Surface geometry of a black hole in a magnetic field

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The Gaussian curvature of the event horizon of a black hole embedded in a magnetic field is examined. It is shown that a zone of negative curvature develops if the magnetic field parameter exceeds a certain limit. This particular limit depends on the radius of the black hole. Further, as the magnetic field increases, the polar circumference increases while the equatorial circumference decreases. Possible ramifications to black-hole physics are discussed.

I. INTRODUCTION

There is much interest in studying black holes under realistic conditions, such as in the presence of matter or external fields.[1-4] Black holes seldom exist in isolation, and in fact may often be found in close binary-star systems where they actively interact with their partner by sucking in mass flowing out of the Roche lobe. This leads to the formation of an accretion disk. The motion of the hot plasma around the black hole may lead to the formation of a dynamo which in turn generates powerful electromagnetic fields near the black hole. The concentration of fields near the black hole not only affects the hole itself but the dynamics of infalling matter.

One model representing a Schwarzschild black hole within a uniform magnetic field has been found by Ernst.[5] This exact solution of the Einstein-Maxwell field equations is axially symmetric and time independent. Although the solution is not asymptotically flat, it may represent a close approximation to physical reality in the near-zone vicinity of the black hole. Exact solutions for the Reissner-Nordström and the Kerr solution embedded in a magnetic field have also been found, though they are mathematically very cumbersome to handle without linearization.[5,6]

We intend to elucidate the effects of an external magnetic field on the event horizon via an analysis of the Gaussian curvature for varying values of the magnetic field parameter in the metric. A similar analysis for the Kerr-Newman metric has been performed by Smarr,[7] where it has been shown that the polar zones exhibit negative Gaussian curvature when the angular momentum parameter a exceeds a certain value. Further, if any portion of the event horizon posesses negative values of K, the Gaussian curvature, it cannot be globally embedded in flat Euclidean three-space but must be embedded in a pseudo-Euclidean three-space with the metric $ds^2 = dx^2 + dy^2 - dz^2$.

II. GAUSSIAN CURVATURE

We shall be concerned with the exact solution of a Schwarzschild black hole within a magnetic field, which in spherical (r, θ, ϕ, t) coordinates is

$$ds^2 = \lambda^2\left[\frac{dr^2}{1-2m/r} + r^2 d\theta^2 - \left(1 - \frac{2m}{r}\right) dt^2\right] + \lambda^{-2} r^2 \sin^2\theta\, d\phi^2 , \tag{1a}$$

where

$$\lambda = 1 + \tfrac{1}{4} B_0{}^2 r^2 \sin^2\theta \tag{1b}$$

and B_0 is the magnetic field parameter $(B_0 \geq 0)$.[5,8] The Cartan components of the external magnetic field are

$$H_r = \lambda^{-2} B_0 \cos\theta , \tag{2a}$$

$$H_\theta = -\lambda^{-2} B_0 (1 - 2m/r)^{1/2} \sin\theta , \tag{2b}$$

where $\theta = 0$ represents the north pole and $\theta = \pi/2$ the equator of the black hole. For $m \to 0$, this solution reduces to Melvin's magnetic geon.[9]

The event horizon of the black hole is at $r = r_0 = 2m$, where m is the geometric mass (units of length). It is mathematically described by the two-dimensional line element

$$ds^2_{\text{event horizon}} = r_0{}^2 \lambda_0{}^2 d\theta^2 + r_0{}^2 \lambda_0{}^{-2} \sin^2\theta\, d\phi^2 , \tag{3}$$

where λ_0 stands for the evaluation of λ at $r = r_0$. In order to calculate the Gaussian curvature, we define the following notation:

$$E(\theta) = r_0 \lambda_0 , \tag{4a}$$

$$G(\theta) = r_0 \lambda_0{}^{-1} \sin\theta . \tag{4b}$$

Then the Gaussian curvature for a line element of the form

$$ds^2 = E^2(\theta)d\theta^2 + G^2(\theta)d\phi^2 \tag{4c}$$

is[10]

$$K = -\frac{1}{2EG}\frac{d}{d\theta}\left(\frac{1}{EG}\frac{d}{d\theta}G^2\right). \tag{5}$$

Upon performing the necessary differentiations and simplifications, we get for K the following rather complicated formula:

$$K = \frac{1}{r_0^2\lambda_0^4}[\lambda_0^2 + 2\beta^2\lambda_0(4\cos^2\theta - \sin^2\theta) - 12\beta^4\sin^2\theta\cos^2\theta]. \tag{6}$$

Here $\beta = mB_0$ is defined as the dimensionless distortion parameter, analogous to the quantity Smarr uses in his analysis of the Kerr solution.[7] As $B_0 \to 0$, for the Schwarzschild black hole Eq. (6) reduces to $K = 1/r_0^2$, the curvature of a sphere of radius r_0.

