2.9) and the ϕ-functional integration is performed, one reconstructs the complete original field theory.

B. Local effective theories at the tree level and mass hierarchies

Although $(\mathcal{L}_\phi)_{\text{eff}}$ is nonlocal, under certain conditions this Lagrangian becomes approximately local. Suppose that there is a characteristic mass scale M associated with the set of fields χ_α. Then it is convenient to write

$$W^{(n)}(k_1, \ldots, k_n) = \frac{1}{(M)^{n-4}} f^{(n)}(k_1, \ldots, k_n) \quad \text{for all } n \,, \tag{2.19}$$

where $f^{(n)}(k_1, \ldots, k_n)$ is a dimensionless function of the momenta, χ_α masses (as computed in the world with all $\phi_a = 0$), and renormalization mass scale. A simplification occurs if the coefficients in (2.18) are free of infrared singularities. A sufficient condition for this is

$$M = \min\{M_{\chi_\alpha}\} \neq 0 \tag{2.20}$$

and

$$\mu^2 = O(M^2), \tag{2.21}$$

where M_{χ_α} is the mass of the field χ_α with all ϕ couplings turned off and μ is the arbitrary mass scale introduced by the renormalization procedure. Then

$$\lim_{\{k_i\}\to 0} f^{(n)}(k_1, \ldots, k_n) = f^{(n)}(0, \ldots, 0) \tag{2.22}$$

exists, and

$$W^{(n)}(k_1, \ldots, k_n) \xrightarrow[k_i \ll M]{} \frac{1}{(M)^{n-4}}[f^{(n)}(0, \ldots, 0) + O(k_i k_j/M^2)]\,. \tag{2.23}$$

If the $W^{(n)}$ coefficients exist in this limit, then $(\mathcal{L}_\phi)_{\text{eff}}$ becomes (approximately) local with

$$(\mathcal{L}_\phi)_{\text{eff}} = \mathcal{L}_\phi + (\Delta\mathcal{L}_\phi)_1 + W[0; \phi]\,, \tag{2.24}$$

where in this kinematic region

$$\begin{aligned} W[0;\phi] \simeq \sum_{n=0}^{\infty} \int d^4k_1 \cdots d^4k_n (2\pi)^4 \delta(k_1 + \cdots + k_n) \\ \times\{ [M^{4-n} f^{(n)}_{a\cdots b}(0)]\,\phi_a(k_1) \cdots \phi_b(k_2) \\ + [Z^{(2)}_{ab,\mu\nu}(0)]\partial^\mu \phi_a(k_1) \partial^\nu \phi_b(k_2) \\ + [g^{(3)}_{abc,\mu}(0)]\partial^\mu \phi_a(k_1)\phi_b(k_2)\phi_c(k_3)\}\,. \end{aligned} \tag{2.25}$$

The coefficients $Z^{(2)}$ and $g^{(3)}$ are required for wave-function and derivative-gauge-coupling renormalization (for example). ($W^{(0)}$ has been absorbed into the overall normalization N.) If one calculates $W[0; \phi]$ to finite order in the χ-loop expansion, we require the renormalization mass $\mu^2 \simeq O(M^2)$ to avoid large logarithms, although the result may be renormalization-group improved with respect to μ to widen the domain of validity. This remark applies to either the nonlocal or local versions of $W[0; \phi]$.

Notice that the construction of an approximately local, tree-level effective field theory is applicable to both natural and unnatural effective theories. Once the approximation (2.25) is adopted, it is no longer obvious that (2.25) inserted into (2.12) or (2.13) gives a good representation of the original field theory in *any* kinematical domain. This question must be studied as an issue independent of those considered up to now.

If m is the characteristic mass associated with the $\{\phi_\alpha\}$ fields and M is the characteristic mass associated with the $\{\chi_\alpha\}$ fields, then we say that there is a mass hierarchy if $m \ll M$. Under the assumption that a mass hierarchy exists, the tree-level Lagrangian can be greatly simplified by omitting terms with dimension greater than four. The reason is that the relative contributions of such higher-order terms to tree-level processes are always suppressed by some positive power of m/M. One can exhibit and analyze the correction terms of order $1/M^2$. Since the overall theory is renormalizable, the renormalization of terms in (2.18) and (2.19) of dimension five and six is not independent of those of four and less. For a natural effective field theory, terms of dimension five and six are made finite by counterterms of the original theory. Further, these terms of $O(1/M^2)$ may be renormalization-group improved. This particular feature of natural effective field theories with a mass hierarchy has been exploited by Kazama and Yao,[7] who give an elegant description of the $(1/M^2)$ effects of muon closed loops on the quantum electrodynamics of electrons, as well as a general discussion of such problems.

C. Domain of validity of approximations

One may ask when a particular approximate $(\mathcal{L})_{\text{eff}}$ reproduces the predictions of the complete theory to a given degree of accuracy, when $(\mathcal{L})_{\text{eff}}$ is inserted into (2.12) or (2.13). Of course, if one uses the exact, nonlocal version of $(\mathcal{L})_{\text{eff}}$, then the predictions of the original theory are obviously obtained. The interesting question involves establishing the domain of validity of a given approximation to $(\mathcal{L})_{\text{eff}}$ so that, for a given set of mass scales and momenta, the effective field theory reproduces the predictions of the original theory to required accuracy. Since the renormalization mass scale μ appears in condition (2.21), the domain of validity of the local approximation to $(\mathcal{L})_{\text{eff}}$ will depend on μ. This domain may be widened by use of the renormalization group.

In subsequent sections of this paper we will present several detailed examples to illustrate the various issues raised in this section, all of which will be drawn from scalar field theories. The methods used and the results, however, apply to theories involving fields of arbitrary spin.

III. A NATURAL EFFECTIVE FIELD THEORY

In this section we present an explicit example of an effective field theory which is natural in the sense of Sec. II. Let ϕ_1 and ϕ_2 be two independent, real scalar fields whose interactions are described by the most general renormalizable Lagrangian invariant under the discrete transformations

$$\phi_1 \to -\phi_1\,,\quad \phi_2 \to \phi_2$$

and (3.1)

$$\phi_1 \to \phi_1\,,\quad \phi_2 \to -\phi_2\,.$$

In terms of bare fields and unrenormalized coupling parameters,

$$\mathcal{L} = \mathcal{L}_{\text{INV}} + \mathcal{L}_{\text{BR}}\,, \tag{3.2}$$

where

$$\mathcal{L}_{\text{INV}} = \tfrac{1}{2}(\partial_\mu\phi)_i(\partial^\mu\phi)_i + \frac{m^2}{2}\phi_i\phi_i - \frac{\lambda}{4!}(\phi_i\phi_i)^2 \tag{3.3}$$

and

$$\mathcal{L}_{\text{BR}} = -\frac{a^2}{2}\phi_2^{\,2} - \frac{2}{4!}b\phi_1^{\,2}\phi_2^{\,2} - \frac{c}{4!}\phi_2^{\,4}\,, \tag{3.4}$$

where $\mathcal{L}_{\text{INV}}$ is invariant under SO(2) group actions which transform ϕ_1 and ϕ_2 as an irreducible doublet, while $\mathcal{L}_{\text{BR}}$ is the most general symmetry-breaking interaction compatible with (3.1). Hence, the subscript BR in (3.4) for "breaking." We will use dimensional regularization throughout this work, with a multiplicative, minimal-subtraction renormalization scheme. In general cases, wave-function renormalization of the fields is required. However, here we will restrict explicit calculations to processes involving one χ loop. It will then be unnecessary to renormalize the ϕ_i in this approximation. With this in mind, define renormalized fields and parameters by

$$\begin{aligned} \phi_i &= \phi_{Ri}\,,\\ m^2 &= m_R^{\,2}(1+D/\epsilon)\,,\\ \lambda &= \lambda_R\mu^{-\epsilon}(1+E/\epsilon)\,,\\ a^2 &= a_R^{\,2}(1+A/\epsilon)\,,\\ b &= b_R\mu^{-\epsilon}(1+B/\epsilon)\,,\\ c &= c_R\mu^{-\epsilon}(1+C/\epsilon)\,, \end{aligned} \tag{3.5}$$

where R stands for renormalized and $\epsilon = n-4$. Since the calculations of this section are limited to at most one χ loop, the Laurent expansions in (3.5) have been truncated at ϵ^{-1}. The field theory generated by the Lagrangian (3.2) is multiplicatively renormalizable if none of the above unrenormalized quantities vanish. In this section we assume this to be the case. To ensure a stable vacuum for the complete theory, one requires that λ_R and c_R are positive, while the sign of b_R is left unspecified. Further, we take $m_R^{\,2}$ and $a_R^{\,2}$ to be positive. Because of the choice of the sign of the mass term in (3.3), spontaneous symmetry breaking occurs. (If we had taken the opposite sign for $m_R^{\,2}$, we would still have an example of a natural effective field theory, since spontaneous symmetry breaking is not a requisite of natural effective field theories. We have chosen an example with broken symmetry because it is less trivial and of greater physical interest.) The tree-level vacuum expectation value of the scalar fields is

$$\vec{v}_R = \begin{pmatrix} v_R \\ 0 \end{pmatrix},\quad v_R = \left(\frac{3!\,m_R^{\,2}}{\lambda_R}\right)^{1/2}. \tag{3.6}$$

One could define $\Phi = \{\phi_{R1}, \phi_{R2}\}$ with Φ as in Sec. II, and partition Φ into the two sets $\{\phi_R = \phi_{R2}\}$ and $\{\chi_R = \phi_{R1}\}$, in which case a natural effective Lagrangian for ϕ_R exists, but with an obscure physical meaning. It is more interesting to let $\Phi = \{\phi'_{R1} = v_R - \phi_{R1}, \phi_{R2}\}$ and to partition Φ into the sets $\{\phi_R = \phi_R\}$ and $\{\chi_R = \phi'_{R1}\}$. In terms of these fields, (3.2) becomes

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\chi + \mathcal{L}_{\phi\chi} + \Delta\mathcal{L}_\phi + \Delta\mathcal{L}_\chi + \Delta\mathcal{L}_{\phi\chi} \tag{3.7}$$

as in (2.4), where (now dropping the subscripts R and powers of μ for notational economy)

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^{\,2}}{2}\phi^2 - \frac{1}{4!}(\lambda+c)\phi^4\,, \tag{3.8}$$

$$\mathcal{L}_\chi = \tfrac{1}{2}(\partial_\mu\chi)^2 - \frac{m_\chi^{\,2}}{2}\chi^2 - \frac{1}{3!}\lambda v\chi^3 - \frac{\lambda}{4!}\chi^4\,, \tag{3.9}$$

$$\mathcal{L}_{\phi\chi} = -\frac{1}{3!}\tilde{\lambda}v\phi^2\chi - \frac{2}{4!}\tilde{\lambda}\phi^2\chi^2\,, \tag{3.10}$$

and

$$\Delta\mathcal{L}_\phi = \left(\frac{m_\chi^{\,2}}{4}(D-E) - \frac{a^2}{2}A - \tfrac{1}{12}bv^2B\right)\frac{\phi^2}{\epsilon} - \frac{1}{4!}(\lambda E + cC)\frac{\phi^4}{\epsilon} + \cdots\,, \tag{3.11}$$

$$\Delta\mathcal{L}_\chi = \frac{m_\chi^{\,2}}{2}v(D-E)\frac{\chi}{\epsilon} + \frac{m_\chi^{\,2}}{4}(D-3E)\frac{\chi^2}{\epsilon} + \cdots\,, \tag{3.12}$$

$$\Delta\mathcal{L}_{\phi\chi} = -\frac{v}{3!}(\lambda E + bB)\frac{\phi^2\chi}{\epsilon} + \cdots\,. \tag{3.13}$$

We have omitted those counterterms not required at the one-loop level, and have defined

$$m_\phi^{\,2} = a^2 + bv^2/3!\,, \tag{3.14}$$

$$m_\chi^{\,2} = 2m^2 = \lambda v^2/3\,, \tag{3.15}$$

$$\tilde{\lambda} = \lambda + b\,. \tag{3.16}$$

Anticipating our discussion of mass hierarchies in Sec. V, we emphasize that there is no natural relationship between $m_\phi^{\,2}$ and $m_\chi^{\,2}$.

