Operator formalism of statistical mechanics of gauge theory in covariant gauges

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An operator formalism of statistical mechanics of a gauge theory is presented in covariant gauges. We derive and propose a simple statistical operator $e^{-\beta H-\pi Q_c}$ instead of the usual form $e^{-\beta H}$ for the physical equilibrium system in a gauge theory, where Q_c is the Faddeev-Popov (FP) ghost charge. The diagrammatic expansion rule for the partition function is discussed in this formalism, and Bernard's suggestion of the strange rule that the FP ghost should be assigned a *periodic* temperature Green's function in spite of its Fermi statistics is confirmed rigorously. The gauge-fixing condition independence of the partition function and other physical quantities are also discussed.

I. INTRODUCTION

Recently, several authors have paid attention to gauge theories at finite temperature and/or densities, and discussed some important problems that may arise in such a situation, e.g., the restoration of spontaneously broken symmetries above a critical temperature,[1] the phase transition from nuclear matter to quark matter inside nuclei or a neutron star,[2] etc. The problem of quantization and renormalization of gauge theories under such situations have also been discussed,[3,4] and calculations of thermodynamic potentials to higher orders have been performed.[4,5]

Usually, these arguments are based on Feynman's path-integral formulation of statistical mechanics.[6] The partition function $Z(\beta)$ is represented as a functional integral over all fields denoted generically as ϕ with suitable boundary conditions [periodic for bosons and Faddeev-Popov (FP) ghosts, antiperiodic for fermions] weighted by the exponential of the action[3,7]:

$$Z(\beta)=N(\beta)\int[d\phi]\exp i\left[\int_0^{-i\beta}dx^0\int d^3x\mathcal{L}(\phi)\right], \tag{1.1}$$

where β is the inverse temperature and $N(\beta)$ is a normalization factor. We use units such that $\hbar=c=k$ (Boltzmann's constant) = 1. A curious point in the formula (1.1) resides in the *periodic* boundary condition for Faddeev-Popov (FP) ghosts, which was first stated by Bernard[3] and used by subsequent authors: The functional integration over FP ghosts must be performed over the periodic orbits in the interval $0\leq x_0\leq -i\beta$ despite the fact that they are *fermions*. Consequently, the free propagator of the FP ghosts that appear in the perturbation expansion of (1.1) is of Bose form, $1/(\vec{k}^2+\omega_n^2)$ with $\omega_n\equiv 2n\pi/\beta$ (n = integer). This apparently strange rule originates from the troublesome fact in gauge theories that the well-known expression $\mathrm{Tr}e^{-\beta H}$ is a gauge-dependent quantity and in general no longer the correct expression for the partition function. This gauge dependence is caused by the appearance (in the state vector space and the Hamiltonian H) of unphysical particles such as FP ghosts and the longitudinal and scalar modes of gauge fields in the unbroken theory. Such unphysical particles could never come to equilibrium with a physical heat bath. Therefore, in order to obtain the correct partition function, we should calculate $\mathrm{Tr}e^{-\beta H}$ either in a special gauge in which no unphysical particles appear (e.g., Coulomb gauge, axial gauge), or by restricting the trace operation to a subspace of states which consists of genuine physical particles alone.

Bernard[3] has obtained the above rule for the FP ghosts by starting from the functional integral in the axial gauge and transforming it to the expression in covariant gauges. However, his argument seems rather heuristic and intuitive. In fact, he explains essentially as follows: In view of 't Hooft's trick introducing FP ghosts,[8]

$$\det M=\int[dc][d\bar{c}]\exp(i\bar{c}Mc), \tag{1.2}$$

it turns out that the FP ghosts c and $\bar{c}$ should be periodic, because M is defined in the space of the periodic gauge and matter fields and hence $\det M$ is the determinant in the functional space of periodic functions. However, it would not be so easy to treat directly such a functional determinant in a rigorous way. So it is desirable to verify his conclusion in another way.

In this paper, we present an operator formalism of statistical mechanics of a gauge theory in covariant gauges. We derive and propose a very simple statistical operator for the physical equilibrium system of gauge theory, and give a rigorous proof for Bernard's suggestion. The starting point of our formalism is the expression of the partition function $Z(\beta)$ correct in any gauges:

$$Z(\beta)=\mathrm{Tr}(Pe^{-\beta H}), \tag{1.3}$$

where P is the projection operator onto a subspace $\mathcal{H}_{\text{phys}}$ of states consisting of physical particles alone. With the help of manifestly covariant canonical operator formalism of non-Abelian gauge theories, presented before by Ojima and one of the present authors,[9,10] we transform (1.3) to another simple expression

$$Z(\beta)=\mathrm{Tr}(e^{-\beta H-\pi Q_c}), \tag{1.4}$$

where Q_c is the Faddeev-Popov ghost charge. The expression (1.4) can be regarded as a partition function of a grand-canonical ensemble having pure-imaginary chemical potential $\mu_{FP}=i\pi/\beta$ for FP ghosts, although we are discussing only a canonical ensemble throughout this paper. By virtue of the form (1.4), the usual diagram technique in ordinary (nongauge) theories[11] becomes applicable.

Section II is devoted to the derivation of (1.4) from (1.3). Some results of Ref. 10 necessary for that purpose are briefly explained. In Sec. III, the diagrammatic expansion of the partition function $Z(\beta)$ is discussed. It is explained in Sec. IV that we can consistently formulate the statistical mechanics of a gauge theory simply by adopting the operator $e^{-\beta H-\pi Q_c}$ as the statistical operator for the equilibrium system from the beginning, instead of starting from the complicated operator $Pe^{-\beta H}$. We prove there the gauge-fixing independence of the partition function and other physical quantities in our formalism.

