New ghost-free gravity Lagrangians with propagating torsion

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A new class of R^2 type of actions without ghosts and tachyons is found in which both the vierbein field $e_\mu{}^a$ and the spin connection $\omega_\mu{}^{ab}$ are independent and propagating fields. A complete set of spin-projection operators for $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ is constructed. The particle content of these actions and other R^2 theories in the literature is given. The relation of this work to the interesting work of Neville is discussed.

I. INTRODUCTION

Einstein's field theory of gravity seems perfect as a classical theory. All recent experimental data have confirmed it and have ruled out several possible alternatives. However, as a quantum theory it is less satisfactory since, as soon as one couples to matter, the first-order quantum corrections lead to a divergent S matrix. These divergences are nonrenormalizable. Owing to the dimension of a length of the gravitational coupling constant, the one-loop divergences have a different *functional* form from the classical action and they cannot be absorbed into the classical action by constant rescaling of the physical parameters. Pure gravity has an S matrix which is finite at the one-loop level and *might* be finite at the two-loop level; this is still not known. One possible alternative is supergravity whose S matrix is not only one-loop but also two-loop finite and *might* be three-loop finite. (In two-loop ordinary gravity and three-loop supergravity dangerous counterterms do exist. The crucial question is whether their coefficients are zero.) There are at present no arguments in favor of these possibilities, but in our opinion they should be investigated.

In this article we consider a class of gravitational theories which contain, in addition to the scalar curvature, extra R^2-type terms. If in these $R+R^2$ theories the only independent field is the metric (or, equivalently, the vierbein field), then such theories are in general higher-derivative theories. Deser and van Nieuwenhuizen[1] argued that such theories would be renormalizable but contain ghosts due to the p^{-4} propagators. Stelle[2] proved rigorously the renormalizability of the most general $R+R^2$ type of action without torsion

$$\mathcal{L} = \alpha R(\omega(e)) + \beta R^2(\omega(e)) + \gamma R_{\mu\nu}{}^2(\omega(e)), \qquad (1)$$

where $\omega(e)$ denotes the spin connection $\omega_\mu{}^{mn}$ expressed in terms of the vierbein field $e_\mu{}^m$. *Actually, not every higher-derivative theory necessarily has ghosts or tachyons.* For example, taking $\gamma=0$ in (1), and using that the kinetic terms of $\sqrt{-g}R$ and $\sqrt{-g}R^2$ are proportional[2] to $\Box(P^2-2P_s^0)$ and $\Box^2 P_s^0$, one finds easily from the orthonormality of the spin projection operators[4] that the propagator is given by the usual spin-2 propagator of the graviton plus the propagator of a spin-0 physical mode with real mass.

The corresponding results for supergravity were derived in Ref. 3. As shown there, physical states and ghost states arrange themselves in whole multiplets of global supersymmetry, and the unitarity properties remain as in the purely bosonic case.

The class of theories we will consider below are not-higher-derivative theories, in which the spin connection is an independent field which is propagating. Thus, there are separate physical modes for $e_\mu{}^m$ and $\omega_\mu{}^{mn}$ and one cannot eliminate $\omega_\mu{}^{mn}$ as an independent field by solving its field equation as in ordinary Einstein gravity (the Palatini formalism). In two recent articles,[5,6] Neville has considered such theories. However, as becomes most clear in his second article, he requires power-counting renormalizability as proposed in Ref. 1, and therefore needs p^{-4} propagators for the graviton and p^{-2} propagators for the spin connection *in all spin sectors*. Whereas in the lower-spin sectors (as for instance in the spin-0 sector in the example above) one can sometimes obtain that the possible ghosts due to $(p^2+m_1{}^2)^{-1}(p^2+m_2{}^2)^{-1}=(m_2{}^2-m_1{}^2)^{-1}[(p^2+m_1{}^2)^{-1}-(p^2+m_2{}^2)^{-1}]$ are compensated by lower-spin terms in the higher-spin projection operators, no such compensation can take place in the spin-2 sector. Thus one expects that any power-counting renormalizable theory will have ghosts in the spin-2 sector and, indeed, that is what Neville finds.

We will restrict our attention from the beginning to theories which are unitary (free from ghosts and tachyons). In particular, we will consider the nine-parameter action which is the most general action for $e_\mu{}^m$ and $\omega_\mu{}^{mn}$ such that there are at most second derivatives. We could have added higher-derivative terms since they sometimes do not destroy unitarity; this we have not done. We could also have taken, instead of $(e_\mu{}^m, \omega_\mu{}^{mn})$ the vierbein contorsion tensor as basis. This would have led to a different nine-parameter action, as the reader may easily verify. We have preferred to work in the $(e_\mu{}^m, \omega_\mu{}^{mn})$ basis, since this is the basis which seems to us preferable for geometrical reasons. The problem we consider in this article is which particular choices of the nine coefficients lead to theories without ghosts or tachyons.

Our nine-parameter action is generally coordinate and local Lorentz invariant. Actually, we believe that this is not a restriction for the following reason. An analysis of pure gravity in terms of symmetric as well as antisymmetric tensor fields of several years ago[4] showed that if one required unitarity, the solutions were invariant under the linearized spacetime symmetries. Also, it was found that the *antisymmetric tensor fields did decouple from the asymmetric ones*. Thus in that case, general coordinate invariance and local Lorentz invariance followed from unitarity, and we believe that the same result would be found in the present case.

Neville found in his first article one ghost- and tachyon-free action which describes, in addition to the graviton, a spin-0^- "torsion," but this solution does not have a p^{-4} propagator in the spin-2 sector, in agreement with our comments above. We will present several new solutions, depending on up to five free parameters. We have completely analyzed the case where there are no extra gauge invariances beyond the spacetime symmetries and no mass degeneracies. It might be interesting to extend our work to the case where the masses of some of the propagating modes become equal and the cases where the action has more gauge invariances in addition to the spacetime symmetries. There is a large class of local symmetries to choose from, which makes a complete investigation difficult. Although our $R+R^2$ theories are one-loop renormalizable as far as power counting is concerned, it is not clear whether they are truly renormalizable at the one-loop level (to decide this, one would need to study Ward identities to show that the one-loop $\Delta\mathfrak{L}$ can be absorbed into $\mathfrak{L}$) and they are nonrenormalizable from the two-loop level on.

A systematic study of the absence or presence of ghosts and tachyons in gravitational theories is virtually impossible if one does not use spin projection operators. They were constructed for the vierbein fields in Ref. 4 and used in Refs. 2 and 4–6. The extension to the spin projection operators for the $(e_\mu{}^m, \omega_\mu{}^{mn})$ space was made by Neville in the gauge $e_{m\mu} = e_{\mu m}$. We consider it one of the main technical achievements of this article that a complete orthonormal set of spin projection operators for the fields $(e_\mu{}^m, \omega_\mu{}^{mn})$ has been constructed valid in any gauge. To make sure that it does not contain algebraic errors, we checked it by using Veltman's algebraic manipulation program SCHOONSCHIP.[7] Having obtained this machinery, we were then able to investigate the existing R^2 theories in the literature on their ghosts and tachyon content. The results of this examination appear in a list in the conclusions.

The article is organized as follows. In Sec. II we discuss the nine-parameter action. In Sec. III we invert its kinetic part by means of the spin projection operators. In Sec. IV we require that this propagator have positive residues at real masses. Interestingly enough, one can completely solve this problem because all algebraic equations factorize when there are no other gauge invariances other than the usual spacetime symmetries and when all masses are different. It is here that we find our new solutions. In the conclusions we discuss how far these results justify the hope for a unitary and power-counting renormalizable theory. Also in the conclusions is the list of the ghost content of existing $R+R^2$ theories in the literature. In the Appendix the spin projection operators are constructed.