We now turn to the construction of an effective Lagrangian according to the procedure of Sec. II. Note that the structure of the Lagrangian (3.7) im-

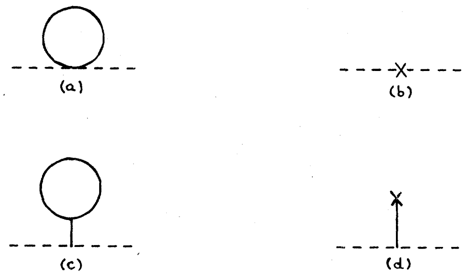

FIG. 1. One-loop graphs contributing to $W^{(2)}(k_1,k_2)$ in a theory with two scalar fields. The dashed line stands for the ϕ field and the solid line for the χ field.

plies that $W^{(n)}(k_1,\ldots,k_n)$ vanishes when n is odd. It is not possible to perform the χ-functional integration in (2.10) exactly, therefore we consider the simplest nontrivial approximation, a single loop in the χ field. To this order, the graphs contributing to $W^{(2)}(k_1,k_2)$, defined by (2.10), (2.12), and (2.18) are shown in Fig. 1. The counterterm Fig. 1(b) will remove the divergence from Fig. 1(a) if the renormalization constants appropriate to the ϕ^2 term in (3.11) satisfy

$$\frac{m_\chi^2}{2}(D-E)-d^2A-\frac{bv^2}{6}B=\frac{\tilde{\lambda} m_\chi^2}{48\pi^2}\,. \tag{3.17}$$

Similarly, the counterterm Fig. 1(d) will remove the divergence from the χ one-point function [Fig. 1(c)] if

$$D-E=\frac{\lambda}{8\pi^2}\,. \tag{3.18}$$

With these minimal subtractions we find that

$$W^{(2)}(k_1,k_2)=\frac{i}{96\pi^2}\tilde{\lambda} m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}^2}\,, \tag{3.19a}$$

where

$$\bar{\mu}^2=4\pi e^{1-\gamma}\mu^2\,, \tag{3.19b}$$

with γ =Euler's constant. The graphs which contribute to $W^{(4)}(k_1,\ldots,k_4)$ to one loop in the χ-functional integral are shown in Fig. 2. Equation (3.18) ensures exact cancellation of the divergences in the first graphs of Figs. 2(b) and 2(a) by the corresponding tadpole counterterms. Cancellation of the divergences in the first two graphs of Fig. 2(c) by the counterterm in Fig. 2(c) will occur if the χ^2 vertex is renormalized by the relation

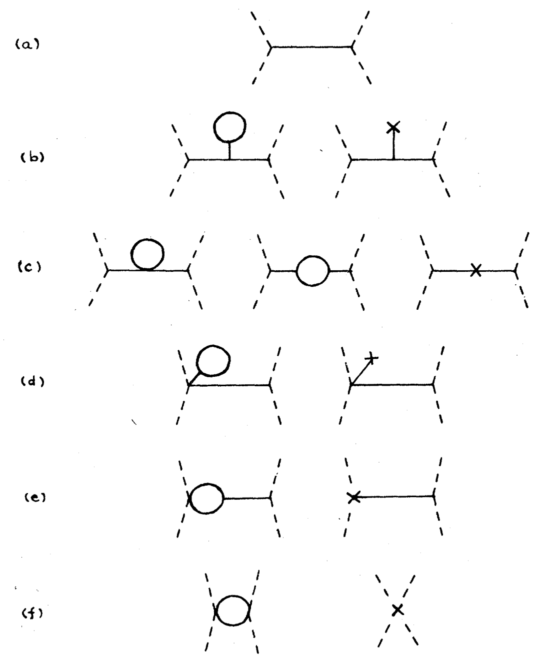

FIG. 2. Tree-level and one-loop graphs contributing to $W^{(4)}(k_1,k_2,k_3,k_4)$ in a theory with two scalar fields. The notation is the same as in Fig. 1.

$$-D+3E=-\frac{\lambda}{2\pi^2}\,. \tag{3.20}$$

Similarly, the divergences in the first graphs of Figs. 2(e) and 2(f) will be canceled by their respective counterterms, if in Fig. 2(c) the counterterm for the 1PI $\phi^2\chi$ vertex satisfies

$$\lambda E+bB=-\frac{\tilde{\lambda}\lambda}{16\pi^2} \tag{3.21}$$

and if the counterterm for the ϕ^4 1PI vertex in Fig. 2(f) satisfies

$$\lambda E+cC=-\frac{\tilde{\lambda}^2}{48\pi^2}\,. \tag{3.22}$$

With these subtractions we find that

$$W^{(4)}(k_1,\ldots,k_4)=\left[-\frac{i}{4!}\frac{\tilde{\lambda}^2}{\lambda}+\frac{i}{16\pi^2}\frac{\tilde{\lambda}^2}{24}\left(1+\frac{m_\chi^2}{p^2-m_\chi^2}\right)\ln\frac{m_\chi^2}{\bar{\mu}^2} -\frac{i}{16\pi^2}\frac{\tilde{\lambda}^2}{36}\left(\frac{p^2-m_\chi^2}{4m_\chi^2}+\frac{3}{2}+\frac{9}{4}\frac{m_\chi^2}{p^2-m_\chi^2}\right)F(p^2,m_\chi^2)\right]\frac{m_\chi^2}{p^2-m_\chi^2}\,, \tag{3.23}$$

where

$$F(p^2,m_\chi^2)=\ln\left(\frac{p^2}{\bar{\mu}^2}\right)+1+\sum_{i=1}^{2}\left[\ln(1-y_i)-y_i\ln\left(\frac{y_i-1}{y_i}\right)-1\right]\text{with } y_{\binom{1}{2}}=\tfrac{1}{2}\pm\tfrac{1}{2}\left(1-\frac{4m_\chi^2}{p^2}\right)^{1/2} \quad \text{for } p^2\neq 0 \tag{3.24}$$

and

$$F(0, m_\chi^2) = \ln\left(\frac{m_\chi^2}{\overline{\mu}^2}\right) + 1\,, \tag{3.25}$$

with $\overline{\mu}^2$ as in (3.19b), and $p = k_1 + k_2 = -(k_3 + k_4)$. Note that $W^{(4)}$ is defined for all p for which $p^2 \neq m_\chi^2$. Unlike $W^{(2)}$, the coefficient $W^{(4)}$ is momentum dependent, so that the effective Lagrangian for ϕ is nonlocal, as discussed in Sec. II. The renormalization equations (3.17), (3.18), and (3.20)–(3.22) provide five equations in five unknowns, with the solution

$$\begin{aligned} D &= -\frac{\lambda}{16\pi^2}\,, \\ E &= -\frac{3\lambda}{16\pi^2}\,, \\ A &= -\frac{m_\chi^2}{32\pi^2 a^2}(\lambda - \tilde{\lambda}/3)\,, \\ B &= \frac{3\lambda}{16\pi^2 b}(\lambda - \tilde{\lambda}/3)\,, \\ C &= \frac{3\lambda}{16\pi^2 c}(\lambda - \tilde{\lambda}^2/9\lambda)\,. \end{aligned} \tag{3.26}$$

(The reader is reminded that a^2, b, and c are all nonzero.) We therefore conclude that the counterterms of the complete Lagrangian $\Delta\mathcal{L}_\phi$, $\Delta\mathcal{L}_\chi$, and $\Delta\mathcal{L}_{\phi\chi}$ are sufficient to subtract all the divergences in $W^{(2)}$ and $W^{(4)}$. In principle, one should also check the consistency of the renormalization procedure for the single χ-loop contributions to $W^{(2n)}$ when $n \geq 3$. However, in the important problem of mass hierarchies this is unnecessary and we presume that no difficulties arise.

In conclusion, it follows from definition 1 of Sec. II that to one-loop order in χ, the effective field theory for ϕ described in this section is natural. The $W^{(2)}$ and $W^{(4)}$ coefficients of the effective Lagrangian (2.12) and (2.18) are given by Eqs. (3.19) and (3.23), respectively.

IV. UNNATURAL EFFECTIVE FIELD THEORIES

Examples of effective field theories which are unnatural in our sense are easily generated from the scalar field theory presented in Sec. III. One example is obtained if ϕ_1 and ϕ_2 transform as an irreducible doublet under SO(2), and we demand that the complete Lagrangian (3.2) be *invariant* under such transformations. Then $\mathcal{L}_{\rm BR} = 0$, and $a^2 = b = c = 0$. With our choice of sign of m_R^2, the model has spontaneous symmetry breaking. One can repeat an analysis in parallel with that of Sec. II. The renormalization Eqs. (3.17) and (3.18) for the ϕ^2 and χ 1PI vertices in (3.8)–(3.13) are now replaced by

$$D - E = \frac{\lambda}{24\pi^2} \tag{4.1}$$

and

$$D - E = \frac{\lambda}{8\pi^2}\,, \tag{4.2}$$

respectively. Clearly there is no consistent solution of this pair of equations. It follows that the divergence in the $W^{(2)}$ coefficient cannot be eliminated using the counterterms supplied by the original SO(2)-invariant Lagrangian. Similarly, the renormalization Eqs. (3.20)–(3.22) for the χ^2, $\phi^2\chi$, and ϕ^4 1PI vertices are replaced by

$$-D + 3E = -\frac{\lambda}{2\pi^2}\,, \tag{4.3}$$

$$E = -\frac{\lambda}{16\pi^2}\,, \tag{4.4}$$

$$E = -\frac{\lambda}{48\pi^2}\,, \tag{4.5}$$

respectively. Again there is no consistent solution of this set of equations, so that the divergences of $\mathcal{L}_{\phi\chi}$ cannot be eliminated using the counterterms of the original SO(2)-invariant Lagrangian. This effective field theory for ϕ is unnatural according to definition 2 of Sec. II, as a consequence of the SO(2) invariance of the counterterms of the original Lagrangian. In this model we encounter one example of a general phenomenon, i.e., the ϕ field is a massless Goldstone boson [see (3.14)]. One concludes from the above and its generalizations that a natural effective field theory for a Goldstone boson never exists because of the symmetry relations imposed on the counterterms of the complete field theory (see Sec. VIII).