As a special case for Eq. (6), notice that the curvature at the equator reduces to

$$K_{\theta=\pi/2} = \frac{1}{r_0^2\lambda_0^3}(1-\beta^2), \tag{7}$$

whereby

$K > 0$ if $\beta < 1$,

$K = 0$ if $\beta = 1$,

$K < 0$ if $\beta > 1$.

For values of the magnetic field parameter greater than $2/r_0$, or for when we have $\beta > 1$, a zone of negative Gaussian curvature develops near the equatorial plane as shown in Fig. 1.

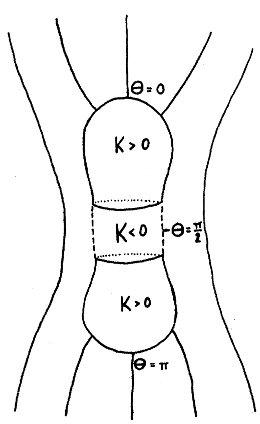

FIG. 1. Graphic illustration showing the influence of the external magnetic field on the event horizon. The region where $K < 0$ cannot be visualized, since the surface cannot be globally embedded in a flat Euclidean three-space.

On the other hand, at the poles $\theta = 0, \pi$, we obtain for the Gaussian curvature

$$K_{\theta=0,\pi} = \frac{1}{r_0^2}(1+8\beta^2), \tag{8}$$

and for all values of B_0, $K > 0$. In fact, as $B_0 \to \infty$, $K \to \infty$ and a cusplike singularity develops in the surface.

III. GEOMETRICAL PROPERTIES OF THE EVENT HORIZON

To get an intuitive idea of what happens to the shape of the event horizon, we can compute the circumferences about the equator and the poles. The relative sizes of these quantities give a measure of the prolateness. The equatorial circumference is given by the definite integral

$$C_e = \int_0^{2\pi} G\,d\phi = \int_0^{2\pi} r_0\lambda_0^{-1}d\phi = 2\pi r_0\lambda_0^{-1}. \tag{9}$$

Therefore as B_0 increases, C_e decreases because of the λ_0 in the denominator. For the polar circumference, we have

$$C_p = \int_0^{2\pi} E\,d\theta = \int_0^{2\pi} r_0\lambda_0 d\theta = 2\pi r_0(1+\tfrac{1}{2}\beta^2). \tag{10}$$

As B_0 increases, the polar circumference increases as B_0^2. In the $B_0 \to 0$ limit, we get the geometry of a simple sphere, $C_e = C_p$.

The surface area of the event horizon is easily evaluated by taking the double integral

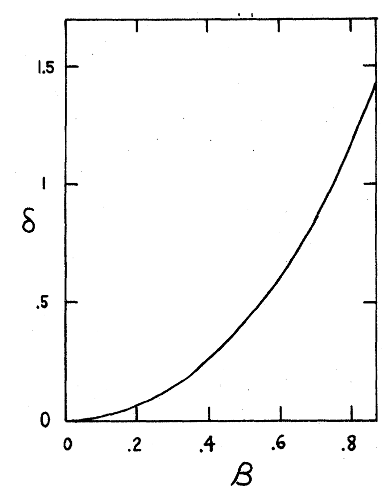

FIG. 2. Plot of the deviation from spherical symmetry. Here the deviation is defined as $\delta = (C_p - C_e)/C_e$ and is the negative of the analogous quantity defined by Smarr. As the magnetic field parameter B_0 increases, the event horizon becomes more prolate.

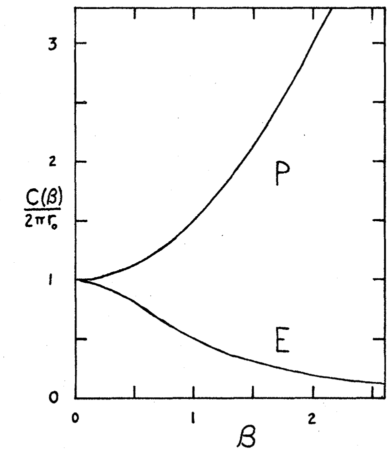

FIG. 3. Behavior of the polar circumference (P) and equatorial circumference (E) of the event horizon as a function of β.

$$A=\int_0^{2\pi}\int_0^{\pi} EG\, d\theta\, d\phi = 4\pi r_0^{\,2}, \tag{11}$$

indicating that this quantity remains invariant. It can also be shown that the surface is topologically a sphere by applying the Gauss-Bonnet theorem. This requires evaluating the double integral[11]

$$\int_0^{2\pi}\int_0^{\pi} KEG\, d\theta\, d\phi = 2\pi\chi, \tag{12}$$

where χ is the Euler characteristic for the compact surface. Calculation yields $\chi=2$, the value for the surface to be homeomorphic to a sphere.

The magnetic field (as well as an electric field, since the two fields are interchangeable through a duality transformation) has the effect of elongating the event horizon into a cigar-shaped object, the long axis being parallel with the magnetic field lines. The magnetic field lines remain perpendicular to all points on the event horizon as seen in Eq. (2b), analogous to electric lines of force about a conductor.

