Size of a bouncing mixmaster universe

John D. Barrow
Department of Astrophysics, Oxford University, South Parks Road, Oxford, United Kingdom

Richard A. Matzner
Relativity Center, University of Texas at Austin, Austin, Texas 78712
(Received 3 October 1979)

An analysis is given of the evolution of a massive scalar field in a closed mixmaster universe of Bianchi type IX. Although the scalar field violates the strong energy condition, the probability of the model "bouncing" at a very early time is infinitesimally small; of the order of the ratio of the minimum to maximum sizes of the universe $\sim 10^{-40}$.

I. INTRODUCTION

The singularity theorems of Hawking and Penrose[1] are among the most precise and far-reaching results of modern theoretical cosmology. They postulate that reasonable restrictions on the causal structure of space-time and the positivity of energy are satisfied in nature and then use the latter condition to guarantee that gravitational focusing will inevitably create a focal point along the congruence of causal geodesic curves in the space-time. Since the development of such a focal point would contradict the very causal-structure assumptions used to derive it, one concludes that geodesics can never reach these focal points. To ensure this, there must exist at least one incomplete geodesic and the end points of all such incomplete geodesics form the singular boundary of the space-time. In some cases approach to these boundary points may also be accompanied by infinities in measurable physical quantities like the material densities or induced tidal forces.[2]

Attempts to avoid the conclusions of these theorems regarding space-time incompleteness generally concentrate attention upon undermining one of its assumptions—the positive-energy condition. Two such positivity conditions are commonly employed in the proofs: the *strong* energy condition requiring that for all causal vectors u^μ

(S) $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu \geqslant 0$,

and the *weak* energy condition which is, accordingly, less restrictive, demanding only

(W) $T_{\mu\nu}u^\mu u^\nu \geqslant 0$.

Typical examples of model universes in which either (S) or (W) may be violated are those incorporating (a) bulk viscosity,[3] (b) a sufficiently large positive cosmological constant,[4] (c) torsion, via an antisymmetric metric connection,[5] (d) spontaneously broken symmetries,[6] (e) spinor fields,[7] (f) negative vacuum stresses and quantum particle production,[8] and (g) massive scalar fields.[9] This last example will be pursued below and is relevant because the massive scalar stress allows a violation of the strong energy condition. For this field

$$(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)u^\mu u^\nu = (\phi_{,\beta}u^\beta)^2 - \tfrac{1}{2}m^2\phi^2 .$$

Whether or not a violation takes place in a particular cosmological model clearly depends upon the space-time evolution of the ϕ field. From dimensional analysis alone we see that violations are only anticipated at times $\lesssim m^{-1}$. The nonlinear dependence of the energy conditions upon the geometry can be seen in a number of cases. For example, although singularities may be avoided in isotropic universes by the devices (a) and (c) cited above, the results of Ref. 10 suggest they remain in the corresponding *anisotropic* universes. This shows the need to examine the most general possible dynamical backgrounds in conjunction with possible energy-condition violations.

We shall examine the evolution of a massive scalar field in a closed, homogeneous, and anisotropic universe of Bianchi type IX, which is among the most general class of spatially homogeneous space-times.[11] We shall show that the probability of a violation of (S) leading to a "bounce" of the scale factor in the quantum era ($t \sim m^{-1}$) of the early universe is of the order of the ratio between the minimum and maximum scales of expansion for the model. Numerically this probability is $\sim 10^{-40}$ for our universe and confirms the results of Starobinskii[12] obtained for the special case of isotropic expansion.

This result provides an interesting example of a point recently stressed by Tipler[6]: The violation of an energy condition in a small region of space or time will not generally be sufficient to avoid the presence of an incomplete geodesic. For the space-time to be singularity-free the energy condition must be violated on the average when integrated over the entire history of a causal geodesic. Our results can be interpreted in this light: The larger the expansion maximum of a closed universe the longer is the period of classical evolution during which the energy condition holds. Therefore the relative time during which a geodesic is dominated by the negative-energy effects and the probability of singularity avoidance both diminish. For a cosmological model to be "bounced" by a scalar field with high probability it must never expand out of the quantum era to a dimension greater than $\sim 10^{-13}$ cm. Although this discussion is able to take no account of quantum gravity effects at $t \sim 10^{-43}$ sec it gives some credence to Wheeler's "principle of unanimity,"[13] which conjectures that, except for a set of measure zero, quantum equations of motion will predict "singularities" whenever their classical counterparts do.