An example of an unnatural effective field theory with a discrete symmetry can also be extracted from (3.2). Now assume that ϕ_1 and ϕ_2 transform irreducibly under the discrete group Z_4 [$\subset$ SO(2)] defined by

$$\phi_1 \to \phi_2\,, \quad \phi_2 \to -\phi_1\,, \tag{4.6}$$

and demand that (3.2) be invariant under this transformation [but not under SO(2)]. This requires $a^2 = c = 0$ in (3.4). The parameter b is nonzero, but b_R should be positive if m_ϕ^2 is to be positive. Now construct $W^{(2)}$ and $W^{(4)}$ as before. The renormalization equations (3.17) and (3.18) become

$$\frac{m_\chi^2}{2}(D - E) - \frac{bv^2}{3!}B = \frac{\tilde{\lambda} m_\chi^2}{48\pi^2} \tag{4.7}$$

and

$$D - E = \frac{\lambda}{8\pi^2}\,, \tag{4.8}$$

respectively, which are mutually consistent. The renormalization equations (3.20)–(3.22) are now

$$-D+3E=-\frac{\lambda}{2\pi^2}\,, \tag{4.9}$$

$$\lambda E+bB=-\frac{\tilde{\lambda}\lambda}{16\pi^2}\,, \tag{4.10}$$

$$\lambda E=-\frac{\tilde{\lambda}^2}{48\pi^2}\,, \tag{4.11}$$

respectively, which are consistent among themselves. However, consistency of the entire set of five equations (4.7)–(4.11) demands that $b=2\lambda$. Assuming for the moment that this is *not* the case, we conclude that the Z_4-invariant counterterms provided by the original Lagrangian cannot remove the divergences in both $W^{(2)}$ and $W^{(4)}$ simultaneously. Again the absence of a natural effective field theory is due to the symmetry constraints imposed on the counterterms (in this case Z_4).

If we now arbitrarily impose the unnatural constraint $b=2\lambda$, then a consistent solution of (4.7)–(4.11) exists with

$$\begin{aligned} D&=-\frac{\lambda}{16\pi^2}\,,\\ E&=-\frac{3\lambda}{16\pi^2}\,,\\ B&=0\,. \end{aligned} \tag{4.12}$$

From (3.14) and (3.15) we note that $a=0$ and $b=2\lambda$ implies that $m_\phi^2=m_\chi^2$. When $b=2\lambda$, all the divergences in the one-χ-loop contributions to $W^{(2)}$ and $W^{(4)}$ can be canceled by the counterterms of the original Lagrangian. It follows that this effective field theory for ϕ is natural in our sense, at least to one-loop order in χ. Of course, in this case one is unable to construct a mass hierarchy because of the constraint $m_\phi^2=m_\chi^2$.

It is clear from the discussion of this section that the question of whether an effective field theory is natural or unnatural is closely related to the symmetry properties of the original Lagrangian $\mathcal{L}_\Phi$ as well as the particular partition chosen for $\Phi=\phi\oplus\chi$. From these results we conjecture the following. Let $\phi_1,\ldots,\phi_n$ $(n\geq 2)$ be a set of scalar fields which transform irreducibly under some group G (continuous or discrete), and let the Lagrangian describing the interaction of these fields exhibit spontaneous symmetry breakdown. Shift the fields to the tree-level vacuum and form any nontrivial partition of these new fields into $\{\phi_a\}$ and $\{\chi_\alpha\}$. Then

(i) if the original Lagrangian breaks the G invariance maximally, then the effective field theory for the fields ϕ_a is natural;

(ii) if the original Lagrangian is G invariant, then the effective field theory for the fields ϕ_a is unnatural (with the possible exception of G a discrete symmetry supplemented by the requirement that the coupling parameters take special values).

V. A LOCAL EFFECTIVE THEORY AND MASS HIERARCHY

A. Local limit

We turn our attention to the natural effective field theory discussed in Sec. III. As remarked earlier, it is not possible to perform the χ-functional integration exactly in any interesting model, so that approximations are required. In Sec. III we considered the χ-functional integral to one-loop order in the χ field. In this section we consider two further approximations, $|p^2|\ll m_\chi^2$ and $m_\phi^2\ll m_\chi^2$, which make the effective Lagrangian for ϕ approximately local, and establish a mass hierarchy, respectively.

The $W^{(2)}$ coefficient, given by (3.19), is momentum independent and hence local. Further, since $W^{(2)}$ is momentum independent to one-loop order in χ, the coefficients $g^{(3)}$ and $Z^{(2)}$ defined in (2.25) vanish to this order in the χ-loop expansion, although they must be considered in higher orders.

Consider the $W^{(4)}$ coefficient, which according to (2.12), (2.13), and (2.18) is required for the description of processes with four external ϕ lines, with $k_1+k_2+k_3+k_4=0$. Define $p=k_1+k_2$ and demand that $|p^2|\ll m_\chi^2$. For momenta satisfying this restriction, (3.23) and (3.24) become

$$\begin{aligned} W^{(4)}(k_1,\ldots,k_4) &=\frac{i}{4!}\left[\frac{\tilde{\lambda}^2}{\lambda}\left(1+\frac{p^2}{m_\chi^2}\right)+\frac{\tilde{\lambda}^2}{16\pi^2}\,\frac{p^2}{m_\chi^2}\,\ln\left(\frac{m_\chi^2}{\bar{\mu}^2}\right)\right.\\ &\qquad\left.-\frac{\tilde{\lambda}^2}{24}\left(1+\frac{4p^2}{m_\chi^2}\right)F(p^2,m_\chi^2)\right]+O\left(\frac{p^4}{m_\chi^4}\right), \end{aligned} \tag{5.1}$$

where

$$F(p^2,m_\chi^2)=\ln\frac{m_\chi^2}{\bar{\mu}^2}+1+\frac{p^2}{m_\chi^2}+O\left(\frac{p^4}{m_\chi^4}\right), \tag{5.2}$$

with $\bar{\mu}^2=4\pi\mu^2e^{1-\gamma}$. When the effective Lagrangian for ϕ is constructed according to (2.12), (2.13), and (2.18), the effective four-point ϕ-field coupling parameter λ_{eff} is given [in the approximation (5.1) and (5.2)] by

$$\begin{aligned} \lambda(\bar{\mu})_{\text{eff}}&=\lambda+c-\frac{\tilde{\lambda}^2}{\lambda}+\frac{\tilde{\lambda}^2}{24\pi^2}\left(\ln\frac{m_\chi^2}{\bar{\mu}^2}+1\right)\\ &\quad+\frac{p^2}{m_\chi^2}\left[\frac{\tilde{\lambda}^2}{\lambda}-\frac{\tilde{\lambda}^2}{48\pi^2}+\frac{11}{3}\,\frac{\tilde{\lambda}^2}{16\pi^2}\left(\ln\frac{m_\chi^2}{\bar{\mu}^2}+1\right)\right]\\ &\quad+O(p^4/m_\chi^4)\,. \end{aligned} \tag{5.3}$$

The logarithmic term multiplying p^2/m_χ^2 can always be neglected relative to the first logarithm in

(5.3) when $|p^2|/m_\chi^2 \ll 1$. A necessary condition for the single χ-loop approximation for $W^{(4)}$ to be valid is that all the coupling parameters be small. Given this, we can ignore $(p^2/m_\chi^2)\lambda^2/48\pi^2$ relative to the constant term in (5.3). To proceed further, an additional restriction must be imposed. Suppose that

$$\left|\lambda + c - \frac{\tilde{\lambda}^2}{\lambda}\right| \gg \left|\left(\frac{p^2}{m_\chi^2}\right)\left(\frac{\tilde{\lambda}^2}{\lambda} - \frac{\tilde{\lambda}^2}{48\pi^2}\right)\right| \simeq \left|\left(\frac{p^2}{m_\chi^2}\right)\frac{\tilde{\lambda}^2}{\lambda}\right|, \tag{5.4}$$

which can be satisfied for a wide range of parameters. We assume that (5.4) is true for the remainder of this paper. We now have shown that both $W^{(2)}$ and $W^{(4)}$ become local if $|p^2|/m_\chi^2 \ll 1$ and (5.4) is satisfied. At this point we should show that the $W^{(2n)}$, $n \geq 3$ coefficients also become local with these same constraints. This is almost obvious, and for the important case of mass hierarchies, unnecessary. We shall assume that these coefficients are in fact local.

The momentum integrals in (2.18) are trivial when the $W^{(n)}$ coefficients are local. Constructing the effective Lagrangian for the ϕ field according to Sec. II, we find

$$(\mathcal{L}_\phi)_{\text{eff}} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2(\bar{\mu})_{\text{eff}}}{2}\phi^2 - \frac{\lambda(\bar{\mu})_{\text{eff}}}{4!}\phi^4 + \sum_{n=3}^{\infty}\frac{f^{(2n)}(\bar{\mu})}{(m_\chi)^{2n-4}}\phi^{2n}, \tag{5.5}$$

where

$$m^2(\bar{\mu})_{\text{eff}} = m_\phi^2 - \frac{\tilde{\lambda}}{48\pi^2} m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}^2} \tag{5.6}$$

and

$$\lambda(\bar{\mu})_{\text{eff}} = \lambda + c - \frac{\tilde{\lambda}^2}{\lambda} + \frac{\tilde{\lambda}^2}{24\pi^2}\left(\ln\frac{m_\chi^2}{\bar{\mu}^2} + 1\right). \tag{5.7}$$

We have not evaluated the $f^{(2n)}$ coefficients, although the restrictions $|p^2| \ll m_\chi^2$ and (5.4) should guarantee that they are local. Note that the terms with $n \geq 3$ cannot be neglected for general values of m_ϕ^2. Nevertheless, locality of $(\mathcal{L}_\phi)_{\text{eff}}$ is associated with momenta small compared to the characteristic mass scale m_χ.