II. THE LAGRANGIAN

We wish to investigate the particle content of the following nine-parameter Lagrangian:

$$\begin{aligned}\mathfrak{L} = &-\lambda R + \tfrac{1}{12}(4a+b+3\lambda)(R_{abc})^2 + \tfrac{1}{6}(-2a+b-3\lambda)R_{abc} \\ &\times R^{bca} + \tfrac{1}{3}(-a+2c-3\lambda)(R_{ab}{}^b)^2 + \tfrac{1}{6}(2p+q)(R_{abcd})^2 \\ &+ \tfrac{1}{6}(2p+q-6r)R_{abcd}R^{cdab} + \tfrac{2}{3}(p-q)R_{abcd}R^{acbd} \\ &+ (s+t)(R_{ab})^2 + (s-t)R_{ab}R^{ba} + e_\mu{}^a\Sigma^\mu{}_a + \omega_{\mu ab}\tau^{\mu ab}\,, \end{aligned} \tag{2}$$

where $\lambda, a, b, c, p, q, r, s, t$ are nine arbitrary constants. $S_\mu{}^A = (\Sigma_\mu{}^a, \tau_\mu{}^{ab})$ are the vierbein field and spin-connection sources, respectively, and $\Sigma_{\mu a}$ is not symmetric. The curvatures of the Poincaré group $R_{\mu\nu}{}^A = (R_{\mu\nu}{}^a, R_{\mu\nu}{}^{ab})$ are

$$R_{\mu\nu a} = \partial_\nu e_{\mu a} + \omega_{\mu ab} e_\nu{}^b - (\mu \leftrightarrow \nu)\,, \tag{3}$$

$$R_{\mu\nu ab} = \partial_\nu \omega_{\mu ab} + \omega_{\mu a}{}^c \omega_{\nu c b} - (\mu \leftrightarrow \nu)\,, \tag{4}$$

where $h_\mu{}^A = (e_\mu{}^a, \omega_\mu{}^{ab})$ are the gauge fields associated with the translation and Lorentz rotation part of the Poincaré group, respectively. The $R_{\mu\nu a}$ are usually called torsions. (We are aware of other geometrical interpretations of the vierbein field.) The contracted curvatures are

$$R_{ab} = R_{\mu\nu c d} e^{\nu d} e^\mu{}_a \eta^c{}_b\,, \quad R = R_{\mu\nu ab} e^{\mu a} e^{\nu b}\,. \tag{5}$$

The choice of coefficients in the Lagrangian is purely for convenience; it will simplify the parameter combinations appearing in the propagator in Sec. III.

On the other hand, we choose $(e_\mu{}^a, \omega_\mu{}^{ab})$ as the basis of our dynamical field variables. This is a natural choice from the gauge-theoretic point of view. Our Lagrangian is the most general metric Lagrangian which is quadratic or less in the derivatives on the gauge fields $(e_\mu{}^a, \omega_\mu{}^{ab})$, without cosmological constant. The R^2 term need not be included due to the use of the Gauss-Bonnet theorem

$$\int d^4x\, e(R_{\mu\nu ab}R^{\mu\nu ab} - 4R_{\mu a}R^{\mu a} + R^2) = 0 \qquad (6)$$

for spactimes topologically equivalent to flat space. A term $R(\omega(e))$, on the other hand, can be related to those terms already in the Lagrangian by using the relation

$$\int d^4x[eR(\omega) - eR(\omega(e))] = \int d^4x(\tfrac{1}{4}R_{abc}R^{abc} - \tfrac{1}{2}R_{abc}R^{bca} - R_{ab}{}^bR^{ac}{}_c), \qquad (7)$$

where $\omega(e) = -\omega_{\mu ab}(e)$ and

$$\omega_{\mu ab}(e) = \tfrac{1}{2}[e^\nu{}_a(e_{\nu b,\mu} - e_{\mu b,\nu}) + e^\lambda{}_a e^\sigma{}_b e_{\lambda c,\sigma} e_\mu{}^c - (a \leftrightarrow b)] . \qquad (8)$$

Terms of the type $R^2(\omega(e))$, $(D_\mu R_{\nu\lambda a})^2$, or $(D_\mu R_{\nu\lambda a})R^{\mu\nu\lambda a}$ are not included because they contain derivatives of the gauge fields higher than two.

The action is invariant under general coordinate and local Lorentz transformations

$$\delta h_\mu{}^A = (D_\mu \epsilon)^A + \epsilon^\lambda R_{\mu\lambda}{}^A , \qquad (9)$$

where $\epsilon^A = (\epsilon^a, \epsilon^{ab})$ are the gauge parameters of the Poincaré group and $(D_\mu\epsilon)^A = D_\mu\epsilon^A + f_{BC}{}^A h_\mu{}^B\epsilon^C$, the sum being over the structure constants of the Poincaré group. The gauge fields of $h_\mu{}^A$ were defined under (4). Owing to these invariances there are two source constraints which can be obtained from (2) and (9) by reading off the coefficients of the independent parameters ϵ^a and ϵ^{ab}:

$$D^\mu(\omega)(e\tau_{\mu ab}) + \Sigma_{[ab]} = 0 , \qquad (10a)$$

$$D^\mu(\omega)(e\Sigma_{\mu a}) = e^\lambda{}_a R_{\mu\lambda}{}^A S^\mu{}_A , \qquad (10b)$$

where $D_\mu(\omega) = \partial_\mu - \frac{1}{2}\omega_{\mu ab}\sigma^{ab}$, $S^\mu{}_A = (\tau^\mu{}_{ab}, \Sigma^\mu{}_a)$, and $\Sigma_{[ab]} = \frac{1}{2}(\Sigma_{ab} - \Sigma_{ba})$. We will obtain these results, at the linearized level, in Sec. III.

III. CALCULATION OF THE PROPAGATOR

We first linearize the Lagrangian by making the weak-field approximation

$$e_{\mu a} = \eta_{\mu a} + h_{\mu a} , \quad |h_{\mu a}| \ll 1 . \qquad (11)$$

From here on we can drop the distinction between the Greek and Latin indices. The Minkowski metric in our convention is $\eta_{ab} = (+---)$. It is convenient to define

$$\varphi_{ab} \equiv h_{(ab)} = \tfrac{1}{2}(h_{ab} + h_{ba}) , \quad \chi_{ab} \equiv h_{[ab]} = \tfrac{1}{2}(h_{ab} - h_{ba}) . \qquad (12)$$

Keeping only the terms bilinear in (ω, φ, χ), the linearized quadratic Lagrangian $\mathcal{L}^{(2)}$ can be written as

$$\mathcal{L}^{(2)} = \sum_{\alpha,\beta} \tfrac{1}{2}\phi_\alpha \mathcal{O}_{\alpha\beta}\phi_\beta , \qquad (13)$$

where $\phi_\beta = (\omega_{cab}, \varphi_{ab}, \chi_{ab})$ and $\mathcal{O}_{\alpha\beta}$ is the wave operator which contains Kronecker δ's and at most two derivatives. The saturated propagator can be written as

$$\Pi = -\sum_{\psi_\alpha, \phi_\beta} S_\alpha \mathcal{O}^{-1}{}_{\alpha\beta} S_\beta , \qquad (14)$$

where $S_\alpha = (\tau_{cab}, \Sigma_{(ab)}, \Sigma_{[ab]})$. In order to invert $\mathcal{O}_{\alpha\beta}$ we use the fact that the fields (ω_{cab}, h_{ab}) are reducible under the Lorentz group. Therefore, they can be decomposed into subspaces of dimension $(2J+1)$ with definite spin-parity J^P. Since under the three-dimensional rotation group a four-vector decomposes into a vector part $(J^P = 1^-)$ and a scalar part $(J^P = 0^+)$, ω_{cab} and h_{ab} have the decomposition

$$\omega_{cab} = (1^- \oplus 0^+) \otimes (1^- \oplus 0^+) \otimes (1^- \oplus 0^+) = (2^+) \oplus 2(1^-) \oplus (0^+) \oplus (2^-) \oplus 2(1^+) \oplus (0^-) , \qquad (15a)$$