Before proceeding to prove these results, it is worth inquiring into the possible *astrophysical* relevance of processes occurring during the first 10^{-23} sec of the universe's life. Until quite recently one would have said there was none. The earliest cosmological time at which accompanying physical processes led to observable predictions was the epoch of[14] primordial nucleosynthesis $\sim 1-10^2$ sec. This may no longer be the case following the new development of grand unified gauge theories of the strong and electro-weak interactions[15] based upon simple non-Abelian gauge groups like SU(5), SO(10), or E_6. All these groups allow the embedding of the SU(3) color group but admit different possible breakdowns for the electro-weak symmetry, in particular SU(2)×U(1). These models of nongravitational interactions predict baryon nonconservation mediated by heavy ($\gtrsim 10^{14}$ GeV) lepto-quark gauge quanta which couple to both charge and color. The dual coupling allows quarks internal to the proton to decay into leptons ($qq \to x \to l\bar{q}$), with the observable consequence that the proton is unstable. These properties could be strongly manifested during the very earliest stages of the big-bang cosmology and might enable an explanation to be provided for the observed baryon asymmetry and specific entropy of the universe.[16] It therefore becomes relevant to observational cosmology whether or not the universe ever got as hot as $10^{14}-10^{16}$ GeV. Universes that are bounced by massive scalar fields at $t_{\min} \sim 10^{-23}$ sec would get no hotter than $\sim 10^9$ GeV and for them the implications of grand unification would be irrelevant.

The only other process occurring at $\sim 10^{-23}$ sec which might conceivably have astrophysical relevance is the process of primordial black-hole formation.[17] If the universe were to bounce at a time $t_{\min}$ then the smallest mini black hole able to form primordially would be roughly the horizon mass $\sim 10^{38}(t_{\min}/1\text{ sec})$ g. If $t_{\min}$ were larger than $\sim 10^{-23}$ sec then no black holes with a Bekenstein-Hawking lifetime (Ref. 18), $\sim 10^{-26}(M/1\text{ g})^3$ sec, equal to the present Hubble age could ever have formed. The discovery of an exploding mini black hole would constrain the possible epoch of a bounce regardless of its cause.

Notation. Metric signature: $(-+++)$; $\hbar \equiv c \equiv 1$; field equations: $\mathsf{G} = (8\pi G/3)\mathsf{T}$; Greek indices μ, $\nu \cdots$ run over 0, 1, 2, 3; summation convention is suspended for bracketed indices $T_{(ii)}$; normalization: $u_\mu u^\mu = -1$.

II. MASSIVE SCALAR FIELD IN THE MIXMASTER UNIVERSE

Starobinskii[12] has argued that quantum effects involving bosons play a negligible role near the singularity of a universe that collapses isotropically from a macroscopic, classical radius. A study of the problem in the Friedmann background involves the solution of the scalar field equation

$$(\Box^2 - m^2)\phi = 0\,, \tag{1}$$

which generates the stress tensor

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}(\phi_{,\mu}\phi^{,\mu} + m^2\phi)\,. \tag{2}$$

We shall consider this field in a closed Bianchi type-IX universe, the most general class of a closed, homogeneous, and anisotropic universe. The restriction to spatial homogeneity ensures ϕ is a function of time alone. The nonzero components of $T_{\mu\nu}$ are

$$T_{00}(\phi) = \tfrac{1}{2}(\dot{\phi}^2 + m^2\phi^2) \tag{3}$$

and

$$T_{ij}(\phi) = -\tfrac{1}{2}g_{ij}(m^2\phi^2 - \dot{\phi}^2)\,. \tag{4}$$

The scalar field exerts an isotropic pressure upon the geometry regardless of any anisotropy in the expanding geometry.

In order to generalize Starobinskii's analysis away from the very special isotropically expanding Friedmann background, we introduce the metric for the type-IX geometry:

$$ds^2 = -dt^2 + a^2(t)(e^{2\beta(t)})_{ij}\omega^i\omega^j\,, \tag{5}$$

where the forms ω^i are defined by their invariance properties under the group action of the spatial Killing vectors and $\exp(2\beta)_{ij}$ is a matrix exponential of the diagonal traceless matrix β_{ij} (see Ref. 11).