The coupling parameters λ, b, and c must be small so as to justify the one-χ-loop approximation for (5.5) and (5.6), and the more general result (3.19) and (3.23)–(3.25). These necessary conditions are not sufficient to ensure the validity of the χ-loop expansion. Multiple-χ-loop contributions to $W^{(n)}$ will introduce powers of $\tilde{\lambda}\ln(m_\chi^2/\bar{\mu}^2)$, unless this quantity is small. If not, the multiple-χ-loop contribution to $W^{(n)}$ will dominate the one-loop approximation. Therefore, in addition to small coupling constants, we require

$$\left|\frac{\tilde{\lambda}}{24\pi^2}\ln\frac{m_\chi^2}{\bar{\mu}^2}\right| \ll 1, \quad \left|\frac{\tilde{\lambda}^2}{48\pi^2} m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}^2}\right| \ll m_\phi^2. \tag{5.8}$$

Equation (5.8) restricts $\bar{\mu}^2$ to be of the order of m_χ^2, although a wide range of values of $\bar{\mu}^2$ centered on m_χ^2 are permitted. Before proceeding, we note that it is possible to go beyond the one-χ-loop approximation to $\mathcal{L}_{\text{eff}}$ by using the renormalization group to improve with respect to $\bar{\mu}^2$, and to eliminate the constraint (5.8). Because of the infrared safety of this particular scaling, one may extend $\bar{\mu}^2$ all the way down to zero. At that stage $0 \lesssim \bar{\mu}^2 \lesssim O(m_\chi^2)$ is allowed.

B. Mass hierarchy

It should be emphasized that the imposition of the local limit is logically independent of the imposition of a mass hierarchy. The contributions of coefficients $f^{(2n)}$ in (5.5) to tree-level processes are of strength m_ϕ^2/m_χ^2 or $m^2{}_{\text{eff}}/m_\chi^2$ (to some positive power). Therefore, without a mass hierarchy, tree-level processes involving ϕ^{2n}, $n \geq 3$ in the Lagrangian cannot be neglected relative to those with ϕ^2 and ϕ^4.

Suppose one restricts the mass parameters so that

$$m_\phi^2/m_\chi^2 \ll 1, \tag{5.9}$$

in which case all terms of dimension greater than four in the effective Lagrangian are suppressed. We define (5.9) as the "mass-hierarchy" constraint, a condition which cannot be imposed by any known natural condition. Assuming (5.9), the effective Lagrangian becomes

$$(\mathcal{L}_\phi)_{\text{eff}} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}(\bar{\mu})_{\text{eff}}\phi^2 - \frac{\lambda(\mu)_{\text{eff}}}{4!}\phi^4, \tag{5.10}$$

where $m^2(\bar{\mu})_{\text{eff}}$ and $\lambda(\bar{\mu})_{\text{eff}}$ are given by (5.6) and (5.7), respectively. At this stage we have obtained a local effective Lagrangian with interactions of dimension four or less.

In the next section we verify that the effective Lagrangian gives predictions which coincide with those of the complete theory as the one-loop level. In Sec. VII we shall renormalization group improve (5.10).

VI. VALIDITY OF THE EFFECTIVE FIELD THEORY

In Secs. III–V we developed an explicit example of a natural effective field theory in order to illustrate the general discussion presented in Sec. II. In particular, in the limit $|p^2| \ll m_\chi^2$ and $m_\phi^2 \ll m_\chi^2$, the effective Lagrangian describing purely

ϕ processes has a local limit of dimension four. To one loop in the χ field this local effective Lagrangian is

$$(\mathcal{L}_\phi)_{\text{eff}} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m_{\text{eff}}^2(\bar{\mu})}{2}\phi^2 - \frac{\lambda_{\text{eff}}(\bar{\mu})}{4!}\phi^4, \quad (6.1)$$

with

$$m_{\text{eff}}^2(\bar{\mu}) = m_\phi^2 - \frac{\tilde{\lambda}}{48\pi^2} m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}^2} \quad (6.2)$$

and

$$\lambda_{\text{eff}}(\bar{\mu}) = \left(\lambda + c - \frac{\tilde{\lambda}^2}{\lambda}\right) + \frac{\tilde{\lambda}^2}{24\pi^2}\left(\ln\frac{m_\chi^2}{\bar{\mu}^2} + 1\right). \quad (6.3)$$

It is important to examine whether the predictions of the effective field theory coincide (to desired accuracy) with those of the complete theory evaluated by the usual method of integrating over both the ϕ and χ fields, i.e., over the Φ field when $|p^2| \ll m_\chi^2$ and $m_\phi^2 \ll m_\chi^2$. Since it is not possible to evaluate the complete theory exactly, we will make our comparison with ϕ processes evaluated to one-loop order in Φ. This will place additional restrictions on mass scales and coupling constants for Φ-loop-wise perturbation theory to be valid. If these are not satisfied, a comparison of the effective field theory with the complete theory, perturbatively evaluated, makes no sense.

With this in mind, we examine the complete theory in one-Φ-loop approximation for low-energy ϕ processes using the minimal subtraction scheme for renormalization, and compare with the predictions of Eqs. (6.1)–(6.3) in single-ϕ-loop perturbation theory. In performing these calculations, different mass scales enter the renormalization of the various loop integrals. The mass scales in question are

(i) $\bar{\mu}_1$ from χ loops,

(ii) $\bar{\mu}_2$ from ϕ loops,

(iii) $\bar{\mu}_3$ from Φ loops.

One can keep $\bar{\mu}_1$ distinct from $\bar{\mu}_2$ in a natural effective field theory, since the χ loops and ϕ loops can be renormalized independently. (This will allow us to renormalization group improve the χ-loop expansion, and then treat the ϕ loops perturbatively, as in Sec. VII.) We now consider $\Gamma^{(2\phi)}$ evaluated different ways, in order to examine these questions.

A. Ordinary perturbation theory–$\Gamma^{(2\phi)}$

Consider $\Gamma^{(2\phi)}$, the 1PI 2-point ϕ function evaluated by ordinary perturbation theory. The Lagrangian is given by (3.7)–(3.16), with the relevant one-loop graphs given by Fig. 3. The regularization of these divergent integrals introduces a mass

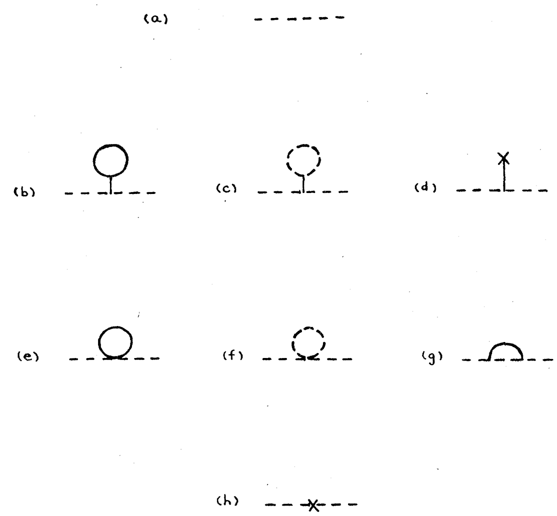

FIG. 3. Tree-level and one-loop graphs contributing to $\Gamma^{(2\phi)}$ in usual, Φ-field perturbation theory. The notation is the same as in Fig. 1.

scale $\bar{\mu}_3$, where we use *dimensional regularization* with *minimal* subtraction, as we did in constructing the effective Lagrangian. The only nontrivial graph is Fig. 3(g), which is momentum dependent. When $|p^2| \ll m_\chi^2$, $m_\phi^2 \ll m_\chi^2$, and $|p^2| \ll m_\phi^2$

$$\text{Graph } 3(\text{g}) = \frac{i}{32\pi^2}\,\frac{1}{(p^2 - m_\phi^2)^2}\left(\frac{2}{3}\right)\frac{\tilde{\lambda}}{\lambda} \times m_\chi^2\left(\frac{2}{\epsilon} + F(p^2, m_\phi^2, m_\chi^2) + O(\epsilon)\right), \quad (6.4)$$

where

$$F(p^2, m_\phi^2, m_\chi^2) = \ln\frac{m_\chi^2}{\bar{\mu}_3^2} - \frac{m_\phi^2}{m_\chi^2}\ln\frac{m_\phi^2}{\bar{\mu}_3^2} - \frac{p^2}{2m_\chi^2} + O\left(\frac{p^4}{m_\chi^4}\right) \quad (6.5)$$

and

$$\bar{\mu}_3^2 = 4\pi e^{1-\gamma}\mu_3^2. \quad (6.6)$$

The counterterms of Figs. 3(d) and 3(h) remove the divergent poles from Figs.[3(b)+3(c)] and Figs. [3(e)+3(f)+3(g)] respectively. The finite part of

$$\text{Graphs } [3(\text{a}) + 3(\text{b}) + 3(\text{e})] = i[p^2 - m_{\text{eff}}^2(\bar{\mu}_3)]^{-1}, \quad (6.7)$$

with $m_{\text{eff}}^2(\bar{\mu})$ given by (6.2). The finite part of

$$\text{Graphs } [3(\text{c}) + 3(\text{f}) + 3(\text{g})] = \frac{i}{32\pi^2}(p^2 - m_\phi^2)^2\left[\left(\lambda + c - \frac{\tilde{\lambda}^2}{\lambda}\right)m_\phi^2 \ln\frac{m_\phi^2}{\bar{\mu}_3^2} + \frac{2}{3}\,\frac{\tilde{\lambda}^2}{\lambda}\left(m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}_3^2} - \frac{p^2}{2}\right)\right] \quad (6.8)$$

to leading order in p^2. Note that the independent contributions $\lambda + c$, $-\frac{1}{3}\tilde{\lambda}^2/\lambda$, and $-\frac{2}{3}\tilde{\lambda}^2/\lambda$ come from Figs. 3(c), 3(f), and 3(g), respectively, adding to $\lambda + c - \tilde{\lambda}^2/\lambda$ which, to the order required in this problem, is $\lambda_{\rm eff}$. Also we may write $(p^2 - m_\phi^{\,2}) = (p^2 - m_{\rm eff}^{\,2})$ in Eq. (6.8), correct to the order we are working.