$$h_{ab} = (1^- \oplus 0^+) \otimes (1^- \oplus 0^+) = (2^+) \oplus 2(1^-) \oplus 2(0^+) \oplus (1^+) , \qquad (15b)$$

where the numbers in front of the parentheses denote the multiplicity of those states. Therefore, considered as a 40×40 matrix in the field space, $\mathcal{O}_{\alpha\beta}$ decomposes into two 1×1 subblocks $(J^P = 2^-, 0^-)$, one 2×2 subblock $(J^P = 2^+)$, two 3×3 subblocks $(J^P = 1^+, 0^+)$, and one 4×4 subblock $(J^P = 1^-)$. In order to achieve this decomposition and subsequently the inversion of $\mathcal{O}_{\alpha\beta}$, we use the spin-projection-operator formalism. In the Appendix we have derived the spin projection operators $P^{\psi\phi}_{ij}(J^P)_{\alpha\beta}$ which connect the fields $\phi_\beta = (\omega_{cab}, \varphi_{ab}, \chi_{ab})$ with the same spin-parity J^P. The indices i,j label the projection operators with multiplicity greater than one. These projection operators are orthonormal and complete in the following sense:

$$P^{\psi\Sigma}_{ik}(I^Q)_{\alpha\sigma} P^{\Pi\phi}_{lj}(J^P)_{\tau\beta} = \delta^{\Sigma\Pi}\delta^{IJ}\delta^{PQ}\delta_{kl}\delta_{\sigma\tau} P^{\psi\phi}_{ij}(J^P)_{\alpha\beta} , \qquad (16a)$$

$$\sum_{\phi,i,J^P} P^{\phi\phi}_{ii}(J^P)_{\alpha\beta} = 1_{\alpha\beta} . \qquad (16b)$$

Since these 40 operators are crucial for our analysis, we have checked their orthonormality on the computer, using the algebraic manipulation program SCHOONSCHIP. We can now expand the wave operator $\mathcal{O}_{\alpha\beta}$ in terms of these projection operators and write $\mathcal{L}^{(2)}$ as

$$\mathcal{L}^{(2)} = \sum_{\psi_\alpha,\phi_\beta,J^P,i,j} a_{ij}^{\psi\phi}(J^P)\psi_\alpha P_{ij}^{\psi\phi}(J^P)_{\alpha\beta}\phi_\beta\,, \qquad (17)$$

where $a_{ij}^{\psi\phi}(J^P)$ denote the coefficient matrices. Once they are calculated, the saturated propagator is (for nonsingular coefficient matrices)

$$\Pi = -\sum_{\psi_\alpha,\phi_\beta,J^P,i,j} a^{-1}{}_{ij}{}^{\psi\phi}(J^P)S_\alpha P_{ij}^{\psi\phi}(J^P)_{\alpha\beta}S_\beta \qquad (18)$$

due to the properties of the projection operators given in (16). For singular coefficient matrices we apply the method of Ref. 8, which we shall briefly discuss below. The calculation of the coefficient matrices is straightforward but tedious. We only present the result here. If the action in (2) is decomposed as in (17), the coefficient matrices $a_{ij}^{\psi\phi}(J^P)$ are given by

$$a^{\omega\omega}(2^-) = pk^2 + \tfrac{1}{2}a\,, \qquad (19a)$$

$$a^{\omega\omega}(0^-) = qk^2 + b\,, \qquad (19b)$$

$$a_{ij}^{\psi\phi}(1^+) = \begin{array}{c} \\ \left[\begin{array}{ccc} (2r+t)k^2+\frac{1}{6}(a+4b) & \frac{-1}{3\sqrt{2}}(2b-a) & \frac{i}{3}(\frac{1}{2}k^2)^{1/2}(2b-a) \\ \frac{-1}{3\sqrt{2}}(2b-a) & \frac{1}{3}(a+b) & \frac{i}{3}(k^2)^{1/2}(a+b) \\ -\frac{i}{3}(\frac{1}{2}k^2)^{1/2}(2b-a) & -\frac{i}{3}(k^2)^{1/2}(a+b) & \frac{1}{3}(a+b)k^2 \end{array}\right] \end{array} \begin{array}{c} \omega \\ \omega \\ \chi \end{array}, \quad \text{columns: } \omega\ \ \omega\ \ \chi \qquad (19c)$$

$$a_{ij}^{\psi\phi}(1^-) = \left[\begin{array}{cccc} (p+s+t)k^2+\frac{1}{6}(a+4c) & \frac{1}{3\sqrt{2}}(2c-a) & \frac{i}{3}(k^2)^{1/2}(2c-a) & \frac{i}{3}(k^2)^{1/2}(2c-a) \\ \frac{1}{3\sqrt{2}}(2c-a) & \frac{1}{3}(a+c) & \frac{i}{3}(k^2)^{1/2}(a+c) & \frac{i}{3}(k^2)^{1/2}(a+c) \\ -\frac{i}{3}(k^2)^{1/2}(2c-a) & -\frac{i}{3}(k^2)^{1/2}(a+c) & \frac{1}{3}(a+c)k^2 & \frac{1}{3}(a+c)k^2 \\ -\frac{i}{3}(k^2)^{1/2}(2c-a) & -\frac{i}{3}(k^2)^{1/2}(a+c) & \frac{1}{3}(a+c)k^2 & \frac{1}{3}(a+c)k^2 \end{array}\right] \begin{array}{c} \omega \\ \omega \\ \varphi \\ \chi \end{array}, \quad \text{columns: } \omega\ \ \omega\ \ \varphi\ \ \chi \qquad (19d)$$

$$a_{ij}^{\psi\phi}(1^+) = \left[\begin{array}{cc} (2p-2r+s)k^2+\frac{1}{2}a & ia(\frac{1}{2}k^2)^{1/2} \\ -ia(\frac{1}{2}k^2)^{1/2} & (a+\lambda)k^2 \end{array}\right] \begin{array}{c} \omega \\ \varphi \end{array}, \quad \text{columns: } \omega\ \ \varphi \qquad (19e)$$

$$a_{ij}^{\psi\phi}(0^+) = \left[\begin{array}{ccc} (2p-2r+4s)k^2+c & ic(2k^2)^{1/2} & 0 \\ -ic(2k^2)^{1/2} & 2(c-\lambda)k^2 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} \omega \\ \varphi \\ \varphi \end{array}. \quad \text{columns: } \omega\ \ \varphi\ \ \varphi \qquad (19f)$$

Our convention is that $k^2 = m^2$ with real m for physical particles, so that the square roots are positive on-shell. The square roots of (k^2) cancel similar square roots in the spin projection operators whereas the factors i cancel similar factors due to an odd number of derivatives in the action. The ϕ sector and χ sector have separately been treated before,[4] but the ω sector has not separately been investigated before. It is obvious that the $J^P = (1^\mp, 0^+)$ sectors are degenerate and of rank two. Therefore, $J^P=(1^+,0^+)$ sectors have one, $J^P=(1^-)$ sector has two left and two right null eigenvectors. Suppose

$$\sum_j a_{ij}(J^P)V_j^{(R,n)}(J^P)=0\,, \tag{20a}$$

$$\sum_j V_j^{(L,n)}(J^P)a_{ji}(J^P)=0\,, \tag{20b}$$

where $V_j^{(R,n)}(J^P)$ is the nth right null eigenvector of the J^P sector and we have suppressed the $(\psi\phi)$ label of the coefficient matrix. Equation (20) implies the existence of the following gauge invariances and source constraints:

$$\delta\phi_\alpha=\sum_{J^P,j,\beta,n}V_j^{(R,n)}(J^P)P_{jk}(J^P)_{\alpha\beta}f_\beta(J^P)\quad\text{for any }k\,, \tag{21a}$$

$$\sum_{j,\beta}V_j^{(L,n)}(J^P)P_{kj}(J^P)_{\alpha\beta}S_\beta=0\quad\text{for any }k\,, \tag{21b}$$