By defining the quantity

$$\delta=\frac{C_p - C_e}{C_e}=\tfrac{1}{2}\beta^2(3+\beta), \tag{13}$$

which is the negative of the analogous parameter defined by Smarr,[7] we can get a picture of the departure of the event horizon from spherical symmetry by plotting δ as a function of the distortion parameter β, shown in Fig. 2. Figure 3 illustrates the effect that the magnetic field has on the polar and equatorial circumferences. Note that C_p increases without bound. Our model allows any value of B_0 and still possesses a nonsingular event horizon, whereas for a realistic Kerr-Newman black hole $a<m$ so that δ remains finite (where we refer to the δ derived by Smarr for a Kerr-Newman black hole).

IV. UNITS

Up to now our study of the effects of the external magnetic field on the event horizon of a static black hole has been rather abstract; all results are specified in terms of the parameters m, the "geometric mass" of the hole, and B_0, the magnetic field strength. Further, the introduction of the distortion parameter β appears to be natural. However, how are these parameters, specifically B_0, related to known quantities (such as Gaussian units) so that one can see how realistic fields affect the hole?

r_0 and hence m have units of length and can be expressed in kilometers. For a black hole that is as massive as the sun, $r_0=2m\simeq 3$ km. The units of B_0 are inverse length, such as km^{-1}. Rather than considering B_0, it is more natural to consider β for physical situations. According to Wald[1]

$$\beta=8.5\times10^{-9}\left(\frac{M}{M_\odot}\right)\left(\frac{B_0}{10^{12}\text{ gauss}}\right), \tag{14}$$

where M is the mass of the black hole in solar masses.

Therefore it can be seen from Eq. (14) that only for very massive black holes and very strong magnetic fields is there an appreciable distortion. Such conditions where $B_0\simeq10^{12}$ Gauss and $M\simeq10^9$ solar masses may exist at the center of some galaxies and perhaps even quasars.

Furthermore, notice that as r_0 decreases, the stronger the field needed to bring about sufficient distortion to make $K<0$ along the equatorial zone. Thus one sees the "soap bubble" analogy of the black hole here in that large black holes are easily distorted, whereas very strong fields are required to influence the small holes. One would expect this to occur on an intuitive level if one adapts the concept of surface tension for a black hole.[12] The larger the soap bubble (black hole), the more easily it is swayed by the wind (external field).

Since our model becomes less realistic for large B_0, it is impossible to conclude that external fields may never destroy black holes from our analysis. More realistic time-dependent solutions must be discovered and investigated. Further, it has been speculated quite often in the literature that very large black holes may reside in the centers of some galaxies. These entities would be expected to have masses on the order of several billion solar masses, or an appreciable fraction of the galactic mass. It has been conjectured that such holes may be surrounded by fairly powerful magnetic fields.[13] Assuming that infalling matter has little effect, one can envision the galactic magnetic field distorting this hole appreciably—a distortion which could affect the distribution of matter falling into the hole. Further observations concerning the magnitude of the galactic fields are necessary to determine if this can indeed be possible.

V. CONCLUSION

Our analysis indicates that black holes are influenced by their environment. They tend to remain spherical if they are small, though they may in general be fairly complex in shape. For spinning black holes in a magnetic field,[6] there are two forces acting to distort the event horizon—the rotation and the external field. According to Smarr's[7] work, the rotation causes $C_e > C_p$, whereas our analysis shows that for magnetic distortion $C_p > C_e$. Exactly what happens when the two effects combine remains unanswered. Further, the influence of external masses via their gravitational fields also has an effect on the geometry of the event horizon.[14]

From Eq. (11), it is evident that the surface area of the event horizon remains constant for any value of B_0. This can be understood by invoking Hawking's theorem, which states that the total surface area of black holes can never decrease (in the classical domain).[15] This must be considered as the most serious flaw in our model, since for a time-dependent situation the surface area of a black hole in an external field will increase with time because the black hole absorbs the energy density of the field. One can envision a black hole moving into a region of space with a large magnetic field: The hole is initially spherical, though once it enters the magnetic field it becomes slightly distorted (the magnitude of distortion dependent on B_0). The black hole desires to remain in its "lowest energy" state, and does so by drawing in the field in its neighborhood—in the process, the hole becomes larger.

In our model the external field is capable of distorting the geometry from spherical symmetry, though the hole does not absorb the field around it; rather, a state of equilibrium occurs.

We have attempted to shed some light on the question of how external fields interact with black holes and have used the simplest model available. Although it is our desire to be as realistic as possible in our research, the addition of physical parameters tends to make the equations not only more difficult to handle, but may disguise interesting phenomena. Perhaps by obtaining a better understanding of existing metrics, we may ultimately be able to find those solutions which more adequately portray reality.

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