When (6.7) is added to (6.8), one obtains

$$\Gamma^{(2\phi)}(p) = p^2\left(1 + \frac{1}{96\pi^2}\,\frac{\tilde{\lambda}^2}{\lambda}\right) - \left[m_{\rm eff}^{\,2}(\bar{\mu}_3) + \frac{\lambda_{\rm eff}}{32\pi^2}\,m_\phi^{\,2}\ln\frac{m_\phi^{\,2}}{\mu_3^{\,2}} + \frac{1}{48\pi^2}\,\frac{\tilde{\lambda}^2}{\lambda}\,m_\chi^{\,2}\ln\frac{m_\chi^{\,2}}{\bar{\mu}_3^{\,2}}\right] \tag{6.9a}$$

$$= p^2\left(1 + \frac{1}{96\pi^2}\,\frac{\tilde{\lambda}^2}{\lambda}\right) - \left[m_\phi^{\,2} + \frac{\lambda_{\rm eff}}{32\pi^2}\,m_\phi^{\,2}\ln\frac{m_\phi^{\,2}}{\mu_3^{\,2}} - \frac{\tilde{\lambda}}{48\pi^2}\left(1 + \frac{\tilde{\lambda}}{\lambda}\right)m_\chi^{\,2}\ln\frac{m_\chi^{\,2}}{\mu_3^{\,2}}\right]. \tag{6.9b}$$

In arriving at (6.9) we have assumed that the one-Φ-loop approximation to $\Gamma^{(2\phi)}$ is a good approximation, which requires

$$\frac{\lambda}{32\pi^2},\ \frac{b}{32\pi^2},\ \frac{c}{32\pi^2},\ \frac{\tilde{\lambda}}{32\pi^2},\ \text{and}\ \frac{\lambda_{\rm eff}}{32\pi^2} \ll 1 \tag{6.10}$$

as well as

$$\left|\frac{\lambda_{\rm eff}}{32\pi^2}\ln\frac{m_\phi^{\,2}}{\bar{\mu}_3^{\,2}}\right| \ll 1 \tag{6.11}$$

and

$$\left|\frac{\tilde{\lambda}}{48\pi^2}\left(1 + \frac{\tilde{\lambda}}{\lambda}\right)\ln\frac{m_\chi^{\,2}}{\bar{\mu}_3^{\,2}}\right| \ll \frac{m_\phi^{\,2}}{m_\chi^{\,2}}\,. \tag{6.12}$$

B. Local effective theory; one ϕ loop

We now compute the one-ϕ-loop contribution to $\Gamma^{(2\phi)}$ using (6.1)–(6.3). Since we are considering a natural effective field theory, the renormalization of the χ-functional integration is independent of that of the ϕ-loop expansion. Hence, the quantities (6.2) and (6.3) are renormalized quantities with respect to the χ field, but unrenormalized quantities with respect to the ϕ perturbation theory. If we are to do ϕ perturbation theory to the one-loop level, we should write

$$\phi = \phi_R\,, \tag{6.13a}$$

$$m_{\rm eff}^{\,2} = m_{R\,{\rm eff}}^{\,2}(1 + D'/\epsilon)\,, \tag{6.13b}$$

and

$$\lambda_{\rm eff} = \lambda_{R\,{\rm eff}}\,\mu_2^{-\epsilon}(1 + E'/\epsilon)\,. \tag{6.13c}$$

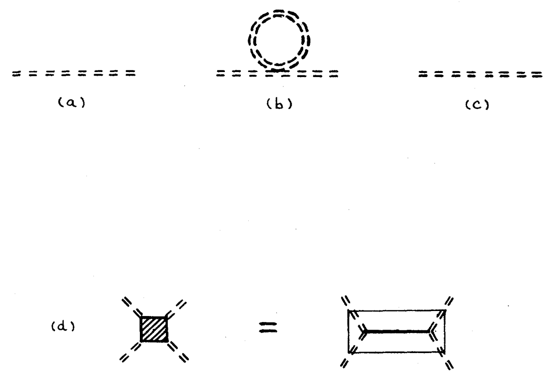

FIG. 4. Tree-level and one-loop graphs contributing to $\Gamma_{\rm eff}^{(2\phi)}$ in effective ϕ-field perturbation theory. The double dashed line stands for the effective ϕ field. The rest of the notation is the same as in Fig. 1. The shaded box represents the nonlocal, four-point interaction.

The renormalized parameters $m^2_{R\,{\rm eff}}$ and $\lambda_{R\,{\rm eff}}$ in (6.13) depend explicitly on $\bar{\mu}_1$ and implicitly on $\bar{\mu}_2$, due to the independent renormalization of χ and ϕ loops.

In terms of renormalized quantities, the effective Lagrangian becomes, suppressing the subscript R and mass-scale dependences for notational convenience,

$$(\mathcal{L}_\phi)_{\rm eff} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m_{\rm eff}^{\,2}}{2}\phi^2 - \frac{\lambda_{\rm eff}}{4!}\phi^4 - \frac{m_{\rm eff}^{\,2}}{2}\,\frac{D'}{\epsilon}\phi^2 - \frac{\lambda_{\rm eff}}{4!}\,\frac{E'}{\epsilon}\phi^4\,. \tag{6.14}$$

The graphs which contribute $\Gamma_{\rm eff}^{(2\phi)}$ to one-loop order in ϕ are shown in Figs. 4(a)–4(c). From Eqs. (6.2), (6.3), and (6.14) one finds

$$\Gamma_{\rm eff}^{(2\phi)}(\text{1-loop}) = p^2 - \left[m_{\rm eff}^{\,2}(\mu_1) + \frac{\lambda_{\rm eff}}{32\pi^2}\,m_\phi^{\,2}\ln\frac{m_\phi^{\,2}}{\mu_2^{\,2}}\right] \tag{6.15a}$$

$$= p^2 - \left[m_\phi^{\,2} + \frac{\lambda_{\rm eff}}{32\pi^2}\,m_\phi^{\,2}\ln\frac{m_\phi^{\,2}}{\bar{\mu}_2^{\,2}} - \frac{\tilde{\lambda}}{48\pi^2}\,m_\chi^{\,2}\ln\frac{m_\chi^{\,2}}{\bar{\mu}_1^{\,2}}\right], \tag{6.15b}$$

and

$$D' = -\frac{\lambda_{\rm eff}}{16\pi^2}\,, \tag{6.16}$$

when $|p^2| \ll m_\chi^{\,2}$ and $m_\phi^{\,2} \ll m_\chi^{\,2}$. Also, it follows from the renormalization of $\Gamma_{\rm eff}^{(4\phi)}$ that

$$E' = -\frac{3\lambda_{\rm eff}}{16\pi^2}\,. \tag{6.17}$$

Expression (6.2) is a valid one-loop approximation to $\Gamma_{\rm eff}^{(2\phi)}$ if, in addition to expression (6.10), we require that

$$\left| \frac{\lambda_{\rm eff}}{32\pi^2} \ln \frac{m_\phi^2}{\bar{\mu}_2^2} \right| \ll 1 \tag{6.18}$$

and

$$\left| \frac{\tilde{\lambda}}{48\pi^2} \ln \frac{m_\chi^2}{\bar{\mu}_1^2} \right| \ll \frac{m_\phi^2}{m_\chi^2} . \tag{6.19}$$

One might attempt to enforce equality of $\Gamma_{\rm eff}^{(2\phi)}$ [Eq. (6.15)] to $\Gamma^{(2\phi)}$ [Eq. (6.9)] by demanding that

$$\begin{aligned} &\bar{\mu}_2 = \bar{\mu}_3 , \\ &\bar{\mu}_1 = \bar{\mu}_3 (\bar{\mu}_3/m_\chi)^{\tilde{\lambda}/\lambda} \end{aligned} \tag{6.20}$$

and (since $\tilde{\lambda}^2/\lambda \ll 1$) by ignoring the nonlogarithmic terms in (6.9) proportional to $\tilde{\lambda}/\lambda$. However, for such an equality to be meaningful, it is necessary that conditions (6.20) lead to equality of $\Gamma_{\rm eff}^{(N\phi)}$ with $\Gamma^{(N\phi)}$ to one-loop level *for every N*. We have checked this possibility for $N=4$ and found that it is *not* the case. We conclude that no relationships between μ_1, μ_2, and μ_3 will allow our *local* effective Lagrangian to reproduce the results of the usual method given above. What has gone wrong? The answer is simply that the effective Lagrangian (6.1) is only local when $|p^2| \ll m_\chi^2$, but the ϕ-loop integration in Fig. 4(b) samples the effective four-point vertex in a momentum range from zero to infinity. We therefore must use the momentum-dependent effective four-point vertex derivable from expression (5.1). Fortunately, to the one-loop level in the effective ϕ field, we can drop all but the first term in (5.1). This term arises from the χ-field tree graph, Fig. 2(a), and, as shown in Fig. 4(d), contributes an extra, heavy propagator to the effective four-point coupling. That is, the nonlocal effective four-point coupling is (speaking loosely) given by

$$\lambda + c - \frac{\tilde{\lambda}^2}{\lambda} \left(\frac{m_\chi^2}{m_\chi^2 - (p-k)^2} \right) . \tag{6.21}$$

With this nonlocal vertex we find, using dimensional regularization and minimal subtraction, that the renormalized two-point vertex function is, for $|p^2| \ll m_\phi^2$, given by

$$\begin{aligned} \Gamma_{\rm eff}^{(2\phi)} = p^2 &\left(1 + \frac{1}{96\pi^2} \frac{\tilde{\lambda}^2}{\lambda}\right) \\ &- \left[m_{\rm eff}^2(\mu_1) + \frac{\lambda_{\rm eff}}{32\pi^2} m_\phi^2 \ln \frac{m_\phi^2}{\bar{\mu}_2^2} \right. \\ &\left. + \frac{1}{48\pi^2} \frac{\tilde{\lambda}^2}{\lambda} m_\chi^2 \ln \frac{m_\chi^2}{\bar{\mu}_2^2} \right] \end{aligned} \tag{6.22}$$

and

$$D' = -\frac{\lambda_{\rm eff}}{16\pi^2} - \frac{1}{24\pi^2} \frac{\tilde{\lambda}^2}{\lambda} \left(\frac{m_\chi^2}{m_\phi^2} + 1 \right) . \tag{6.23}$$

The E' renormalization coefficient can be determined from the effective one-loop four-point function (with nonlocal vertices) and is found to be

$$E' = -\frac{3\lambda_{\rm eff}}{16\pi^2} - \frac{\tilde{\lambda}^2}{4\lambda\lambda_{\rm eff}} \left(\lambda + c - \frac{2}{3} \frac{\tilde{\lambda}^2}{\lambda} + \frac{\tilde{\lambda}}{4} \frac{m_\phi^2}{m_\chi^2} \right) . \tag{6.24}$$

Clearly $\Gamma_{\rm eff}^{(2\phi)}$ in (6.22) is identical to $\Gamma^{(2\phi)}$ in (6.9) if we choose $\mu_1 = \mu_2 = \mu_3$ as the relation between mass scales. Also D' and E' given by (6.31) and (6.32), respectively, are precisely the results attained by the usual method (with dimensional regularization and minimal subtraction) for the coefficients of ϵ^{-1} in the one-loop pole expansion of bare quantities $a^2 + bv^2/6$ and $\lambda + c - \tilde{\lambda}^2/\lambda$, respectively. We conclude that our method, which uses dimensional regularization and minimal subtraction in both the χ- and the effective ϕ-field renormalizations, is completely identical to the usual $\{\Phi\}$-field method where dimensional regularization and minimal subtraction are employed, provided we use nonlocal effective parameters (where necessary) and choose $\mu_1 = \mu_2 = \mu_3$. Also, because of the necessity of using nonlocal couplings, we must include in the effective Lagrangian terms of dimension greater than four.