where we have suppressed the $(\psi\phi)$ label of the projection operators and $f_\beta(J^P)$ is arbitrary. Applying these general observations to the $J^P=(1^\mp,0^+)$ sectors, one finds the invariances

$$\delta h_{ab}=\epsilon_{ab}+\epsilon_{b,a}\,, \tag{22a}$$

$$\delta\omega_{cab}=-\epsilon_{ab,c}\,, \tag{22b}$$

where ϵ_{ab} and ϵ_a are functions of f parameters in (21a). Clearly, these results are the linearized general coordinate and local Lorentz transformations; in particular, there is no term with ϵ_b in $\delta\omega$, since this term is nonlinear, being of the form $\epsilon_b\delta\omega+(\delta\epsilon_b)\omega$. (Note that in h_{ab} the first index is curved.) One also finds the source constraints

$$\partial_a\Sigma_{ab}=0\,, \tag{23a}$$

$$\partial_c\tau_{cab}+\Sigma_{[ab]}=0\,. \tag{23b}$$

Equations (22) and (23) are in agreement with the linearized form of the previously written gauge invariances and source constraints given in (9) and (10). Another check on (19) is that for $\lambda=(8\pi G)^{-1}$, $a=b=c=p=q=r=s=t=0$; it reduces to $(8\pi G)^{-1}k^2[P_{22}^{\varphi\varphi}(2^+)-2P_{22}^{\varphi\varphi}(0^+)]$ which is the familiar Einstein Lagrangian [see (7)]. Another way of deriving (23) is to consider Einstein-Cartan theory. Going from first- to second-order formalism,

$$\tau_{\mu ab}\omega_{\text{lin}}^{\mu ab}=e_{\mu a}(\partial_\lambda\tau^\lambda{}_{a\mu}-\partial_\lambda\tau_{\mu a}{}^\lambda-\partial_\lambda\tau_{a\mu}{}^\lambda) \tag{24}$$

and adding $e_{\mu a}\Sigma_{\mu a}$, the requirement that the antisymmetric part of the total $e_{\mu a}$ source vanishes and its symmetric part is conserved leads to (23).

We now proceed to invert these matrices to obtain the propagator. In Ref. 8 it is shown that the correct propagator is obtained by taking the inverse of any largest submatrix with nonzero determinant and saturating it with sources. It is gauge invariant. If such a matrix is $m\times m$, the deletion of $(n-m)$ rows and columns merely amounts to choosing $(n-m)$ gauge conditions, and due to the $(n-m)$ source constraints the saturated propagator will not depend on the choice of gauge. We choose to invert the upper-left 2×2 matrices which we denote by $b_{ij}^{\psi\phi}(J^P)$ in the $J^P=(1^\mp)$ sectors. Together with the inversion of the other sectors one obtains the following result for the inverted coefficient matrices:

$$a^{-1\omega\omega}(2^-)=(pk^2+\tfrac{1}{2}a)^{-1}\,, \tag{25a}$$

$$a^{-1\omega\omega}(0^-)=(qk^2+b)^{-1}\,, \tag{25b}$$

$$b_{ij}^{-1\psi\phi}(1^+)=\begin{pmatrix}\dfrac{(a+b)}{3} & \dfrac{-(2b-a)}{3\sqrt{2}}\\[2ex] \dfrac{-(2b-a)}{3\sqrt{2}} & (2r+t)k^2+\dfrac{(a+4b)}{6}\end{pmatrix}[\tfrac{1}{3}(a+b)(2r+t)k^2+\tfrac{1}{2}ab]^{-1}\,, \tag{25c}$$

$$b_{ij}^{-1\psi\phi}(1^-)=\begin{pmatrix}\dfrac{(a+c)}{3} & \dfrac{(a-2c)}{3\sqrt{2}}\\[2ex] \dfrac{(a-2c)}{3\sqrt{2}} & (p+s+t)k^2+\dfrac{(a+4c)}{6}\end{pmatrix}[\tfrac{1}{3}(a+c)(p+s+t)k^2+\tfrac{1}{2}ac]^{-1}\,, \tag{25d}$$

$$a_{ij}^{-1\psi\phi}(2^+)=\begin{pmatrix}(a+\lambda)k^2 & -ia(\tfrac{1}{2}k^2)^{1/2}\\[2ex] ia(\tfrac{1}{2}k^2)^{1/2} & (2p-2r+s)k^2+\dfrac{a}{2}\end{pmatrix}k^{-2}[(a+\lambda)(2p-2r+s)k^2+\tfrac{1}{2}a\lambda]^{-1}\,, \tag{25e}$$

$$b_{ij}^{-1\psi\phi}(0^+)=\begin{pmatrix}(c-\lambda) & -ic(\frac{1}{2}k^2)^{1/2}\\ ic(\frac{1}{2}k^2)^{1/2} & (p-r+2s)k^2+\dfrac{c}{2}\end{pmatrix}k^{-2}[2(c-\lambda)(p-r+2s)k^2-c\lambda]^{-1}. \tag{25f}$$

It should be noted that these results only hold if the only gauge invariances are those assumed from the beginning, namely, general coordinate and local Lorentz invariance. However, if there are additional gauge invariances one can start from this point and take limits that certain parameters vanish. If one then takes smaller nonsingular submatrices and adds the extra new source constraints, then one can analyze these cases as well. For example, if one takes the limit that all parameters except λ tend to zero, one clearly regains Einstein's theory with propagator $[P_{22}^{\varphi\varphi}(2^+)-\frac{1}{2}P_{22}^{\varphi\varphi}(0^+)]k^{-2}$. In that limit several new gauge invariances appear, corresponding to the fact that $\omega_\mu{}^{ab}$ has disappeared from the action.

The result in (25) is equivalent to a similar result by Neville, except that he uses the tetrad-contortion basis. Also, he chooses the gauge $e_{\mu a}-e_{a\mu}=0$ whereas we did not work in a specific gauge (which forced us to find more spin projection operators).

IV. CONSTRAINTS ON THE PARAMETERS FOR GHOST- AND TACHYON-FREE GRAVITY LAGRANGIANS

We start our search for Lagrangians without ghosts and tachyons by considering the most general case wherein there are no extra gauge invariances in addition to general coordinate and local Lorentz invariances with all mass parameters nonzero (except the graviton, of course). The saturated propagator $\Pi(J^P)$ which was calculated in Sec. III can be written in the form [for $J^P=(1^\mp,2^+,0^+)$]

$$\Pi(J^P)=\sum_{\psi_\alpha,\phi_\beta,i,j}A_{ij}^{\psi\phi}(J^P)S_\alpha P_{ij}^{\psi\phi}(J^P)_{\alpha\beta}S_\beta(-k^2+m^2)^{-1}, \tag{26}$$

where $A_{ij}^{\psi\phi}(J^P)$ is a 2×2 matrix which is degenerate at the pole. The constraints for not having ghosts and tachyons follow from the requirement of having real mass and positive-definite residue matrix at the pole. Since A has one nonzero eigenvalue at the pole which is equal to the trace of A, and since there are 3 (2) θ's in the odd- (even-) parity operators, while $\eta_{ab}=(1,-1,-1,-1)$, we have

$$m^2>0, \tag{27a}$$

$$(-1)^P\,\mathrm{tr}A|_{k^2=m^2}>0. \tag{27b}$$

Before applying formulas (27) we decompose the $J^P=2^+$ and 0^+ sectors into partial fractions and extract the part with $1/\lambda$ which corresponds to the graviton