The scalar wave equation (1) becomes

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi = 0\,, \tag{6}$$

and the Einstein equations written in an orthonormal frame reduce to

$$\left(\frac{\dot a}{a}\right)^2=\frac{8\pi G}{3}[T_{00}(\phi)+\rho] + \tfrac{1}{6}\sigma_{ij}\sigma_{ij}+a^{-2}[V(\beta)-1]\,, \tag{7}$$

$$2\frac{\ddot a}{a}=-\left(\frac{\dot a}{a}\right)^2-8\pi G[T_{(11)}(\phi)+p] -\tfrac{1}{2}\sigma_{ij}\sigma_{ij}+a^{-2}[V(\beta)-1]\,. \tag{8}$$

Here ρ and p give the density and pressure of any additional classical matter fields and $\sigma_{ij}\equiv\dot\beta_{ij}$ is the shear of the timelike normals to the surfaces of homogeneity. The last term on the right-hand side of equations (7) and (8) is proportional to the negative of the three-curvature of the $t=$constant spacelike sections. Note that the shear-free, isotropic expansion of the Friedmann models corresponds to the vanishing of the curvature potential, $V\equiv 0$. For our case $V(\beta)$ may become larger than unity and the three-curvature can change its sign as the expansion evolves.

III. CONDITIONS FOR BOUNCE

In order that the negative stress of the massive scalar field produce a singularity-avoiding bounce at high density it is necessary that the scale factor pass through a turning point. By (3) and (7) this occurs at $a=a_{\min}$, where $\dot a(a_{\min})=0$, and so:

$$\dot\phi^2+m^2\phi+2\rho+\frac{1}{4\pi G}\{\tfrac{1}{2}\sigma_{ij}\sigma_{ij} + 3[V(\beta)-1]a_{\min}^{-2}\}=0\,. \tag{9}$$

Without solving any equations we may see qualitatively how difficult it is for the quantum stress to effect a bounce in a classical geometry. The possibility of bounce depends on that very small positive part of the three-curvature which is responsible for the existence of the expansion maximum $a_{\max}$ at a later classical epoch. Since any physically realistic equation of state generates densities which diverge at least as rapidly as the scalar stress $\rho\sim a^{-3}$–a^{-6}, there appears very little chance of a turning point in the quantum era. This unlikely situation can only occur if the ϕ field is anomalously small. At a minimum we also require $\ddot a>0$, and so by (8) this implies

$$-T_{(11)}(\phi)>[\tfrac{1}{3}T_{00}(\phi)+\tfrac{1}{3}\rho+p]+\frac{\sigma_{ij}\sigma_{ij}}{12\pi G}\,, \tag{10}$$

where we recall that $T_{(11)}(\phi)$ may be negative. Using (3) and (4) this may be rewritten in the form of an inequality governing the behavior of the field

$$m^2\phi^2>2\dot\phi^2+\rho+3p+\frac{\sigma_{ij}\sigma_{ij}}{4\pi G}\,. \tag{11}$$

In summary, (9) and (11) provide the *necessary* conditions for a bounce to occur in the evolution of the geometry. Clearly (9) and (11) further require the following inequalities to hold:

$$V(\beta)<1\,, \tag{12}$$

$$\tfrac{1}{6}\sigma_{ij}\sigma_{ij}<a_{\min}^{-2}\,, \tag{13}$$

$$\frac{8\pi G}{3}\rho<a_{\min}^{-2}\,, \tag{14}$$

$$\dot\phi^2<\tfrac{1}{2}m^2\phi^2\,. \tag{15}$$

Condition (12) requires the three-curvature to be approximately isotropic and in combination with (13) says that the anisotropy energy must be dominated by the isotropic part of the three-curvature. All anisotropies must be small when a bounce occurs and (9) shows that both the classical matter and shear energy must be dominated by the very small scalar stress $T_{00}(\phi)$. Finally we note that (15) is the condition that the strong energy condition be violated and demands that the bounce occur at an epoch $t_{\min}\lesssim m^{-1}$.

IV. THE EXPANSION MAXIMUM AND MINIMUM

In the classical regime, $m\gg\dot a/a$ and the WKB solution of (6) is the sum of the two elementary solutions $\bar\phi_1$, $\bar\phi_2$:

$$\phi\equiv\bar\phi_1+\bar\phi_2=a^{-3/2}(C\cos mt+D\sin mt)\,, \tag{16}$$

where the amplitudes C and D are constants to be determined. It is clear that the bounce condition (15) can only hold in the nonclassical region. The expansion maximum is given by $\dot a=0$ and $\ddot a<0$ and lies in the classical region where (16) holds; using (9) and (10) we find $a_{\max}$ to satisfy

$$\frac{4\pi G}{3}a_{\max}^{-3}m^2(C^2+D^2)+\frac{8\pi G}{3}\rho +\frac{\sigma_{ij}\sigma_{ij}}{6}+a_{\max}^{-2}[V(\beta)-1]=0\,. \tag{17}$$