It would be disturbing if our effective Lagrangian had to be a nonlocal, infinite polynomial in order to reproduce the results of the usual perturbation theory in a domain where *both* methods are supposed to be valid. Inherent in the concept of an effective field theory is the hope that the local effective Lagrangian, with terms up to dimension four only, can be used. (This is necessary if we want the Weinberg-Salam model and QCD to arise as the low-energy manifestations of a grand unified theory.) From the above analysis this would not seem to be the case. However, one must be *careful*. Expression (6.22) for $\Gamma_{\rm eff}^{(2\phi)}$ differs from expression (6.15) for $\Gamma_{\rm eff}^{(2\phi)}$ (derived from the *local* effective Lagrangian) by a finite wave-function renormalization and by a logarithm involving a heavy mass. The finite wave-function renormalization clearly has no physical content. Furthermore, the heavy logarithm can always be absorbed into a redefinition of the effective mass parameter. Not only does this redefinition establish the formal equality of expression (6.22) and (6.15), but in addition, the new effective mass renormalizes according to the *local* renormalization equation (6.16) and not (6.23). It follows that the nonlocal *effective* Lagrangian is, physically, completely equivalent to the local *effective* Lagrangian for the computation of the one-loop, two-point function, at least when $|p^2| \ll m_\phi^2$. That is, natural effective field theories defined by *minimal* subtraction give predictions which coincide with those of the complete theory with *nonminimal* subtractions, when $|p^2| \ll m_\phi^2$. A complete discussion of this issue appears in a subsequent publication.[8]

This result reestablishes the utility of our effective Lagrangian method. It is *much* simpler to use a local effective Lagrangian with minimal subtraction to calculate $\{\phi_a\}$-field processes then it is to calculate the same processes with the $\{\Phi_A\}$-field method. Furthermore, it is obvious from our approach that the effective $\{\phi_a\}$ fields can be renormalized by an independent, light field renormalization scheme, a result which is difficult to establish (and to understand) using the usual $\{\Phi_A\}$-field method.

VII. THE χ-FIELD RENORMALIZATION GROUP

It should be clear from the discussions in Secs. III, V, and VI that our effective field theory method involves two independent renormalization schemes. First, it is necessary to renormalize the divergences occurring in the $\{\chi\}$-field loop expansion which leads to the effective Lagrangian. For natural effective field theories, this can be done using the counterterms available in the original Lagrangian, with a minimal subtraction procedure. Once a finite, effective $\{\phi\}$-field Lagrangian is obtained perturbation theory can be carried out in the effective $\{\phi\}$ field. Divergences in $\{\phi\}$-field loops must be removed, which can be done using counterterms available in the effective Lagrangian, again with minimal subtraction. Clearly, this second renormalization procedure is independent of the first. It follows that for each of the above renormalization schemes, there is associated an independent renormalization group. The $\{\phi\}$-field renormalization group scales the effective field theory according to the independent, light field renormalization scheme. This "decoupled," light field scaling has been widely discussed in the literature.[1-3,9] We will not discuss it here other than to remark once again that the existence of such scaling follows trivially from our effective Lagrangian method. This method also reveals a new kind of scaling based on the $\{\chi\}$-field renormalization. In this section we will study the $\{\chi\}$-field renormalization group within the context of the natural effective field theory introduced in Secs. III and V.

The renormalization constants for the χ-field renormalization are given in (3.26). From these constants, the β functions corresponding to parameters λ, b, and c are easily derived and are given by

$$\beta_\lambda(\lambda)=\frac{3\lambda^2}{16\pi^2}, \tag{7.1}$$

$$\beta_b(\lambda,b)=-\frac{3\lambda}{16\pi^2}\left(\lambda-\frac{\tilde\lambda}{3}\right), \tag{7.2}$$

$$\beta_c(\lambda,b)=-\frac{3\lambda}{16\pi^2}\left(\lambda-\frac{\tilde\lambda^2}{9\lambda}\right), \tag{7.3}$$

respectively. Similarly, the γ functions for m and a are found to be

$$\gamma_m(\lambda)=-\frac{\lambda}{32\pi^2}, \tag{7.4}$$

$$\gamma_a(\lambda,b,a,v)=-\frac{m_\chi{}^2}{64\pi^2a^2}\left(\lambda-\frac{\tilde\lambda}{3}\right). \tag{7.5}$$

The renormalization-group scaling equations for arbitrary coupling constant $\mathcal{C}$ and mass parameter $\mathfrak{M}$ are given by

$$\frac{d\mathcal{C}(t)}{dt}=\beta_{\mathcal{C}}(t), \tag{7.6}$$

$$\frac{d\mathfrak{M}(t)}{dt}=-\gamma_{\mathfrak{M}}(t)\mathfrak{M}(t), \tag{7.7}$$

respectively. These equations are to be solved subject to the boundary conditions that $\mathcal{C}(0)=\mathcal{C}$ and $\mathfrak{M}(0)=\mathfrak{M}$. It follows from (7.6) and (7.1), (7.2), and (7.3) that

$$\lambda(t)=\lambda\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1}, \tag{7.8}$$

$$\tilde\lambda(t)=\tilde\lambda\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1/3}, \tag{7.9}$$

$$b(t)=\tilde\lambda\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1/3}-\lambda\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1}, \tag{7.10}$$

$$c(t)=c-\frac{3\lambda^2t}{16\pi^2}\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1}-\frac{1}{3}\frac{\tilde\lambda^2}{\lambda}\left[\left(1-\frac{3\lambda t}{16\pi^2}\right)^{1/3}-1\right]. \tag{7.11}$$

Similarly, from (7.7) and (7.4) we find that

$$m^2(t)=m^2\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1/3}. \tag{7.12}$$

To proceed, we must determine the scaling behavior of $m_\chi{}^2$ and v^2. This is done by demanding the persistence of the relation

$$m_\chi{}^2=\frac{\lambda v^2}{3}=2m^2 \tag{7.13}$$

under the renormalization group. It follows that

$$m_\chi{}^2(t)=m_\chi{}^2\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1/3}, \tag{7.14}$$

$$v^2(t)=v^2\left(1-\frac{3\lambda t}{16\pi^2}\right)^{2/3}. \tag{7.15}$$

Using (7.7), (7.5), and (7.14) we find that

$$a^2(t)=a^2+\frac{m_\chi{}^2}{2}\left[\left(1-\frac{3\lambda t}{16\pi^2}\right)^{-1/3}-1\right]+\frac{m_\chi{}^2}{6}\frac{\tilde\lambda}{\lambda}\left[\left(1-\frac{3\lambda t}{16\pi^2}\right)^{1/3}-1\right]. \tag{7.16}$$

The above solutions are only valid for values of t

for which the running parameters are sufficiently small that the one-loop approximations to the β and γ functions are valid. It is clear from the above that this will be true only when

$$-\infty \lesssim t \lesssim 0 . \tag{7.17}$$

The fact that t can be continued to a large negative number follows from the infrared safety of the χ sector of the theory. We can now turn to the χ-renormalization-group improvement of the effective Lagrangian. The action of the effective Lagrangian can be written as

$$\Gamma_{\text{eff}}[\phi]=\sum_{N=1}^{\infty}\int d^4k_1\cdots d^4k_{2N}(2\pi)^4\delta(k_1+\cdots+k_{2N}) \times \tilde{W}^{(2N)}(k_1,\ldots,k_{2N})\phi(k_1)\cdots\phi(k_{2N}) , \tag{7.18}$$

where

$$\tilde{W}^{(2)}(k_1,k_2)=\tfrac{1}{2}(-k_{1\mu}k_2{}^{\mu}-m_\phi{}^2)-iW^{(2)}(k_1,k_2) , \tag{7.19}$$

$$\tilde{W}^{(4)}(k_1,k_2,k_3,k_4)=-\frac{(\lambda+c)}{4!}-iW^{(4)}(k_1,k_2,k_3,k_4) , \tag{7.20}$$

$$\tilde{W}^{(2N)}(k_1,\ldots,k_{2N})=-iW^{(2N)}(k_1,\ldots,k_{2N}), \quad N\geq 3 . \tag{7.21}$$

The action (7.18) can be viewed as the connected, zero-point function of the χ-field theory, where the ϕ field plays the role of an external potential. We emphasize that the pure ϕ-field Lagrangian $\mathcal{L}_\phi$ [see (3.8)] must be included as part of the χ-field zero-point function, since counterterms from $\mathcal{L}_\phi$ are necessary in the renormalization of $W[0;\phi]$. Now $\Gamma_{\text{eff}}[\phi]$ contains both implicit (in the renormalized parameters) and explicit dependence on the mass scale μ_1 introduced in the χ-field renormalization procedure. However, the basic tenet of the renormalization group is that

$$\frac{d}{d\mu_1}\Gamma_{\text{eff}}[\phi]=0 . \tag{7.22}$$

Inserting (7.18) into (7.22) and realizing that $\phi(k)$ is arbitrary, we conclude that

$$\frac{d\tilde{W}^{(2N)}}{d\mu_1}(k_1,\ldots,k_{2N})=0 \tag{7.23}$$

for each $N\geq 1$. The solution of Eq. (7.23) is well known and is given by

$$\tilde{W}^{(2N)}(k_1,\ldots,k_{2N};\lambda,\ldots,m_\chi{}^2,\ldots;\mu_1{}^2) =\tilde{W}^{(2N)}[k_1,\ldots,k_{2N};\lambda(t),\ldots,m_\chi{}^2(t),\ldots;e^{2t}\mu_1{}^2] , \tag{7.24}$$

where $\lambda(t)$, $m_\chi{}^2(t)$, etc., are the "running," renormalization-group parameters. There is no factor involving anomalous dimensions since we are considering components of a zero-point function. Note that we are *not* scaling the momenta, but rather the mass parameters μ_1. The utility of the right-hand side of Eq. (7.24) is that it is a valid approximation to $W^{(2N)}$ for a much wider range of mass scale ($e^t\mu_1$) than is the left-hand side (mass scale μ_1). Insertion of the right-hand side of (7.24) into the action (7.18) constitutes the renormalization-group improvement of the action, and hence, the effective Lagrangian. Note that renormalization-group improvement of the effective Lagrangian is entirely independent of the momenta k_i and any assumptions about mass parameters.