II. DERIVATION OF THE FORM (1.4) FOR THE PARTITION FUNCTION

We consider a multiparticle system described by a (broken or unbroken) gauge theory in thermal equilibrium. The gauge group G may be any compact Lie group, and the Lagrangian density is given by[10,12,13]

$$\mathcal{L}=\mathcal{L}_s+\mathcal{L}_{\text{GF}}+\mathcal{L}_{\text{FP}}, \tag{2.1a}$$

$$\mathcal{L}_s=-\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}+\mathcal{L}_{\text{matter}}, \tag{2.1b}$$

$$\mathcal{L}_{\text{GF}}=-\partial^\mu B\cdot A_\mu+\tfrac{1}{2}\alpha B\cdot B, \tag{2.1c}$$

$$\mathcal{L}_{\text{FP}}=-i\partial^\mu\bar{c}\cdot D_\mu c, \tag{2.1d}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+gA_\mu\times A_\nu, \qquad D_\mu c=(\partial_\mu+gA_\mu\times)c. \tag{2.2}$$

$\mathcal{L}_{\text{matter}}$ in (2.1b) is the gauge-invariant Lagrangian density of matter fields, $\mathcal{L}_{\text{GF}}$ (2.1c) is a covariant gauge-fixing term with Lagrange multiplier field $B(x)$ included,[14] and $\mathcal{L}_{\text{FP}}$ is the corresponding Faddeev-Popov (FP) ghost term. Our gauge-fixing term (2.1c) is equivalent to the more familiar covariant gauge-fixing term $-(1/2\alpha)(\partial_\mu A^\mu)^2$, and the gauge parameter choice $\alpha=0$ ($\alpha=1$) corresponds to the Landau (Feynman) gauge. The multiplier B is necessary in order to ensure the nilpotency of the BRS (Becchi-Rouet-Stora) transformation[15] off the mass shell as well as to treat the Landau gauge (α=0) unifiedly with nonzero α gauges.

As stated in Sec. I, the total state vector space denoted as $\mathcal{V}$ of a gauge theory consists of many unphysical particles as well as physical ones, and the "trace" operation in the calculation of the partition function $Z(\beta)$ should be performed only over the states in the subspace $\mathcal{H}_{\text{phys}}$ consisting solely of physical particles (e.g., the transverse modes of gauge fields and other matter states in the case of unbroken theory):

$$Z(\beta)=\sum_{\alpha,\beta}\langle\alpha|Pe^{-\beta H}|\beta\rangle\,\eta^{-1}{}_{\beta\alpha}\equiv\mathrm{Tr}(Pe^{-\beta H}), \tag{2.3}$$

where P is the projection operator onto the subspace $\mathcal{H}_{\text{phys}}$, H is the (total) Hamiltonian derived from the Lagrangian (2.1), and the metric matrix η is defined by

$$\eta_{\alpha\beta}=\langle\alpha|\beta\rangle, \quad \sum_\beta\eta_{\alpha\beta}\eta^{-1}{}_{\beta\gamma}=\delta_{\alpha\gamma}. \tag{2.4}$$

In (2.3), the summations are performed over *all* independent states $|\alpha\rangle$ and $|\beta\rangle$ in $\mathcal{V}$.

Equation (2.3), as it stands, is difficult to evaluate because of the presence of the projection operator P. However, the canonical operator formalism of non-Abelian gauge theories of Ref. 10 enables us to rewrite the quantity (2.3) in another simple form which permits an easy perturbation calculation. For this purpose, we need two conserved charges of the system (2.1); BRS charge Q_B and FP ghost charge Q_c. Q_B is the generator of the BRS transformation:

$$[iQ_B,A_\mu]=D_\mu c, \quad [iQ_B,B]=0, \quad [iQ_B,\varphi]=ig(T\cdot c)\varphi,$$
$$\{iQ_B,c\}=-\tfrac{1}{2}g(c\times c), \quad \{iQ_B,\bar{c}\}=iB, \tag{2.5}$$

where φ is the generic notation for the matter fields and the T^a's are the generators of the group G in the representation to which φ belongs. The transformation generated by Q_c is a scale transformation of FP ghosts:

$$[Q_c,c]=-ic, \quad [Q_c,\bar{c}]=i\bar{c}, \qquad [Q_c,A_\mu]=[Q_c,B]=[Q_c,\varphi]=0, \tag{2.6}$$

and its explicit form is given by

$$Q_c=i\int d^3x[\bar{c}\cdot(D_0c)-\partial_0\bar{c}\cdot c] = -\int d^3x(\bar{c}\cdot\pi_{\bar{c}}+\pi_c\cdot c), \tag{2.7}$$

where π_c ($=i\partial_0\bar{c}$) and $\pi_{\bar{c}}$ ($=-iD_0c$) are canonical momenta conjugate to c and $\bar{c}$, respectively. The commutation relation between these two charges is given by

$$[Q_c, Q_B] = -iQ_B\,. \tag{2.8}$$

Note that the FP ghost charge Q_c, although Hermitian ($Q_c^\dagger = Q_c$), has pure-imaginary eigenvalues

$$Q_c|n\rangle = -in|n\rangle \tag{2.9}$$

for the state $|n\rangle$ with FP ghost number n. (We assign the FP ghost number +1 and −1 for ghost c and antighost $\bar{c}$, respectively.) That is, FP-ghost-number operator $N_{\rm FP}$ is given by $N_{\rm FP} = iQ_c$. This unfamiliar feature is caused by the fact that the transformation generated by Q_c is a *scale* transformation and not a phase transformation[16] of FP ghosts.