$$\begin{aligned}a_{ij}^{-1\psi\phi}(2^+)=&-\lambda^{-1}k^{-2}\begin{pmatrix}-2k^2 & i(2k^2)^{1/2}\\ -i(2k^2)^{1/2} & -1\end{pmatrix}+a^{-1}k^{-2}\begin{pmatrix}2k^2 & 0\\ 0 & 0\end{pmatrix}\\ &+(a\lambda)^{-1}[k^2+\tfrac{1}{2}a\lambda(a+\lambda)^{-1}(2p-2r+s)^{-1}]^{-1}\begin{pmatrix}-2(a+\lambda)k^2 & ia(2k^2)^{1/2}\\ -ia(2k^2)^{1/2} & -a^2(a+\lambda)^{-1}\end{pmatrix},\end{aligned} \tag{28a}$$

$$\begin{aligned}b_{ij}^{-1\psi\phi}(0^+)=&-\tfrac{1}{2}\lambda^{-1}k^{-2}\begin{pmatrix}2k^2 & -i(2k^2)^{1/2}\\ i(2k^2)^{1/2} & 1\end{pmatrix}+c^{-1}k^{-2}\begin{pmatrix}k^2 & 0\\ 0 & 0\end{pmatrix}\\ &+(c\lambda)^{-1}[-2k^2+c\lambda(c-\lambda)^{-1}(p-r+2s)^{-1}]^{-1}\begin{pmatrix}-2(c-\lambda)k^2 & ic(2k^2)^{1/2}\\ -ic(2k^2)^{1/2} & -c^2(c-\lambda)^{-1}\end{pmatrix}.\end{aligned} \tag{28b}$$

We now apply the formulas (27) to all the massive sectors and obtain the following conditions on the parameters of the Lagrangian for not having ghosts and tachyons at the massive poles (assuming that these are not degenerate):

$$\begin{aligned}&(2^-):\ p<0,\quad a>0,\quad (1^-):\ (p+s+t)<0,\quad ac(a+c)>0,\\ &(0^-):\ q<0,\quad b>0,\quad (2^+):\ (2p-2r+s)>0,\quad a\lambda(a+\lambda)<0,\\ &(1^+):\ (2r+t)>0,\quad ab(a+b)<0,\\ &(0^+):\ (p-r+2s)>0,\quad c\lambda(c-\lambda)>0,\end{aligned} \tag{29}$$

where for the $J^P = 2^+$ and 0^+ sectors we have written

$$p_{11}^{\omega\omega}(2^+) = p_{22}^{\varphi\varphi}(2^+)\frac{\overleftrightarrow{\partial\partial}}{k^2}\,,\ \text{etc.}$$

and integrated partially. It is immediately clear that for general values of the parameters the $J^P = (2^-, 1^+, 0^-)$ sector has a tachyon (assuming that it does not have a ghost) and $J^P = (2^\mp, 1^\mp)$ sector has a ghost (assuming that it does not have a tachyon).

We now investigate the massless sectors which require somewhat more care. Since the building blocks for the spin projection operators are

$$\theta_{ab} = \eta_{ab} - k_a k_b k^{-2}\,, \tag{30a}$$

$$\omega_{ab} = k_a k_b k^{-2}\,, \tag{30b}$$

$$\not{k}_a = k_a (k^2)^{-1/2}\,, \tag{30c}$$

one now finds k^{-2n} singularities coming from all projection operators. One must show that the k^{2n} singularities cancel for $n > 1$ and that the residue for $n = 1$ is positive. First of all, k^{-6} singularities obviously vanish since expressions with three derviatives acting on τ_{cab} automatically vanish. As for the k^{-4} singularities, several J^P sectors contribute, however, by using the source constraints it is straightforward to show that their sum vanishes. On the other hand, the k^{-2} singularities from all the J^P sectors except the contribution from the part which is originated by the $-\lambda R(\omega)$ term in the Lagrangian [the first 2×2 matrices in (28a) and (28b)] sum up to zero. The total result for the massless sector, then, can be written as

$$\Pi(2^+, 0^+)_{\text{massless}} = -\lambda^{-1}k^{-2}(\tau_{ca}\Sigma_{ca})\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix} \times [P_{22}^{\psi\psi}(2^+,\eta) - \tfrac{1}{2}P_{22}^{\varphi\varphi}(0^+,\eta)]_{ca}^{c'a'}\begin{pmatrix}\tau_{c'a'}\\ \Sigma_{c'a'}\end{pmatrix}, \tag{31}$$

where we have defined $\tau_{ca} \equiv \partial_b \tau_{cab}$ and

$$P_{ij}^{\psi\phi}(J^P,\eta)_{\alpha\beta} \equiv P_{ij}^{\psi\phi}(J^P)_{\alpha\beta}|_{\partial\to 0}\,. \tag{32}$$

The expression in the square brackets in (31) is the familiar combination which arises in ordinary gravity. Therefore, it follows that the no-ghost condition for the massless sector is just

$$\lambda > 0\,. \tag{33}$$

For general parameters, this contradicts the results of (29) for the 2^+ and 2^- sectors, since if a and λ are positive, so should $a\lambda(a+\lambda)$ be positive.

We now turn to the question of eliminating these ghosts and tachyons from the theory by choosing the parameters appropriately. It turns out that there are five solutions if we consider the case of no extra gauge invariances and nondegenerate mass parameters. We present and examine in some detail those solutions in the following:

(i) $\lambda = (8\pi G)^{-1}$, $a = -b = -c = -\lambda$, $p = q = r = s = t = 0$. The Lagrangian corresponding to this choice of parameters is in the Einstein-Cartan Lagrangian

$$\mathcal{L} = -\lambda R(\omega)\,. \tag{34}$$

Using (25) and the relevant projection operator given in the Appendix for the complete saturated propagator of this theory we find in addition to (31) only two nonzero nonpropagating terms

$$\Pi(2^-) = 2\lambda^{-1}\,\tau\cdot P^{\omega\omega}(2^-,\eta)\cdot\tau\,, \tag{35a}$$

$$\Pi(0^-) = -\lambda^{-1}\,\tau\cdot P^{\omega\omega}(0^-,\eta)\cdot\tau\,. \tag{35b}$$

There is only one propagating mode in this theory since the 2×2 matrix in (31) is degenerate. Actually, this theory has been studied extensively in the literature, and it has been shown to be equivalent to Einstein's theory if one does not couple to fermionic matter.

(ii) $\lambda = (8\pi G)^{-1}$, $a = -b = -c = -\lambda$, $p = r = s = t = 0$, $q < 0$. The corresponding Lagrangian is the one found by Neville,[5]

$$\mathcal{L}_N = -\lambda R + \tfrac{1}{6}\,qR_{abcd}(R^{abcd} + R^{cdab} - 4R^{acbd})\,. \tag{36}$$

This is indeed the only possibility if the torsion-squared terms are not included in the theory. The particle content of this theory is the graviton [Eq. (31)] and a massive spin-0^- excitation (torsion):

$$\Pi(0^-) = -(qk^2 + \lambda)^{-1}\tau\cdot P^{\omega\omega}(0^-, m^2)\cdot\tau\,, \tag{37}$$

where

$$P_{ij}^{\psi\phi}(J^P,\ m^2)_{\alpha\beta} \equiv P_{ij}^{\psi\phi}(J^P)|_{k^2 = m^2}\,. \tag{38}$$

The remaining part of the complete saturated propagator is (35a). [This theory reduces to Einstein's as the torsion is set equal to zero, but this has no dynamical meaning since $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$ is not a field equation.]