To be more specific about the χ-renormalization-group procedure, let us now consider the one-loop, local, $m_\phi{}^2 \ll m_\chi{}^2$, effective Lagrangian given by (5.10). For such a theory, the action is given by Eq. (7.12), where

$$\tilde{W}^{(2)}(k_1,k_2)=-\tfrac{1}{2}k_{1\mu}k_2{}^{\mu}-\frac{m^2(\mu_1)_{\text{eff}}}{2} , \tag{7.25}$$

$$\tilde{W}^{(2)}(k_1,k_2,k_3,k_4)=-\frac{\lambda(\mu_1)_{\text{eff}}}{4!} , \tag{7.26}$$

$$\tilde{W}^{(2N)}(k_1,\ldots,k_{2N})\approx 0 , \quad N\geq 3 , \tag{7.27}$$

and $m_{\text{eff}}{}^2$, λ_{eff} are as defined in (5.6) and (5.7), respectively. As discussed in Sec. V the one-loop approximation to $m_{\text{eff}}{}^2$ and λ_{eff} are only valid if μ_1 is restricted to be of $O(m_\chi{}^2)$. Let μ_{01} be such a mass scale. Then from Eq. (7.24) it follows that the renormalization-group improved $W^{(2)}$ coefficient is given by

$$-\tfrac{1}{2}k_{1\mu}k_2^{\mu}-\tfrac{1}{2}m_{\text{eff}}{}^2(t) , \tag{7.28}$$

where

$$m_{\text{eff}}{}^2(t)=a^2(t)+\frac{b(t)v(t)^2}{6} -\frac{\bar{\lambda}(t)}{48\pi^2}m_\chi{}^2(t)\ln\frac{m_\chi{}^2(t)}{e^{2t}\bar{\mu}_{01}{}^2} . \tag{7.29}$$

The running parameters in (7.29) are given in Eqs. (7.18)–(7.16). From Eq. (7.17) we see that (7.28) and (7.29) are valid for t such that $-\infty < t < 0$ and, therefore, for mass scale $e^t\mu_{01}$ such that

$$0 \lesssim e^t\mu_{01} \lesssim \mu_{01} . \tag{7.30}$$

A check on the correctness of expression (7.29) can be made by inserting the one-loop running parameters in (7.29) and power series expanding them in t. We find that

$$m_{\text{eff}}{}^2(t)=m_{\text{eff}}{}^2+O(t) , \tag{7.31}$$

where $O(t)$ can be ignored to the one-loop level of our calculation. This is the correct behavior if Eq. (7.24) is to be satisfied. Similarly, the renormalization-group improved $W^{(4)}$ coefficient is given by

$$-\frac{\lambda_{\rm eff}(t)}{4!}\,, \qquad (7.32)$$

where

$$\lambda_{\rm eff}(t)=\lambda(t)+c(t)-\frac{\tilde\lambda^2(t)}{\lambda(t)} + \frac{\tilde\lambda^2(t)}{24\pi^2}\left(\ln\frac{m_\chi^{\,2}(t)}{e^{2t}\bar\mu_{01}^{\,2}}+1\right) \qquad (7.33)$$

and the running parameters are as defined in Eqs. (7.8)–(7.16). As above, this expression is valid in the expanded mass-scale range (7.30). Inserting the running parameter into (7.33) and expanding them in t we find that, to the one-loop level,

$$\lambda_{\rm eff}(t)=\lambda_{\rm eff}\,, \qquad (7.34)$$

which is the correct behavior. We emphasize that it is necessary to include $\mathcal{L}_\phi$ in the χ zero-point function in order to arrive at Eqs. (7.31) and (7.34). To conclude, we have shown that the renormalization-group improved, one-loop, local effective Lagrangian for $m_\phi^2 \ll m_\chi^{\,2}$ is given by

$$(\mathcal{L}_\phi)_{\rm eff}=\tfrac{1}{2}(\partial_\mu\phi)^2-\frac{m_{\rm eff}^{\;2}(t)}{2}\phi^2-\frac{\lambda_{\rm eff}(t)}{4!}\phi^4\,, \qquad (7.35)$$

where $m_{\rm eff}^2(t)$ and $\lambda_{\rm eff}(t)$ are given by (7.29) and (7.33), respectively. If we redefine $\mu_1=e^t\mu_{01}$, then the range of mass scale for which (7.35) is valid is given by

$$0\lesssim\mu_1\lesssim\mu_0[=O(m_\chi)]\,. \qquad (7.36)$$

Not only is (7.35) a better approximation to the effective Lagrangian then (5.10) but it is also valid over a much wider range of mass scale [expression (7.36)] than is (5.10). The fact that μ_1 runs from $O(m_\chi)$ *down* to zero follows from the infrared safety of the χ sector of the theory. The heavy-loop renormalization-group improvement is potentially useful in theories where mass scales are so far separated that the inequalities [such as (6.11) and (6.12)] necessary to guarantee the validity of the one-loop approximation cannot be simultaneously maintained by a single choice of mass scale.

VIII. A NATURAL EFFECTIVE FIELD THEORY WITH A GOLDSTONE BOSON

In Sec. IV we showed explicitly that if a field theory contains a Goldstone boson, and if that Goldstone boson is included in the $\{\phi_a\}$ partition, then the effective field theory for fields ϕ_a is *unnatural.* We conjecture that if the Goldstone field were placed instead in the set $\{\chi_\alpha\}$, then the ϕ_a effective field theory would be *natural.* Since Goldstone bosons occur in many field theories of interest, this question is an important one. In this section we will show that this conjecture is correct in a spontaneously broken SO(2) theory. Let ϕ_1, ϕ_2, and ϕ_3 be three real scalar fields, and assume that under SO(2) group actions ϕ_1 and ϕ_2 transform as an irreducible doublet and ϕ_3 as a singlet. The most general SO(2) symmetric Lagrangian for these fields, which is invariant under the discrete transformations $\phi_i\to-\phi_i$ (the other two fields unchanged) for $i=1,2,3$, is given in terms of *bare* fields and parameters by

$$\mathcal{L}_{\rm SO(2)}=\mathcal{L}_{\rm SO(3)}+\mathcal{L}_{\rm BR}\,, \qquad (8.1)$$

where

$$\mathcal{L}_{\rm SO(3)}=\tfrac{1}{2}(\partial_\mu\phi)_i(\partial^\mu\phi)_i+\frac{m^2}{2}\phi_i\phi_i-\frac{\lambda}{4!}(\phi_i\phi_i)^2\,, \qquad (8.2)$$

$$\mathcal{L}_{\rm BR}=\frac{a^2}{2}\phi_3^{\,2}+\frac{2b}{4!}(\phi_1^{\,2}+\phi_2^{\,2})\phi_3^{\,2}+\frac{c}{4!}\phi_3^{\,4}\,. \qquad (8.3)$$

$\mathcal{L}_{\rm SO(3)}$ is invariant under SO(3) group actions which transform ϕ_1, ϕ_2, and ϕ_3 as an irreducible triplet, while $\mathcal{L}_{\rm BR}$ breaks this SO(3) invariance down to SO(2). The sign of the mass term in (8.2) is chosen such that symmetry breakdown occurs spontaneously. We define the renormalized parameters by expressions (5.5), and expand the renormalized fields around the nonzero vacuum expectation value (chosen without loss of generality to be)

$$\vec v_R=\begin{pmatrix} v_R\\ 0\\ 0\end{pmatrix}\,,\quad v_R=\left(\frac{3!\,m_R^{\,2}}{\lambda_R}\right)^{1/2}\,. \qquad (8.4)$$

Denote the shifted one- (Higgs), two- (Goldstone), and three-direction fields by χ_R, ψ_R, and ϕ_R, respectively. Partition the fields into $\{\phi_a\}=\{\phi_R\}$ and $\{\chi_\alpha\}=\{\chi_R,\psi_R\}$. In terms of these renormalized fields and parameters, partitioned as above, the Lagrangian is given by

$$\mathcal{L}=\mathcal{L}_{\{\phi\}}+\mathcal{L}_{\{\chi\}}+\mathcal{L}_{\{\phi\}-\{\chi\}} +\Delta\mathcal{L}_{\{\phi\}}+\Delta\mathcal{L}_{\{\chi\}}+\Delta\mathcal{L}_{\{\phi\}-\{\chi\}}\,, \qquad (8.5)$$

where (dropping subscript R and powers of μ)

$$\mathcal{L}_{\{\phi\}}=\tfrac{1}{2}(\partial_\mu\phi)^2-\frac{m_\phi^{\,2}}{2}\phi^2-\frac{(\lambda+c)}{4!}\phi^4\,, \qquad (8.6)$$

$$\mathcal{L}_{\{\chi\}}=\tfrac{1}{2}(\partial_\mu\chi)^2-\frac{m_\chi^{\,2}}{2}\chi^2-\frac{\lambda v}{3!}\chi^3-\frac{\lambda}{4!}\chi^4 \qquad (8.7)$$

$$+\tfrac{1}{2}(\partial_\mu\psi)^2-\frac{m_\psi^{\,2}}{2}\psi^2-\frac{\lambda}{4!}\psi^4\,, \qquad (8.8)$$

$$\mathcal{L}_{\{\phi\}-\{\chi\}}=-\frac{\tilde\lambda v}{3!}\phi^2\chi-\frac{2}{4!}\tilde\lambda\phi^2\chi^2 \qquad (8.9)$$

$$-\frac{\lambda v}{3!}\psi^2\chi-\frac{2}{4!}\lambda\psi^2\chi^2-\frac{2}{4!}\tilde\lambda\,\phi^2\psi^2\,, \qquad (8.10)$$

and

$$\Delta\mathcal{L}_{\{\phi\}} = \left(\frac{m_\chi^2}{4}(D-E) - \frac{a^2A}{2} - \frac{bv^2B}{12}\right)\frac{\phi^2}{\epsilon} - (\lambda E + cC)\frac{\phi^4}{4!\,\epsilon}\,, \tag{8.11}$$

$$\Delta\mathcal{L}_{\{\chi\}} = \frac{m_\chi^2 v}{2}(D-E)\frac{\chi}{\epsilon} + \frac{m_\chi^2}{4}(D-3E)\frac{\chi^2}{\epsilon} + \cdots\,, \tag{8.12}$$

$$\Delta\mathcal{L}_{\{\phi\}\text{-}\{\chi\}} = -v(\lambda E + bB)\frac{\phi^2\chi}{3!\,\epsilon} + \cdots\,. \tag{8.13}$$

We have ignored eight counterterms that are not needed to renormalize the theory at the one-loop level. Also,

$$m_\phi^2 = a^2 + \frac{bv^2}{3!}\,, \tag{8.14}$$

$$m_\chi^2 = \frac{\lambda v^2}{3}\,, \tag{8.15}$$

$$\tilde{\lambda} = \lambda + b\,. \tag{8.16}$$

Note that we have given the Goldstone boson a mass m_ψ^2 in (8.8). The Goldstone boson mass is, of course, zero and we will take the $m_\psi^2 \to 0$ limit at the end of the calculation. We now turn to the construction of the effective Lagrangian for ϕ. Note from the structure of Lagrangian (8.5) that $W^{(N)}(k_1, \ldots, k_N)$ vanishes when N is odd. To one-loop approximation, the graphs contributing to $W^{(2)}(k_1, k_2)$ are shown in Fig. 5. Counterterm 5(c) will subtract the divergences in graphs 5(a) and 5(b) if the renormalization constants satisfy

$$\frac{m_\chi^2}{2}(D-E) - a^2A - \frac{bv^2B}{6} = \frac{\tilde{\lambda}}{48\pi^2}(m_\chi^2 + m_\psi^2)\,. \tag{8.17}$$