Before proceeding, we must quote another important formula which we need soon below and it becomes the key equation. Let P' be the projection operator onto the subspace of states containing more than one unphysical particle. Then, the following completeness relation clearly holds in the total state vector space $\mathcal{V}$:

$$1 = P + P'\,. \tag{2.10}$$

The nontrivial statement is that the projection operator P' is written in the following particular form in terms of the BRS charge Q_B and some suitable operator R with FP ghost number −1:

$$P' = \{Q_B, R\}\,. \tag{2.11}$$

This was proved in perturbation theory by constructing P' and R explicitly.[9,10] There is also a general argument that proves the relation (2.10) beyond perturbation theory (as long as the physical content of the theory does not depend on the gauge-fixing condition[10,17]).

Now we are ready to rewrite Eq. (2.3). First, notice that in (2.3) we can insert $e^{-\pi Q_c}$ between P and $e^{-\beta H}$ without causing any changes:

$$Z(\beta) = \mathrm{Tr}(Pe^{-\pi Q_c} e^{-\beta H})\,. \tag{2.12}$$

This holds because all the states in the subspace $\mathcal{H}_{\rm phys}$ onto which P projects have vanishing FP ghost number $iQ_c = 0$. Then, by using $P = 1 - \{Q_B, R\}$ owing to (2.10) and (2.11), Eq. (2.12) becomes

$$Z(\beta) = \mathrm{Tr}(e^{-\pi Q_c} e^{-\beta H}) - \mathrm{Tr}(\{Q_B, R\} e^{-\pi Q_c} e^{-\beta H})\,. \tag{2.13}$$

The second term of (2.13) can be shown to vanish:

$$\begin{aligned}\mathrm{Tr}(\{Q_B, R\} e^{-\pi Q_c} e^{-\beta H}) &= \mathrm{Tr}(Re^{-\pi Q_c} e^{-\beta H} Q_B) + \mathrm{Tr}(RQ_B e^{-\pi Q_c} e^{-\beta H}) \\ &= \mathrm{Tr}(R\{e^{-\pi Q_c}, Q_B\} e^{-\beta H}) = 0\,,\end{aligned} \tag{2.14}$$

where use has been made of the cyclic invariance of the trace $[\mathrm{Tr}(AB) = \mathrm{Tr}(BA)]$ and the (anti) commutation relations

$$[Q_B, H] = 0\,, \tag{2.15}$$

$$\{e^{-\pi Q_c}, Q_B\} = 0\,. \tag{2.16}$$

[Equation (2.16) follows from (2.8).] Thus, we finally obtain a very simple formula[18]:

$$Z(\beta) = \mathrm{Tr}(e^{-\pi Q_c} e^{-\beta H}) = \mathrm{Tr}\, e^{-\beta H - \pi Q_c}\,. \tag{2.17}$$

This is the desired result (1.4).

This Eq. (2.17) differs from Eq. (2.3) only in the point that the projection operator P is replaced by $e^{-\pi Q_c}$, but this is a great difference for practical calculations. The projection operator P has a very complicated nonlocal form which is generally difficult to write down explicitly, while the FP-ghost-charge operator Q_c has a very simple form (2.7). As was already noted in the Introduction, Eq. (2.17) can be regarded as a partition function of grand-canonical ensemble with pure-imaginary chemical potential $\mu_{\rm FP} = i\pi/\beta$ for FP ghosts. (Recall that the FP-ghost-*number* operator $N_{\rm FP}$ is not Q_c but iQ_c.) So the usual method of diagrammatic expansion is applicable to the evaluation of it. This will be discussed in the next section.

It is clear from the above derivation of (2.17) that the replacement $P \to e^{-\pi Q_c}$ is allowed always in the trace calculation of any operator A commutative with Q_B; i.e.,

$$\mathrm{Tr}(PA) = \mathrm{Tr}(e^{-\pi Q_c} A) \quad \text{when } [Q_B, A] = 0\,. \tag{2.18}$$

Thus, since any physical observable $\mathcal{O}$ should be gauge-invariant and has vanishing FP ghost number,[19] i.e.,

$$[Q_B, \mathcal{O}] = [Q_c, \mathcal{O}] = 0\,, \tag{2.19}$$

the expectation value of the physical observable defined by

$$\langle \mathcal{O} \rangle \equiv \mathrm{Tr}(Pe^{-\beta H}\mathcal{O}) / \mathrm{Tr}(Pe^{-\beta H}) \tag{2.20}$$

coincides with that given by our "pseudo-grand-canonical" ensemble

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(e^{-\beta H - \pi Q_c}\mathcal{O}) / \mathrm{Tr}(e^{-\beta H - \pi Q_c})\,. \tag{2.21}$$

III. DIAGRAMMATIC EXPANSION OF THE PARTITION FUNCTION

In this section we derive the diagrammatic expansion rule for $Z(\beta)$ [Eq. (2.17)] in the operator formalism following the usual argument of ordinary (nongauge) theory.[11] Although the discussions of the previous section are based on the Heisenberg picture, we turn to the Schrödinger and interaction pictures in this section.