(iii) $\lambda = (8\pi G)^{-1}$, $a = -b$, $a > 0$, $c(a+c) > 0$, $p = r$, $s = q = 0$, $p > 0$, $p + t > 0$. This is one of our new Lagrangians without ghosts and tachyons. To verify that these parameters satisfy (29), one must look at (25) to see when certain modes are nonpropagating. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{I}} = &-\lambda R + \tfrac{1}{4}(a+\lambda)R_{abc}\,(R^{abc} - 2R^{bca})\\ &+ \tfrac{1}{3}(2c - a - 3\lambda)(R_{ab}{}^{b})^2\\ &+ \tfrac{1}{3}r R_{abcd}\,(R^{abcd} - 2R^{cdab} + 2R^{acbd})\\ &+ tR_{ab}\,(R^{ab} - R^{ba})\,.\end{aligned} \tag{39}$$

The graviton propagator is, again, as given in (31). The rest of the complete saturated propagator is

$$\Pi(2^-) = -(rk^2+\tfrac{1}{2}a)^{-1}\,\tau\cdot P^{\omega\omega}(2^-,m^2)\cdot\tau\,, \qquad (40a)$$

$$\Pi(0^-) = -b^{-1}\,\tau\cdot P^{\omega\omega}(0^-,\eta)\cdot\tau\,, \qquad (40b)$$

$$\Pi(1^+) = 2a^{-2}(2r+t)\Sigma\cdot P_{33}^{XX}(1^+,\eta)\cdot\Sigma\,, \qquad (40c)$$

$$\Pi(1^-) = [\tfrac{1}{3}(a+c)(r+t)k^2+\tfrac{1}{2}ac]^{-1} \times\sum_{i,j}(\tau\;\tau)\cdot\begin{pmatrix}-\dfrac{(a+c)}{3} & +\dfrac{(2c-a)}{3\sqrt2}\\ +\dfrac{2c-a}{3\sqrt2} & -\dfrac{(2c-a)^2}{6(a+c)}\end{pmatrix} \times P_{ij}^{\omega\omega}(1^-,m^2)\cdot\begin{pmatrix}\tau\\ \tau\end{pmatrix}, \qquad (40d)$$

where we have used (25), the source constraints, and the explicit forms of the projection operators which are given in the Appendix. As is apparent from (40), this Lagrangian propagates, in addition to the graviton, a massive spin-2^- and massive spin-1^- tordions. It has five parameters and reduces to Einstein's theory as torsion vanishes. An interesting special case of $\mathcal{L}_{\rm I}$ is for $2c-a=3\lambda$, $t=0$, $r\equiv 3\beta<0$, and $a+\lambda=4\alpha>0$:

$$\mathcal{L} = -\lambda R+\alpha R_{abc}(R^{abc}-2R^{bca}) +\beta R_{abcd}(R^{abcd}-2R^{cdab}+2R^{acbd})\,. \qquad (41)$$

This action reduces to the Einstein action if one sets the (propagating) torsion equal to zero.

(iv) $\lambda=(8\pi G)^{-1}$, $a=-c=-\lambda$, $b>\lambda>0$, $p=0$, $s=-t$, $q<0$, $(2r+t)>0$. This is our second ghost- and tachyon-free Lagrangian

$$\mathcal{L}_{\rm II} = -\lambda R+\tfrac{1}{12}(b-\lambda)R_{abc}(R^{abc}+2R^{bca}) +\tfrac{1}{6}qR_{abcd}(R^{abcd}+R^{cdab}-4R^{acbd})-rR_{abcd}R^{cdab} -2t\,R_{ab}R^{ba}\,. \qquad (42)$$

From (25), by using the explicit form of the projection operators, the source constraints, and summing the kinematical singularities one obtains the following complete saturated propagator:

$$\Pi(2^-) = -2a^{-1}\tau\cdot P^{\omega\omega}(2^-,\eta)\cdot\tau\,, \qquad (42a)$$

$$\Pi(0^-) = -(qk^2+b)^{-1}\tau\cdot P^{\omega\omega}(0^-,\eta)\cdot\tau\,, \qquad (42b)$$

$$\Pi(1^+) = [\tfrac{1}{3}(b-\lambda)(2r+t)k^2-\tfrac{1}{2}b\lambda]^{-1}\sum_{i,j}(\tau\;\;\tau)\cdot\begin{pmatrix}\dfrac{-(b-\lambda)}{3} & \dfrac{-(2b+\lambda)}{3\sqrt2}\\ \dfrac{-(2b+\lambda)}{3\sqrt2} & \dfrac{-(2b+\lambda)^2}{6(b-\lambda)}\end{pmatrix}P_{ij}^{\omega\omega}(1^+,m^2)\cdot\begin{pmatrix}\tau\\ \tau\end{pmatrix}, \qquad (42c)$$

$$\Pi(2^+,0^+)_{\rm massless} = -\lambda^{-1}k^{-2}(\tau\;\Sigma)\cdot\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix}[P_{22}^{\varphi\varphi}(2^+,\eta)-\tfrac{1}{2}P_{22}^{\varphi\varphi}(0^+,\eta)]\cdot\begin{pmatrix}\tau\\ \Sigma\end{pmatrix} +\Sigma\cdot[-2(2r+t)\lambda^{-2}P_{22}^{\varphi\varphi}(2^+,\eta)-(r+2t)\lambda^{-2}P_{22}^{\varphi\varphi}(0^+,\eta)]\cdot\Sigma\,, \qquad (42d)$$

where we have suppressed the indices of τ_{cab}, Σ_{ab}, and $\tau_{ca}\equiv\partial_b\tau_{cab}$. Note that if we eliminate the massive spin-0^- excitation by setting q equal to zero and also choose $r=0$, we obtain a particularly simple special case of $\mathcal{L}_{\rm II}$:

$$\mathcal{L} = -\lambda R+\alpha R_{abc}(R^{abc}+2R^{bca})+\beta R_{ab}R^{ba}\,, \qquad (43)$$

where we have relabeled the original parameters so that $\alpha>0$ and $\beta<0$. This theory propagates a massive spin-1^+ with mass $m^2=-\lambda(\lambda+12\alpha)/4\alpha\beta$ in addition to the graviton. It is also interesting to note that if we set the torsion equal to zero by hand, $\mathcal{L}_{\rm II}$ or $\mathcal{L}$ given in (43) reduce to Lagrangians $(\alpha R+\beta R_{\mu\nu}{}^2)$ which contain ghosts. Of course, as we will see later, the opposite of this is possible, namely, of one replaces $\omega(e)$ by ω in a ghost-free gravity Lagrangian, the resulting Lagrangian usually contains ghosts.

(v) $\lambda=(8\pi G)^{-1}$, $a\neq 0$, $b>0$, $c>\lambda$, $p=0$, $s=-t=2r$, $q<0$, $r>0$. This is the third new ghost- and tachyon-free Lagrangian that we find. From (2) it is

$$\mathcal{L}_{\rm III} = -\lambda R+\tfrac{1}{12}(4a+b-\lambda)(R_{abc})^2+\tfrac{1}{6}(-2a+b-3\lambda)R_{abc}R^{bca} +\tfrac{1}{3}(-a+2c-3\lambda)(R_{ab}{}^{b})^2+\tfrac{1}{6}qR_{abcd}(R^{abcd}+R^{cdab}-4R^{acbd})+r(-R_{abcd}R^{cdab}+4R_{ab}R^{ba})\,. \qquad (44)$$

The terms with λ and q constitute the solution under (ii).

This theory, in addition to the graviton [see (31)], propagates a massive spin-0^+ and a massive spin-0^- excitation given by

$$\Pi(0^-) = -(qk^2+b)^{-1}\tau\cdot P^{\omega\omega}(0^-,m^2)\cdot\tau\ ,$$

$$\Pi(0^+) = [-k^2+\tfrac{1}{6}c\lambda r^{-1}(c-\lambda)^{-1}]^{-1} \tag{45a}$$

$$\times\sum_{ij}(\tau\ \ \Sigma)\cdot(c\lambda)^{-1}\begin{pmatrix}2(c-\lambda) & c\\ c & c^2(c-\lambda)^{-1}\end{pmatrix} \times P^{\varphi\varphi}_{ij}(0^+,m^2)\cdot\begin{pmatrix}\tau\\ \Sigma\end{pmatrix}. \tag{45b}$$

For the nonpropagating part of the complete saturated propagator one finds

$$\Pi(1^-) = -\tfrac{1}{6}(ac)^{-1}(a+c)\tau\cdot P^{\omega\omega}_{11}(1^-,\eta)\cdot\tau\ . \tag{46}$$

In obtaining these results we have used (25) and (28), the explicit forms of the projection operators together with the source constraints and we have summed all the kinematical singularities. The torsion-free limit of this theory turns out to be $\mathcal{L} = -\lambda R(\omega(e)) + r\,R^2(\omega(e))$ with $r>0$ which is known to be ghost-free. Elimination of the spin-0^- state, together with a suitable choice of parameters, as before, leads to a very simple Lagrangian:

$$\mathcal{L} = -\lambda R + \alpha(R_{abc})^2 + \beta(-R_{abcd}R^{abcd} + 4R_{ab}R^{ba})\ , \tag{47}$$

where $\alpha>0$, $\beta>0$. The $[(R_{abc})^2 - 2(R_{ab}{}^{b})^2]$ combination alone for the torsion-squared part of the Lagrangian, which has been used by Hehl *et al.*[9] in their Lagrangian, is not allowed here as a special case since it requires $c=\lambda$, which disobeys the constraint $c>\lambda$ we have imposed on c.