From equations (12)–(14) we see that if a bounce does occur then the last three terms in (17) must be negligible. The equations governing the evolution of these terms guarantee their rapid decay $\sim a^{-(3+\epsilon)}$, $\epsilon>0$, as a increases. We therefore see that if a quantum bounce does occur, only the terms involving the scalar field and the curvature will be important near the maximum. We have therefore, by (17), that

$$4\pi G m^2(C^2+D^2)\simeq 3a_{\max}\,. \tag{18}$$

If a solution has a bounce at $a_{\min}$ then we can specify this solution by the amplitudes of the two elementary solutions at $a_{\min}$. When $t<m^{-1}$ Eqs. (6) and (7) yield $\phi\sim A+Ba^3\sim A+Bt$ to first order for the two elementary solutions

$$\hat{\phi}_1 = A\ , \tag{19}$$

$$\hat{\phi}_2 = Bt\ . \tag{20}$$

For a bounce to be possible (15) tells us that

$$B = \delta A m\ , \quad |\delta| < 2^{-1/2} \tag{21}$$

and the expansion minimum is then given by (9), (19), (20) as

$$4\pi(1+\delta^2)Gm^2A^2 = 3a_{\min}^{-2}[1 - V(\beta)] - 8\pi G\rho - \tfrac{1}{2}\sigma_{ij}\sigma_{ij}\ . \tag{22}$$

We can now connect the conditions for an expansion minimum to the value of the maximum scale $a_{\max}$ by finding a relation between amplitudes of the quantum (A,B) and classical (C,D) solutions. The Wronskian W of both solutions has the exact scaling property

$$W[\phi_1,\phi_2] \equiv (\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1) \propto a^{-3}; \tag{23}$$

and so we may evaluate W at the two extrema from (16), (19), (20), and (23),

$$W[\hat{\phi}_1,\hat{\phi}_2]\,|_{a_{\min}} = \delta m A^2 = a_{\min}^{-3}\gamma m C^2\ , \quad \text{where } \gamma \equiv D/C\ . \tag{24}$$

Using (18) and (22) we have that

$$\gamma(1+\gamma^2)^{-1} \simeq \delta(1+\delta^2)^{-1}\left\{\left(\frac{a_{\min}}{a_{\max}}\right) - \left(\frac{a_{\min}{}^3}{a_{\max}}\right) \times\left[8\pi G\rho + \frac{\sigma_{ij}\sigma_{ij}}{2} + \frac{3V(\beta)}{a_{\min}{}^2}\right]\right\}\ . \tag{25}$$

The inequality (21) implies that $\delta(1+\delta^2)^{-1} \sim \delta$ and the invariance of $\gamma(1+\gamma^2)^{-1}$ under $\gamma \to \gamma^{-1}$ allows us to choose $\gamma < 1$. Equation (25) therefore shows that one of the two classical solutions $\bar{\phi}_1$, $\bar{\phi}_2$ must differ from the other by a factor $\gamma \sim \delta(a_{\min}/a_{\max})$, which is $\sim 10^{-40}$ in the actual universe if $a_{\max} \sim 10^{17}$ sec and $a_{\min} \sim 10^{-23}$ sec. A quantum bounce requires domination by the $\hat{\phi}_1$ mode in the quantum epoch and so we may use γ as a measure of the improbability of any classical solution undergoing a bounce. Therefore the probability that the negative stress in the quantum region outweighs the positive stresses in the classical region to give a bounce is

$$(\text{probability of bounce}) \lesssim \left(\frac{a_{\min}}{a_{\max}}\right). \tag{26}$$

V. DISCUSSION

A single massive scalar may bounce the universe at times earlier than $\sim m^{-1}$ but only with the minute probability (26) which is $\lesssim 10^{-40}$ in practice and must be small in any universe large enough and old enough to evolve observers. To offset this result one might conceive of a whole ensemble of scalar fields[19] extending to higher and higher masses and whose total number is large enough to give a bounce probability ~ 1. However, the higher-mass fields allow the possibility of bounce only at earlier times and smaller values of $a_{\min}$ and by (26) this gives them a correspondingly lower probability of being effective.

In practice one anticipates that, even before the quantum gravity era is reached, additional factors will lower the probability (26) even further. For example: (a) The ordinary matter terms in (8) tend to produce singularities (in the case of stiff matter $p=\rho$, and this simulates the scalar field evolution in the nonbouncing $\hat{\phi}_2$ mode where the massive component is negligible). (b) The anisotropy terms will grow strongly on approach to a singularity in the collapse phase of a closed universe, and violation of the constraints (12) and (13) will inevitably occur unless the model is meticulously regular. (c) A generic inhomogeneous cosmological model would have an even smaller probability for bounce simply because as we move from region to region in the space-time the distribution of anisotropies and curvatures would make a violation of the requirements (12) and (13) in some region very much more likely, and a singularity would appear on the past light cone of that region.