First, we decompose the total Hamiltonian H into the unperturbed and the interaction parts

$$H = H_0 + H_I , \tag{3.1a}$$

$$H_0 = \int d^3x(-i\pi_c \cdot \pi_{\bar{c}} - i\partial_i \bar{c} \cdot \partial_i c + \cdots) , \tag{3.1b}$$

$$H_I = g \int d^3x[\pi_c \cdot (A_0 \times c) - i\partial_i \bar{c} \cdot (A_i \times c) + \cdots] , \tag{3.1c}$$

where the repeated index i means the summation over the space indices 1, 2, and 3, and the dots $(\cdots)$ represent terms not containing FP ghosts. (We are mainly interested in the "behavior" of FP ghosts and often make the same abbreviations in what follows.) In (3.1), operators are those in the Schrödinger picture and their Fourier expansions are given by

$$\begin{aligned} c(\vec{x}) &= \int \frac{d^3k}{[(2\pi)^3 2\omega]^{1/2}}[c(\vec{k})e^{i\vec{k}\cdot\vec{x}} + c^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}] \quad (\omega \equiv |\vec{k}|) , \\ \bar{c}(\vec{x}) &= \int \frac{d^3k}{[(2\pi)^3 2\omega]^{1/2}}[\bar{c}(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \bar{c}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}] , \\ \pi_c(\vec{x}) &= \int \frac{\omega d^3k}{[(2\pi)^3 2\omega]^{1/2}}[\bar{c}(\vec{k})e^{i\vec{k}\cdot\vec{x}} - \bar{c}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}] , \\ \pi_{\bar{c}}(\vec{x}) &= \int \frac{\omega d^3k}{[(2\pi)^3 2\omega]^{1/2}}[-c(\vec{k})e^{i\vec{k}\cdot\vec{x}} + c^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}] , \end{aligned} \tag{3.2}$$

where the creation and annihilation operators $c^\dagger(\vec{k})$, $\bar{c}^\dagger(\vec{k})$, $c(\vec{k})$, and $\bar{c}(\vec{k})$ satisfy the anticommutation relations

$$\begin{aligned} &\{c(\vec{k}), \bar{c}^\dagger(\vec{k}')\} = i\delta^3(\vec{k} - \vec{k}') , \\ &\{c(\vec{k}), c^\dagger(\vec{k}')\} = \{\bar{c}(\vec{k}), \bar{c}^\dagger(\vec{k}')\} = 0 . \end{aligned} \tag{3.3}$$

Now we rewrite (2.15) as

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H_0 - \pi Q_c} e^{\beta H_0} e^{-\beta H})$$

and apply Dyson's formula to $e^{\beta H_0} e^{-\beta H}$,

$$\begin{aligned} e^{\beta H_0} e^{-\beta H} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \cdots \int_0^\beta d\tau_1 \cdots d\tau_n \times T_\tau[\hat{H}_I(\tau_1) \cdots \hat{H}_I(\tau_n)] \\ &\equiv T_\tau\left[\exp\left(-\int_0^\beta d\tau \hat{H}_I(\tau)\right)\right] , \end{aligned} \tag{3.4}$$

where T_τ is the τ-ordering operator[20] and $\hat{H}_I(\tau)$ is defined by

$$\hat{H}_I(\tau) = e^{\tau H_0} H_I e^{-\tau H_0} . \tag{3.5}$$

Then we obtain the perturbation formula for $Z(\beta)$:

$$\begin{aligned} Z(\beta)/Z_0(\beta) &= \mathrm{Tr}\left\{e^{-\beta H_0 - \pi Q_c} T_\tau \left[\exp\left(-\int_0^\beta d\tau \hat{H}_I(\tau)\right)\right]\right\} \Big/ \mathrm{Tr}(e^{-\beta H_0 - \pi Q_c}) \\ &= \left\langle T_\tau \exp\left(-\int_0^\beta d\tau \hat{H}_I(\tau)\right)\right\rangle_0 , \end{aligned} \tag{3.6}$$

where $Z_0(\beta)$ is the partition function of the free theory

$$Z_0(\beta) = \mathrm{Tr}\, e^{-\beta H_0 - \pi Q_c} \tag{3.7}$$

and the thermodynamic average in the unperturbed system $\langle \cdots \rangle_0$ is defined by

$$\langle \cdots \rangle_0 = \mathrm{Tr}(e^{-\beta H_0 - \pi Q_c} \cdots)/Z_0 . \tag{3.8}$$

Now, the Wick-Bloch-De Dominicis theorem[11] is applicable to the thermodynamic average of the T_τ product $\langle T_\tau \cdots \rangle_0$. (This is obvious if we remember that the present perturbation expansion is identical to the usual one for the grand-canonical ensemble having FP-ghost chemical potential $\mu_{FP} = i\pi/\beta$.) We obtain from (3.6) the diagrammatic expansion for $Z(\beta)$ with free propagators $D^\beta{}_{ij}$ given by

$$D^\beta{}_{ij}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2) = -i\langle T_\tau \hat{\phi}_i(\vec{x}_1, \tau_1) \hat{\phi}_j(\vec{x}_2, \tau_2)\rangle_0 , \tag{3.9}$$

where $\hat{\phi}$ denotes the generic fields in the interaction picture with the "τ-development" operator H_0:

$$\begin{aligned} &\hat{\phi}_i(\vec{x}, \tau) \equiv e^{H_0 \tau} \phi_i(\vec{x}) e^{-H_0 \tau} , \\ &\phi_i = A_\mu, c, \bar{c}, \text{ or } \varphi . \end{aligned} \tag{3.10}$$