We now briefly discuss two special cases which we have avoided so far. One of these cases is $\lambda=0$, but without extra gauge invariances. This is an interesting case to look at because this Lagrangian is very much Yang-Mills type. In that case, the $J^P=(2^-,\ 1^\mp,\ 0^-)$ sectors do not change but the $J^P=(2^+,0^+)$ sectors take the form

$$\Pi(2^+) = -(2p-2r+s)^{-1}k^{-4}(\tau\ \ \Sigma)\cdot\begin{pmatrix}2 & 1\\ 1 & \tfrac{1}{2}\end{pmatrix} \times P^{\varphi\varphi}_{22}(2^+,\eta)\cdot\begin{pmatrix}\tau\\ \Sigma\end{pmatrix}$$
$$-a^{-1}k^{-2}\Sigma\cdot P^{\varphi\varphi}_{22}(2^+,\eta)\cdot\Sigma$$
$$+2a^{-1}k^{-2}\tau_{ca}P^{\varphi\varphi}_{22}(2^+,\eta)^{ca'}_{ca}\,\tau_{c'a'}$$
$$-4a^{-2}k^{-2}(2p-2r+s)(\tau_b)^2\ , \tag{48a}$$

$$\Pi(0^+) = -\tfrac{1}{2}(p-r+2s)^{-1}k^{-4}(\tau\ \ \Sigma)\cdot\begin{pmatrix}2 & 1\\ 1 & \tfrac{1}{2}\end{pmatrix} \times P^{\varphi\varphi}_{22}(0^+,\eta)\cdot\begin{pmatrix}\tau\\ \Sigma\end{pmatrix}$$
$$-\tfrac{1}{2}c^{-1}k^{-2}\Sigma\circ P^{\varphi\varphi}_{22}(0^+,\eta)\cdot\Sigma$$
$$+c^{-1}k^{-2}\tau_{ca}P^{\varphi\varphi}_{22}(0^+,\eta)^{c'a'}_{ca}\,\tau_{c'a'}\ , \tag{48b}$$

where $\tau_{ca}\equiv\partial_b\tau_{cab}$ as before and $\tau_b\equiv\partial_c\partial_a\tau_{cab}$. In obtaining this result, we have again summed all the massless pole contributions from all the sectors and we have used the source constraints together with the explicit form of the spin projection operators provided in the Appendix. We observe that no extra gauge invariance exists which leads to a source constraint such that τ_{ca} and Σ_{ca} become related. Therefore, the only way to eliminate the k^{-4} singularities is to choose $p=r$ and $s=0$. The Lagrangian, then, has a new gauge invariance and that implies a new source constraint, as it is manifest from (19). This is an example to the second special case, namely, when the Lagrangian has extra gauge invariances. Unfortunately, explicit calculation shows that the $J^P=(2^+,0^+)$ sector is completely eliminated by this extra gauge invariance, when (as usual) we sum up all the massless pole contributions to the saturated propagator. The absence of the graviton in this model makes it uninteresting. One can of course design gauge invariances in several ways by just inspecting (25), and hope to eliminate some of the troublesome constraints on the Lagrangian parameters [see (29)] by using the new source constraints. However, due to the proliferation of the massless states, the analysis of the residue matrices and the spurious spin-projection-operator singularities becomes very complicated. We will be content, at the present, with the results obtained here, but needless to mention, a complete analysis of all possible gauge invariances and/or massless states (in addition to the graviton) is of considerable interest. It is not unmanageable if the present machinery is used.

V. CONCLUSIONS

In this article we have considered the most general action depending on the vierbein field $e_\mu{}^m$ and the spin connection $\omega_\mu{}^{mn}$, with the following properties:

(I) It is invariant under spacetime symmetries (general coordinate and local Lorentz invariance).

(II) It contains at most two derivatives. Requiring unitarity, i.e., first-order poles with positive residues at real masses of the propagators sandwiched between sources which satisfy only those source constraints which follow from the field equations, we completely solved the case that

(A) there are no other gauge invariances besides those in I,

(B) all masses are nondegenerate, i.e., no two masses become accidentally equal by a particular choice of parameters. In particular, the only massless mode is the graviton.

We did not require power-counting renormalizability as done in Refs. 5 and 6 since this leads in our opinion to ghosts and tachyons. Under these conditions the following actions were found:

(i) Einstein-Cartan theory with action $R(\omega, e)$ and Einstein theory with action $R(e)$;

(ii) a solution previously found by Neville,[5]

$$\mathcal{L}_N = -\lambda R(e,\omega) + \alpha R_{abcd}(R^{abcd} + R^{cdab} - 4R^{acbd}) , \tag{49}$$

where $\lambda > 0$ and $\alpha < 0$ are free parameters and all R^2 terms here and below depend on (e, ω) but not on $(e, \omega(e))$, as required by II;

(iii) a new five-parameter ghost- and tachyon-free action given by

$$\begin{aligned}\mathcal{L}_{\rm I} = &-\lambda R(e,\omega) + \alpha R_{abc}(R^{abc} - 2R^{bca}) + \beta (R_{ab}{}^{b})^2 \\ &+ \gamma R_{abcd}(R^{abcd} - 2R^{cdab} + 2R^{acbd}) \\ &+ \delta R_{ab}(R^{ab} - R^{ba}) ,\end{aligned} \tag{50}$$

where $\lambda > 0$, $\alpha > \lambda/4$, $(\beta + 4\alpha)(2\lambda + 3\beta + 4\alpha) > 0$, $\gamma < 0$, and $3\gamma + \delta < 0$;

(iv) a new five-parameter ghost- and tachyon-free action given by

$$\begin{aligned}\mathcal{L}_{\rm II} = &-\lambda R(e,\omega) + \alpha R_{abc}(R^{abc} + 2R^{bca}) \\ &+ \beta R_{abcd}(R^{abcd} + R^{cdab} - 4R^{acbd}) + \gamma R_{abcd}R^{cdab} \\ &+ \delta R_{ab}R^{ba} ,\end{aligned} \tag{51}$$

where $\lambda > 0$, $\alpha > 0$, $\beta < 0$, $4\gamma + \delta > 0$;

(v) a new six-parameter ghost- and tachyon-free action given by

$$\begin{aligned}\mathcal{L}_{\rm III} = &-\lambda R(e,\omega) + \alpha (R_{abc})^2 + \beta R_{abc}R^{bca} + \gamma (R_{ab}{}^{b})^2 \\ &+ \delta R_{abcd}(R^{abcd} + R^{cdab} - 4R^{acbd}) \\ &+ \epsilon (R_{abcd}R^{cdab} + 4R_{ab}R^{ba}) ,\end{aligned} \tag{52}$$

where $2\alpha - \beta \neq \lambda$, $4(\alpha + \beta) + \lambda > 0$, $2\alpha - \beta + 3\gamma > 0$, $\delta < 0$, $\epsilon > 0$.