Similarly, counterterm (f) will subtract the divergences in graphs 5(d) and 5(e):

$$D - E = \frac{\lambda}{8\pi^2}\left(1 + \frac{1}{3}\,\frac{m_\psi^2}{m_\chi^2}\right)\,. \tag{8.18}$$

With these subtractions, we find that

$$W^{(2)}(k_1, k_2) = \frac{i\tilde{\lambda}}{96\pi^2}\left(m_\chi^2 \ln\frac{m_\chi^2}{\bar{\mu}^2} + \frac{2m_\psi^4}{m_\chi^2}\ln\frac{m_\psi^2}{\bar{\mu}^2}\right), \tag{8.19}$$

where $\bar{\mu}^2 = 4\pi\mu^2 e^{1-\gamma}$. To one-loop level, the graphs contributing to $W^{(4)}(k_1, \ldots, k_4)$ are shown in Fig. 6. Renormalization equation (8.18) is sufficient to allow the divergences in graphs 6(c) and 6(d) to be canceled by their respective counterterms. The divergences in graphs 6(b) will be canceled by the χ mass counterterm if

$$D - 3E = \frac{\lambda}{24\pi^2}\left(13 + \frac{m_\psi^2}{m_\chi^2}\right). \tag{8.20}$$

Similarly, the divergences in graphs 6(e) and 6(f) will be canceled by their counterterms if

$$\lambda E + bB = -\frac{\tilde{\lambda}\lambda}{12\pi^2}\,, \tag{8.21}$$

$$\lambda E + cC = -\frac{\tilde{\lambda}^2}{24\pi^2}\,, \tag{8.22}$$

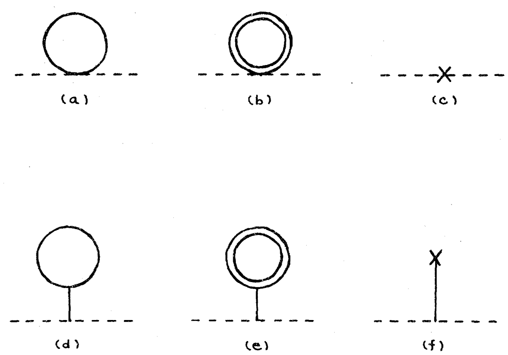

FIG. 5. One-loop graphs contributing to $W^{(2)}(k_1,k_2)$ in a theory with three scalar fields. The double solid line stands for the ψ field. The rest of the notation is the same as in Fig. 1.

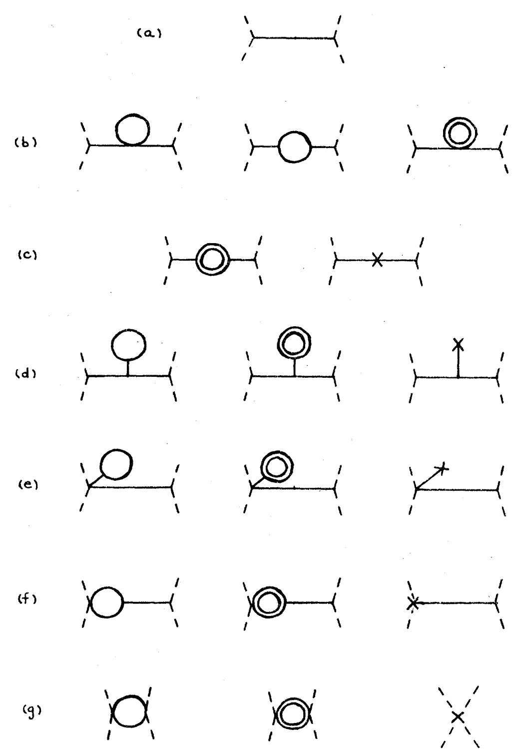

FIG. 6. Tree-level and one-loop graphs contributing to $W^{(4)}(k_1,k_2,k_3,k_4)$ in a theory with three scalar fields. The notation is the same as in Fig. 5.

respectively. Taking the limit $m_\psi{}^2 \to 0$, the five renormalization equations are given by

$$\frac{m_\chi{}^2}{2}(D-E) - a^2 A - \frac{bv^2 B}{6} = \frac{\bar{\lambda}}{48\pi^2} m_\chi{}^2\,, \qquad (8.23)$$

$$D - E = \frac{\lambda}{8\pi^2}\,, \qquad (8.24)$$

$$D - 3E = \frac{13\lambda}{24\pi^2}\,, \qquad (8.25)$$

$$\lambda E + bB = -\frac{\bar{\lambda}\lambda}{12\pi^2}\,, \qquad (8.26)$$

$$\lambda E + cC = -\frac{\bar{\lambda}^2}{24\pi^2}\,. \qquad (8.27)$$

Equations (8.23)–(8.27) are consistent with each other and constitute five equations in five unknowns, which can be solved to give renormalization constants A, B, C, D, and E. We therefore conclude that counterterms $\Delta\mathcal{L}_{\{\phi\}}$, $\Delta\mathcal{L}_{\{\chi\}}$, and $\Delta\mathcal{L}_{\{\phi\}-\{\chi\}}$ are sufficient to subtract all the divergences in $W^{(2)}$ and $W^{(4)}$. We assume that these counterterms are also sufficient to subtract one-loop divergences in $W^{(2n)}$, $n>3$ coefficients. Then from definition 1 in Sec. II it follows that, to one-loop in the $\{\chi_\alpha\}$ fields, the effective field theory for ϕ is natural.

The $W^{(4)}$ coefficient is, of course, momentum dependent. In the limit $m_\psi{}^2 \to 0$ there is no momentum regime for which $W^{(4)}$ becomes local. It follows that there is no momentum regime for which the effective ϕ-field Lagrangian can be approximated by a local Lagrangian. Secondly, some contributions to $W^{(2n)}$, $n>3$ are proportional to $(m_\psi{}^2)^{2-n}$ and become large in the $m_\psi{}^2 \to 0$ limits. It is therefore impossible to approximate the effective Lagrangian with the terms of dimension two and four only. We conclude that the partition $\{\phi_a\} = \{\phi\}$, $\{\chi_\alpha\} = \{\chi, \psi\}$ (that is, the Goldstone boson included in $\{\chi_\alpha\}$) leads to a natural effective field theory for ϕ, but since $m_\psi{}^2 = 0$, the effective Lagrangian is a nonlocal, infinite power series in ϕ.

Now let us assume for a moment that $m_\psi{}^2$ is not zero. Renormalization equations (8.17), (8.18), and (8.20)–(8.22) remain five *consistent* equations in five unknowns. It follows that the effective ϕ-field Lagrangian is still natural. Furthermore, if we let $M = \min\{m_\psi, m_\chi\}$, then for momenta $p = k_1 + k_2$ such that $|p^2| \ll M^2$, the effective Lagrangian is (approximately) local. Finally, assuming $m_\phi{}^2 \ll M^2$, we can drop terms of dimension six and larger and find that the local, ϕ-field effective Lagrangian is given by

$$(\mathcal{L}_\phi)_{\rm eff} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m_{\rm eff}{}^2}{2}\phi^2 - \frac{\lambda_{\rm eff}}{4!}\phi^4\,, \qquad (8.28)$$

where

$$m_{\rm eff}{}^2 = m_\phi{}^2 - \frac{\bar{\lambda}}{48\pi^2}\left(m_\chi{}^2 \ln\frac{m_\chi{}^2}{\bar{\mu}^2} + 2\frac{m_\psi{}^4}{m_\chi{}^2}\ln\frac{m_\psi{}^2}{\bar{\mu}^2}\right), \qquad (8.29)$$

$$\lambda_{\rm eff} = \lambda + c - \frac{\bar{\lambda}^2}{\lambda} + \frac{\bar{\lambda}^2}{24\pi^2}\left[\ln\frac{m_\chi{}^2}{\bar{\mu}^2} - \left(1 + \frac{1}{4}\frac{m_\psi{}^2}{m_\chi{}^2}\right)\ln\frac{m_\psi{}^2}{\bar{\mu}^2}\right]. \qquad (8.30)$$

This result indicates that, for any theory where the Goldstone boson has a large, nonzero mass parameter in the Lagrangian, the above partition (Goldstone boson included in $\{\chi_\alpha\}$) leads to a natural effective field theory for the ϕ field. Furthermore, the effective Lagrangian has a local limit and can be approximated by the dimension two and four terms. This is precisely the situation that occurs when the above scalar fields are minimally coupled to an SO(2) gauge field using the R_ξ gauge. In this case the Goldstone boson has nonzero mass parameter $m_\psi{}^2 = g^2v^2/\xi$, where g is the gauge coupling constant, v is the scalar vacuum expectation value, and ξ is the gauge parameter. The massive gauge field and ghosts *slightly* modify the renormalization equations and Eqs. (8.29) and (8.30), but the effective field theory for ϕ remains natural. Unfortunately, in such a theory both $m_{\rm eff}{}^2$ and $\lambda_{\rm eff}$ are ξ dependent. It is therefore necessary to modify the gauge-fixing procedure to ensure that $m_{\rm eff}{}^2$ and $\lambda_{\rm eff}$ are ξ independent. This problem has been solved by S. Weinberg.[10] The modification does not affect the above results. We conclude that in spontaneously broken gauge theories, the Goldstone bosons can (and must) be included in the $\{\chi_\alpha\}$ partition in order to ensure that the effective $\{\phi_a\}$ field theory is natural.

IX. CONCLUSIONS

We have presented a systematic method for the construction of effective field theories and their renormalization-group improvement with respect to the arbitrary mass scales of the renormalization. Two classes of effective field theories were distinguished, natural and unnatural, according to certain independence criteria placed on the counterterms. Natural effective field theories may be separately renormalization-group improved with respect to the two independent mass scales which enter the renormalization procedure.

It was shown how restrictions on masses and external momenta lead to a local effective field theory of dimension four. The predictions of the local effective field theory were compared with those of the complete theory in a domain where both were valid. It was argued, by example, that the predictions of a natural effective field theory, minimally subtracted, coincide with those of the complete theory only if the complete theory is nonminimally subtracted. It was also argued that Goldstone bosons must be placed in the set of fields "integrated out" if the effective field theory is to be natural, and that the local limit was not attainable unless the scalar fields were coupled to gauge fields.

The various issues raised in this paper were illustrated by examples from scalar field theories. In subsequent work we will extend our analysis to unnatural effective theories and to gauge theories.

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