It is well known in nongauge theories[11] that the temperature Green's function $D^\beta(\vec{x}_1 - \vec{x}_2, \tau_2 - \tau_2)$ for a particle with chemical potential μ satisfies the following boundary condition if we do not include $-\mu N$ (N is the number operator for that particle) to the τ-development operator H_0:

$$D^\beta(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2)|_{\tau_1 = 0} = \pm e^{\beta\mu} D^\beta(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2)|_{\tau_1 = \beta} , \tag{3.11}$$

where the signs + and − correspond to bosons and fermions, respectively. Therefore in the present case, since we are discussing a canonical ensemble with vanishing chemical potentials except for the FP ghosts, the propagators (3.9) of bosons and matter fermions commutative with Q_c satisfy periodic and antiperiodic boundary conditions in the interval $0 \leq \tau_1 \leq \beta$, respectively. On the other hand, the FP ghost fields, for which the "imaginary chemical potential" term $-\mu_{FP}N_{FP} = -(i\pi/\beta)(iQ_c)$ is present, have a *periodic* propagator in spite of their Fermi statistics because of the additional factor $e^{\beta\mu_{FP}} = e^{i\pi} = -1$ in (3.11). Indeed, recapitulating the derivation of (3.11), we can see this explicitly. The FP ghost propagator $D^{\beta}_{c\bar{c}}$, defined as

$$D^{\beta}_{c\bar{c}}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2) = -i\langle T_\tau \hat{c}(\vec{x}_1, \tau_1)\hat{\bar{c}}(\vec{x}_2, \tau_2)\rangle_0 = +i\langle T_\tau \hat{\bar{c}}(\vec{x}_2, \tau_2)\hat{c}(\vec{x}_1, \tau_1)\rangle_0, \tag{3.12}$$

according to Fermi statistics of FP ghosts,[20] where

$$\begin{aligned}\hat{c}(\vec{x}, \tau) &= \int \frac{d^3k}{[(2\pi)^3 2\omega]^{1/2}}[c(\vec{k})e^{-\omega\tau + i\vec{k}\cdot\vec{x}} + c^\dagger(\vec{k})e^{\omega\tau - i\vec{k}\cdot\vec{x}}],\\ \hat{\bar{c}}(\vec{x}, \tau) &= \int \frac{d^3k}{[(2\pi)^3 2\omega]^{1/2}}[\bar{c}(\vec{k})e^{-\omega\tau + i\vec{k}\cdot\vec{x}} + \bar{c}^\dagger(\vec{k})e^{\omega\tau - i\vec{k}\cdot\vec{x}}]\end{aligned} \tag{3.13}$$

satisfies

$$\begin{aligned}D^{\beta}_{c\bar{c}}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2)|_{\tau_1=0} &= +i\langle \hat{\bar{c}}(\vec{x}_2, \tau_2)\hat{c}(\vec{x}_1, 0)\rangle_0\\ &= i\,\mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}\hat{\bar{c}}(\vec{x}_2, \tau_2)\hat{c}(\vec{x}_1, 0)]/Z_0\\ &= i\mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}e^{\beta H_0 + \pi Q_c}\hat{c}(\vec{x}_1, 0)e^{-\beta H_0 - \pi Q_c}\hat{\bar{c}}(\vec{x}_2, \tau_2)]/Z_0\\ &= -i\mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}\hat{c}(\vec{x}_1, \beta)\hat{\bar{c}}(\vec{x}_2, \tau_2)]/Z_0\\ &= D^{\beta}_{c\bar{c}}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2)|_{\tau_1=\beta},\end{aligned} \tag{3.14}$$

where use has been made of the cyclic invariance of the trace and the relation

$$e^{\beta H_0 + \pi Q_c}\hat{c}(\vec{x}_1, 0)e^{-\beta H_0 - \pi Q_c} = -\hat{c}(\vec{x}_1, \beta). \tag{3.15}$$

The minus sign on the rhs of (3.15) is due to the commutation relation (2.6).

The explicit form of $D^{\beta}_{c\bar{c}}$ can also easily be calculated with the help of the following formulas:

$$\begin{aligned}\langle c(\vec{k})\bar{c}^\dagger(\vec{k}')\rangle_0 &= -\langle \bar{c}(\vec{k})c^\dagger(\vec{k}')\rangle_0 = \frac{i}{1 - e^{-\beta\omega}}\delta^3(\vec{k} - \vec{k}'),\\ \langle c^\dagger(\vec{k})\bar{c}(\vec{k}')\rangle_0 &= -\langle \bar{c}^\dagger(\vec{k})c(\vec{k}')\rangle_0 = -\frac{i}{1 - e^{\beta\omega}}\delta^3(\vec{k} - \vec{k}').\end{aligned} \tag{3.16}$$

Here, for example, the first of Eqs. (3.16) is derived as follows:

$$\begin{aligned}\langle c(\vec{k})\bar{c}^\dagger(\vec{k}')\rangle_0 &= \mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}c(\vec{k})\bar{c}^\dagger(\vec{k}')]/Z_0\\ &= \mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}e^{\beta H_0 + \pi Q_c}\bar{c}^\dagger(\vec{k}')e^{-\beta H_0 - \pi Q_c}c(\vec{k})]/Z_0\\ &= -e^{\beta\omega'}\mathrm{Tr}[e^{-\beta H_0 - \pi Q_c}\bar{c}^\dagger(\vec{k}')c(\vec{k})]/Z_0\\ &= -e^{\beta\omega'}\mathrm{Tr}\{e^{-\beta H_0 - \pi Q_c}[-c(\vec{k})\bar{c}^\dagger(\vec{k}') + i\delta^3(\vec{k} - \vec{k}')]\}/Z_0\\ &= e^{\beta\omega'}[\langle c(\vec{k})\bar{c}^\dagger(\mathbf{k}')\rangle_0 - i\delta^3(\vec{k} - \vec{k}')],\end{aligned}$$

where we have used the anticommutation relation (3.3) and the relation

$$e^{\beta H_0 + \pi Q_c}\bar{c}^\dagger(\vec{k})e^{-\beta H_0 - \pi Q_c} = -e^{\beta\omega}\bar{c}^\dagger(\vec{k}). \tag{3.17}$$