In the range of models discussed, despite a violation of the energy condition, it is overwhelmingly improbable that a bounce could occur when the universal scale exceeded the Planck length $\sim 10^{-33}$ cm. On length scales smaller than this, a quantized theory of gravity must be employed. Some model calculations relevant to this era have been performed in minisuperspace quantum cosmologies where the homogeneous gravitational wave modes are quantized.[20] The results which emerge are, unfortunately, dependent upon the boundary conditions and factor-ordering prescriptions employed. For example, Misner's early work[20] on a similar type-IX geometry to ours considered the evolution of a classical state with a narrow spread of eigenmodes centered upon a large (classical) quantum number. The conclusion of his analysis was that the model followed an essentially classical path to a quantum analog of the classical space-time singularity. By way of contrast, Demaret[21] has shown that a different quantization scheme leads to a vanishing of the wave function as $a \to 0$ in anisotropic Bianchi type-I and type-IX models with $p < \rho$ perfect fluid matter sources. Clearly an unambiguous investigation of this question must await the long-promised synthesis of quantum theory and gravity.

ACKNOWLEDGMENTS

J. D. B. would like to thank Christ Church, Oxford, for their support. R. M. acknowledges support under NSF grants PHY77-07619 and INT 7916661, and thanks the Institute of Astronomy, Cambridge, U. K. and the Department of Astrophysics, Oxford, U. K. for their hospitality during the summer, 1979.

---

[1]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, London, 1973).

[2]C. Clarke, Commun. Math. Phys. 41, 65 (1975).

[3]G. L. Murphy, Phys. Rev. D 8, 4231 (1973); M. Heller, Z. Klimek, and L. Suszycki, Astrophys. Space Sci. 20, 205 (1973).

[4]W. Rindler, *Essential Relativity*, 2nd ed. (Springer, New York, 1977).

[5]A. Trautmann, Nature (London), Phys. Sci. 242, 7 (1973).

[6]F. J. Tipler, Phys. Rev. D 17, 2521 (1978).

[7]N. S. Baaklini, Phys. Lett. 66A, 357 (1978).

[8]H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento 36, 1016 (1965); Y. B. Zeldovich and A. A. Starobinskii, Zh. Eksp. Teor. Fiz. 61, 2161 (1972) [Sov. Phys.—JETP 34, 1159 (1972)].

[9]L. Parker and S. A. Fulling, Phys. Rev. D 7, 2357 (1973); J. D. Bekenstein, Ann. Phys. (N.Y.) 82, 535 (1974); P. Candelas, Nature (London) 252, 554 (1974).

[10]V. A. Belinskii and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 72, 3 (1977) [Sov. Phys.—JETP 45, 1 (1977)]; J. Stewart and P. Hajicek, Nature (London), Phys. Sci. 244, 96 (1973).

[11]M. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, N. J., 1974).

[12]A. A. Starobinskii, Pis'ma Astron. Zh. 4, 155 (1978) [Sov. Astron. Lett. 4, 82 (1978)].

[13]J. A. Wheeler, Gen. Relativ. Gravit. 8, 713 (1977).

[14]D. N. Schramm and R. V. Wagoner, Annu. Rev. Nucl. Sci. 27, 37 (1977).

[15]A. J. Buras, J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135, 66 (1978); H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974). For a scheme based on unification via a semisimple group with baryon nonconservation see J. C. Pati and A. Salam, Phys. Rev. D 8, 1240 (1973).

[16]J. Ellis, M. Gaillard, and D. V. Nanopoulos, Phys. Lett. 80B, 360 (1979); 80B, 464 (1979); S. Weinberg, Phys. Rev. Lett. 42, 850 (1979).

[17]S. W. Hawking, Mon. Not. R. Astron. Soc. 152, 75 (1971).

[18]J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973); S. W. Hawking, Nature (London) 248, 30 (1974).

[19]K. C. Jacobs, Ann. N. Y. Acad. Sci. 262, 462 (1975).

[20]C. W. Misner, in *IAU Symposium No. 63*, edited by M. S. Longair (Reidel, Dordrecht, 1973). M. A. H. MacCallum, in *Quantum Gravity*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, Oxford, 1975).

[21]J. Demaret, Nature (London) 277, 199 (1979); report, 1979 (unpublished).