Thus, we obtain from (3.12)–(3.14) and (3.16)

$$D^{\beta}_{c\bar{c}}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2) = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\frac{1}{\vec{k}^2 + \omega_n^2}\exp[i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) + i\omega_n(\tau_1 - \tau_2)], \tag{3.18}$$

where

$$\omega_n \equiv 2\pi n/\beta \quad (n = 0, \pm 1, \pm 2, \dots). \tag{3.19}$$

Note that the momentum-space representation of $D^{\beta}_{c\bar{c}}$ is of Bose form, $1/[\vec{k}^2 + (2\pi n/\beta)^2]$, as claimed by Bernard.[3] This result is also easily understood if we recall that the momentum-space representation of the (free) temperature Green's function in nongauge theory for fermions with chemical potential μ is of the form $1/\{\vec{k}^2 + [(2n+1)\pi/\beta - i\mu]^2\}$

and that the FP ghosts have effectively the chemical potential $\mu_{FP}=i\pi/\beta$ in our case.

As for the interaction Hamiltonian $\hat{H}_I(\tau)$ [Eq. (3.1)], it takes the form

$$\hat{H}_I(\tau)=g\int d^3x[\hat{\pi}_c\cdot(\hat{A}_0\times\hat{c})-i\partial_i\hat{\bar{c}}\cdot(\hat{A}_i\times\hat{c})+\cdots]$$

$$=g\int d^3x[-\partial_\tau\hat{\bar{c}}\cdot(\hat{A}_0\times\hat{c})-i\partial_i\hat{\bar{c}}\cdot(\hat{A}_i\times\hat{c})+\cdots], \quad (3.20)$$

where use has been made of the relation

$$\hat{\pi}_c(\vec{x},\tau)\equiv e^{\tau H_0}\pi_c(\vec{x})e^{-\tau H_0}=-\partial_\tau\hat{\bar{c}}(\vec{x},\tau). \quad (3.21)$$

Then, in view of (3.18), we see that $i\omega_n(k_i)$ is multiplied at the A_0-c-$\bar{c}$ (A_i-c-$\bar{c}$) interaction vertex.

Thus we obtain the following Feynman rules which exactly agree with those of Bernard[3]; that is, to make the following replacement in the usual zero-temperature rules[21]:

$$k_0\to i\omega_n,$$

$$\int\frac{d^4k}{(2\pi)^4}\to\frac{i}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}, \quad (3.22)$$

$$(2\pi)^4\delta^4(k_1+k_2+\cdots)\to\frac{1}{i}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\cdots)\times\beta\delta_{\omega_{n_1}+\omega_{n_2}+\cdots},$$

where $\omega_n=2n\pi/\beta$ for bosons and FP ghosts, and $\omega_n=(2n+1)\pi/\beta$ for matter fermions.

For completeness, we comment on an alternative method for evaluating $Z(\beta)$ which is slightly different from but leads to the same Feynman rules as the above method. We can take $H_0+(\pi/\beta)Q_c$ as the τ-development operator instead of H_0 in the previous method, and apply Dyson's formula to $e^{\beta H_0+\pi Q_c}e^{-\beta H-\pi Q_c}$. The equation corresponding to (3.6) is given by

$$\frac{Z(\beta)}{Z_0(\beta)}=\frac{\mathrm{Tr}(e^{-\beta H_0-\pi Q_c}e^{\beta H_0+\pi Q_c}e^{-\beta H-\pi Q_c})}{\mathrm{Tr}(e^{-\beta H_0-\pi Q_c})}$$

$$=\frac{\mathrm{Tr}\left[e^{-\beta H_0-\pi Q_c}T_\tau\exp\left(-\int_0^\beta d\tau\check{H}_I(\tau)\right)\right]}{\mathrm{Tr}(e^{-\beta H_0-\pi Q_c})}$$

$$=\left\langle T_\tau\exp\left(-\int_0^\beta d\tau\check{H}_I(\tau)\right)\right\rangle_0, \quad (3.23)$$

where, in contrast to (3.5), $\check{H}_I(\tau)$ is defined by

$$\check{H}_I(\tau)=\exp\left[\tau\left(H_0+\frac{\pi}{\beta}Q_c\right)\right]H_I\exp\left[-\tau\left(H_0+\frac{\pi}{\beta}Q_c\right)\right]. \quad (3.24)$$

We obtain the diagrammatic expansion for $Z(\beta)/Z_0(\beta)$ in the same way as in the above. In this case, the FP ghost propagator $\tilde{D}^\beta_{c\bar{c}}$ and the interaction Hamiltonian $\check{H}_I(\tau)$ are given by

$$\tilde{D}^\beta_{c\bar{c}}(\vec{x}_1-\vec{x}_2,\tau_1-\tau_2)\equiv-i\langle T_\tau\check{c}(\vec{x}_1,\tau_1)\check{\bar{c}}(\vec{x}_2,\tau_2)\rangle_0$$

$$=e^{-(i\pi/\beta)(\tau_1-\tau_2)}D^\beta_{c\bar{c}}(\vec{x}_1-\vec{x}_2,\tau_1-\tau_2)$$

$$=\frac{1}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\frac{1}{\vec{k}^2+\omega_n^2}\exp[i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+i\tilde{\omega}_n(\tau_1-\tau_2)], \quad (3.25)$$

$$\check{H}_I(\tau)=g\int d^3x[\check{\pi}_c\cdot(\check{A}_0\times\check{c})-i\partial_i\check{\bar{c}}\cdot(\check{A}_i\times\check{c})+\cdots]$$

$$=g\int d^3x[-(\partial_\tau-i\pi/\beta)\check{\bar{c}}\cdot(\check{A}_0\times\check{c})-i\partial_i\check{\bar{c}}\cdot(\check{A}_i\times\check{c})+\cdots], \quad (3.26)$$

where

$$\tilde{\omega}_n\equiv\omega_n-\pi/\beta=(2n-1)\pi/\beta,$$

$$\check{\phi}_i(\vec{x},\tau)\equiv e^{\tau(H_0+(\pi/\beta)Q_c)}\phi_i(\vec{x})e^{-\tau(H_0+(\pi/\beta)Q_c)}$$

for $\phi_i=A_\mu, c, \bar{c}, \pi_c, \pi_{\bar{c}}$, etc.

In (3.25) and (3.26), we have used the following relations:

$$\check{c}(\vec{x},\tau)=e^{-(i\pi/\beta)\tau}\hat{c}(\vec{x},\tau),\quad \check{\bar{c}}(\vec{x},\tau)=e^{(i\pi/\beta)\tau}\hat{\bar{c}}(\vec{x},\tau),$$
$$\check{\pi}_c(\vec{x},\tau)=-(\partial_\tau-i\pi/\beta)\check{\bar{c}}(\vec{x},\tau). \quad (3.27)$$

We should note the peculiar property of $\tilde{D}^\beta_{c\bar{c}}$. Contrary to $D^\beta_{c\bar{c}}$, it satisfies the *antiperiodic* boundary condition [note that $\tilde{\omega}_n=(2n-1)\pi/\beta$ in (3.25)], but its Fourier transform is the same as that of $D^\beta_{c\bar{c}}$, i.e., $1/(\vec{k}^2+\omega_n^2)$. Corresponding to this fact, in the A_0-c-$\bar{c}$ interaction Hamiltonian (3.26), $\bar{c}$ appears in the form $(\partial_\tau-i\pi/\beta)\bar{c}$, so that it is not $\tilde{\omega}_n$ but ω_n that is multiplied at this vertex. Thus, all the Feynman rules in momentum space for this alternative method are completely the same as those for the previous one.

IV. DISCUSSION–GAUGE-FIXING INDEPENDENCE

We have shown in this paper that the physically meaningful statistical operator is not $e^{-\beta H}$ but $Pe^{-\beta H}$, and that the following remarkable replacement is allowed for this statistical operator in the evaluations of the partition function and the physical observable expectation values:

$$Pe^{-\beta H} \to e^{-\beta H - \pi Q_c}. \tag{4.1}$$

In the evaluations of gauge-variant quantities, of course, such a replacement is no longer possible. However, since we are interested only in physical quantities, we can change our standpoint here and *adopt* the operator $e^{-\beta H-\pi Q_c}$ from the beginning as the statistical operator for an equilibrium system in gauge theory, completely forgetting about the complicated operator $Pe^{-\beta H}$. Then, the Green's functions, which are generally gauge-variant, are simply defined by

$$\langle T_\tau \phi_1 \cdots \phi_n \rangle \equiv \frac{\mathrm{Tr}(e^{-\beta H - \pi Q_c} T_\tau[\phi_1 \cdots \phi_n])}{\mathrm{Tr}(e^{-\beta H - \pi Q_c})}, \tag{4.2}$$

where

$$\phi_i = \phi_i(\vec{x}_i, \tau_i) \equiv \begin{cases} e^{\tau_i H}\phi(\vec{x}_i)e^{-\tau_i H}, \\ \text{or} \\ e^{\tau_i [H + (\pi/\beta)Q_c]}\phi_i(\vec{x}_i)e^{-\tau_i[H+(\pi/\beta)Q_c]}. \end{cases} \tag{4.3}$$

Although these Green's functions themselves depend on the gauge-fixing condition imposed and

$$\langle T_\tau \phi_1 \cdots \phi_n \rangle \neq \mathrm{Tr}(Pe^{-\beta H} T_\tau[\phi_1 \cdots \phi_n])/\mathrm{Tr}(Pe^{-\beta H}),$$

the physical quantities calculated by using these Green's functions do not depend on the gauge-fixing choice. This situation on our trick of the replacement (4.1) is very similar to that encountered in the famous Faddeev-Popov trick of gauge fixing.[21] In fact, the factorization of infinite volume of the gauge group in fixing a gauge by the FP trick was possible only in the evaluations of gauge-invariant quantities, although that was sufficient for physical purposes.

The gauge-fixing-condition independence of the partition function and other physical quantities defined through our statistical operator $e^{-\beta H - \pi Q_c}$ can be proved explicitly in the same way as in the usual proof[10,21] of gauge-fixing independence in the zero-temperature field theory. First we notice that all the physically measureable quantities can be brought into the form

$$\mathrm{Tr}(e^{-\beta H - \pi Q_c}\mathcal{O}), \tag{4.4}$$

with some gauge- (BRS-) invariant $\mathcal{O}$,

$$[Q_B, \mathcal{O}] = 0, \tag{4.5}$$

even though they are actually evaluated by utilizing gauge-variant Green's functions (4.2). (The partition function and the expectation value of physical observables really have such a form.) Hence it is sufficient to prove the gauge-fixing independence of the quantity (4.4). Next, following the usual proof in the zero-temperature case, we consider the response of the quantity (4.4) under an infinitesimal change of gauge-fixing condition. Take a gauge-fixing condition with arbitrary gauge-fixing function F [$F = \partial_\mu A^\mu$ for our covariant gauge-fixing choice (2.1c)] in which the gauge-fixing and corresponding FP-ghost Lagrangian density is given by

$$\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} = B\cdot F + \frac{\alpha}{2}B\cdot B - \bar{c}\cdot[Q_B, F], \tag{4.6}$$

and consider an arbitrary infinitesimal change of the gauge-fixing function $F \to F + \Delta F$, which causes the following infinitesimal change on the total Lagrangian density $\mathcal{L}$:

$$\Delta\mathcal{L} = \Delta\mathcal{L}_{\mathrm{GF}} + \Delta\mathcal{L}_{\mathrm{FP}} = B\cdot\Delta F - \bar{c}\cdot[Q_B, \Delta F]. \tag{4.7}$$

Noting that this $\Delta\mathcal{L}$ can be written in a particular form[10]

$$\Delta\mathcal{L} = \{Q_B, \bar{c}\cdot\Delta F\}, \tag{4.8}$$

by using (2.5) we can easily convince ourselves that the change of the Hamiltonian H is also given in a similar form

$$\Delta H = \{Q_B, \Delta G\}, \tag{4.9}$$

with a suitable operator ΔG of FP ghost number -1. For instance, the change of gauge parameter α to $\alpha + \Delta\alpha$ is realized by taking $\Delta F = \Delta\alpha \cdot B/2$, and the change of the parameter γ to $\gamma + \Delta\gamma$ in the R_ξ-like gauge fixing $F = \partial_\mu A^\mu + \gamma\phi$ (ϕ = Goldstone mode of Higgs field) corresponds to $\Delta F = \Delta\gamma\cdot\phi$. In such cases as these examples in which ΔF involves no time derivatives of fields, the Hamiltonian change ΔH is nothing but $-\Delta L = -\int d^3x\, \Delta\mathcal{L}$ and hence $\Delta G = -\int d^3x\, \bar{c}\cdot\Delta F$. Now, the form (4.9) of ΔH immediately leads to the desired gauge-fixing independence of the quantity (4.4). Indeed the change of (4.4) is confirmed to vanish as follows:

$$\begin{aligned}\Delta\,\mathrm{Tr}(e^{-\beta H - \pi Q_c}\mathcal{O}) &= \mathrm{Tr}\left(-\int_0^\beta d\lambda\, e^{-\lambda H}\Delta H e^{\lambda H} e^{-\beta H - \pi Q_c}\mathcal{O}\right) \\ &= -\int_0^\beta d\lambda\, \mathrm{Tr}(e^{-\lambda H}\{Q_B, \Delta G\} e^{\lambda H} e^{-\pi Q_c} e^{-\beta H}\mathcal{O}) \\ &= -\int_0^\beta d\lambda\, \mathrm{Tr}(e^{-\lambda H}\Delta G e^{\lambda H}\{e^{-\pi Q_c}, Q_B\} e^{-\beta H}\mathcal{O}) = 0,\end{aligned} \tag{4.10}$$

where we have used the cyclic invariance of the trace, Eqs. (2.15), (2.16), (4.5), and (4.9), and the formula

$$\Delta e^{-\beta H} = e^{-\beta (H+\Delta H)} - e^{-\beta H} = -\int_0^\beta d\lambda\, e^{-\lambda H} \Delta H e^{\lambda H} . \quad (4.11)$$

We notice that Eq. (4.10) in this proof is quite similar to Eq. (2.14) showing the decoupling of unphysical particles in the justification of the replacement $P \to e^{-\pi Q_c}$ in Sec. II. This fact reflects two aspects of the *gauge invariance* implied by the BRS symmetry, the decoupling of unphysical particles under a fixed gauge condition on the one hand, and the gauge-fixing independence of physical contents on the other.

Finally we add a comment: All the temperature Green's functions in the present formalism of statistical gauge theory smoothly continue to the Euclidian Green's functions in ordinary (zero-temperature) field theory in the limit $\beta \to \infty$. The reason is as follows: First, although we are working in an indefinite-metric formulation of gauge theory in covariant gauges, the spectrum of the Hamiltonian is *bounded below* and the lowest-energy eigenvalue E_0 is realized by the unique vacuum as is known since the Gupta-Bleuler formalism.[22] Hence, only the vacuum state $|0\rangle$ dominates in the trace evaluation of operators including $e^{-\beta H}$ in the limit $\beta \to \infty$; that is, for any operator A not depending explicitly on β,

$$\lim_{\beta \to \infty} \mathrm{Tr}(e^{-\beta H} A) = \lim_{\beta \to \infty} e^{-\beta E_0} \langle 0 | A | 0 \rangle . \quad (4.12)$$

Thus, noticing $e^{-\pi Q_c}|0\rangle = |0\rangle$, we obtain the desired result

$$\begin{aligned} \lim_{\beta \to \infty} \{\mathrm{Tr}(e^{-\beta H - \pi Q_c} T_\tau[\phi_1 \cdots \phi_n]) / \mathrm{Tr}(e^{-\beta H - \pi Q_c})\} &= \lim_{\beta \to \infty} \{e^{-\beta E_0} \langle 0 | T_\tau[\phi_1 \cdots \phi_n] |0\rangle / e^{-\beta E_0} \langle 0 | 0 \rangle\} \\ &= \langle 0 | T_\tau[\phi_1 \cdots \phi_n] |0\rangle / \langle 0 | 0 \rangle . \quad (4.13) \end{aligned}$$

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[13] The meaning of the abbreviated notation is as follows:

$$A \cdot B \equiv A^a B^a, \quad (A \times B)^a \equiv f^{abc} A^b B^c, \quad (gA)^a \equiv g^{ab} A^b,$$

where f^{abc} is the structure constant of the group G, and the coupling-constant matrix g is diagonal and constant for each simple (or Abelian) factor group contained in G.

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