The particle content is, in addition to a massless spin-2^+ state (the graviton),

(a) no other state,

(b) a massive spin-0^- state,

(c) a massive spin-2^- and massive spin-1^- states,

(d) a massive spin-1^+ and massive spin-0^- states, and

(e) a massive spin-0^+ and massive spin-0^- states.

These results were obtained by noting that at the massive poles the traces of the separate spin-block coefficients always factorize into squares with positive coefficients, so that the conditions of absence of tachyons and ghosts could be obtained easily and in simple form. For the singularities at $k^2 = 0$, partially due to the singularities in the spin projection operators, a remarkable and probably general result was found: The sum of all singularities at $k^2 = 0$ cancels except for a term proportional to the graviton propagator of Einstein theory divided by the coefficient of $R(\omega(e))$

$$\begin{aligned}(\text{residue at } k^2 = 0) = &-\lambda^{-1}(\partial_c \tau_{abc}, \Sigma_{ab}) \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \\ &\times (\eta_{aa'}\eta_{bb'} + \eta_{ab'}\eta_{bd'} - \eta_{ab}\eta_{a'b'}) \\ &\times \begin{pmatrix} \partial_{c'}\tau_{a'b'c'} \\ \Sigma_{a'b'} \end{pmatrix} .\end{aligned} \tag{53}$$

The absence of ghosts for $\lambda > 0$ was already discussed in Ref. 4. Hence, the whole $k^2 = 0$ sector is acceptable provided $\lambda > 0$.

It is interesting in itself that these solutions given above exist at all, but future work should proceed in three ways:

(i) The cases not covered under (A) and (B) should be investigated. This will need our spin projection operators. Quite possibly, new solutions with extra gauge invariances (and hence new source constraints) might lead to surprises. These new gauge invariances can be easily found, since they arise each time a spin block has a new zero eigenvalue. Note that criterion I on the (e, ω) basis is not satisfied on another basis such as vierbein field plus contortion.

(ii) One should analyze whether our new solutions have better ultraviolet properties than Einstein's theory. At first sight the only hopeful direction is that of finiteness of the S matrix, since our unitary solutions are not in general power-counting renormalizable.

(iii) An even wider class of gravitational theories is possible if one relaxes the constraint $D_\lambda g_{\rho\sigma} = 0$ and examines the physics of propagating torsion.

Various models of $(R + R)^2$-type theories have been proposed in the past based on various geometrical and other points of view. With our set of projection operators we were able to analyze them to decide whether they have ghosts and/or tachyons. We now give this list:

(a) Hehl, Ne'eman, Nitsch, and von der Heyde's[9] action

$$\mathcal{L} = \alpha[(R_{abc})^2 - 2(R_{ab}{}^{b})^2] + \beta (R_{abcd})^2 , \tag{54}$$

where $\alpha > 0$ and $\beta < 0$, has a 1^+ ghost and double poles.

(b) Yang's[10] action

$$\mathcal{L} = \alpha (R_{abcd})^2 \tag{55}$$

has dipole ghosts at $k^2 = 0$.

(c) Fairchild's[11] action

$$\mathcal{L} = -\lambda R + \alpha (R_{abcd})^2 , \tag{56}$$

where $\lambda > 0$ and $\alpha < 0$, has a 2^- tachyon.

(d) Mansouri and Chang's[12] action

$$\mathcal{L} = -\lambda R(e, \omega(e)) + \alpha [R_{abcd}(e, \omega(e))]^2 \tag{57}$$

is a higher-derivative theory (see II).

(e) Carmeli's[13] action has the same field equations as Einstein theory and is thus nonrenormalizable.

(f) Nieh's[14] recent action

$$\mathcal{L} = -\lambda R + \alpha (R_{abc})^2 + \beta (R_{abcd})^2 , \tag{58}$$

where $2\alpha = \lambda > 0$ and $\beta < 0$, has a 0^+ ghost. (In addition, there are massless poles which might lead to dipole or new ghosts. These were not considered in this paper but cannot eliminate the massive ghost.)

We would like to hope that our new unitary theories have interesting ultraviolet properties. Grisaru, van Nieuwenhuizen, and Wu[15] pointed out several years ago that if Einstein gravity does conserve helicity, it would have a finite S matrix at higher-loop levels. Recently Grisaru and Zak[16] showed that in Einstein gravity, helicity is not conserved. It is not excluded that our new solutions conserve helicity and have better loop behavior.

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APPENDIX

In this appendix we present the spin projection operators $P_{ij}^{\psi\phi}(J^P)_{\alpha\beta}$ which are used to decompose the fields h_{ab} and ω_{cab} into components with definite spin-parity. Following Neville's notation we often keep the primed indices in a fixed order and suppress them. For example,

$$\begin{aligned} &(\theta\theta\theta)_{cab} = \theta_{cc'}\,\theta_{aa'}\,\theta_{bb'} , \\ &(\theta\omega\theta)_{bca} = \theta_{bc'}\,\omega_{ca'}\,\theta_{ab'} , \\ &(\omega\theta)_{ba} = \omega_{ba'}\,\theta_{ab'} , \\ &(\theta\theta)_{ab} = \theta_{aa'}\,\theta_{bb'} , \text{ etc.} \end{aligned} \tag{A1}$$

We also defind $\hat{k}_a = k_a/\sqrt{k^2}$. The complete set of projection operators satisfying the relations given in (16) are

$$P^{\omega\omega}(2^-) = \tfrac{2}{3}(\theta\theta\theta)_{cab} + \tfrac{2}{3}(\theta\theta\theta)_{acb} - \theta_{cb}\theta_{aa'}\theta_{c'b'} , \tag{A2a}$$

$$P^{\omega\omega}(0^-) = \tfrac{1}{3}(\theta\theta\theta)_{cab} + \tfrac{2}{3}(\theta\theta\theta)_{abc} , \tag{A2b}$$

$$P_{ij}^{\psi\phi}(1^+) = \begin{pmatrix} (\theta\theta\omega)_{cab} + (\theta\omega\theta)_{abc} & -\sqrt{2}\,(\omega\theta\theta)_{abc} & \sqrt{2}\,\hat{k}_a(\theta\theta)_{bc} \\ -\sqrt{2}\,(\theta\omega\theta)_{bca} & (\omega\theta\theta)_{cab} & \tfrac{1}{2}\hat{k}_c(\theta\theta)_{ab} \\ \sqrt{2}\,\hat{k}_{a'}(\theta\theta)_{b'c'} & \tfrac{1}{2}k_{c'}(\theta\theta)_{a'b'} & 2(\theta\theta)_{ab} \end{pmatrix} , \tag{A2c}$$

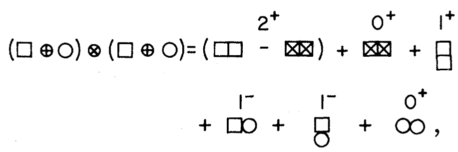
$$P_{ij}^{\psi\phi}(1^-) = \begin{pmatrix} \theta_{cb}\theta_{aa'}\theta_{a'b'} & \sqrt{2}\,\theta_{cb}\theta_{aa'}\omega_{c'b'} & \sqrt{2}\,k_{b'}\theta_{cb}\theta_{aa'} & \sqrt{2}\,\hat{k}_{b'}\theta_{cb}\theta_{aa'} \\ \sqrt{2}\,\omega_{cb}\theta_{aa'}\theta_{c'b'} & 2\omega_{cb}\theta_{aa'}\omega_{c'b'} & \sqrt{2}\,\hat{k}_b(\theta\theta)_{ac} & \sqrt{2}\,\hat{k}_b(\theta\theta)_{ac} \\ \sqrt{2}\,\hat{k}_b\theta_{c'b'}\theta_{a'a} & \sqrt{2}\,\hat{k}_{b'}(\theta\theta)_{a'c'} & 2(\theta\omega)_{ab} & 2(\theta\omega)_{ab} \\ \sqrt{2}\,\hat{k}_b\theta_{c'b'}\theta_{a'a} & \sqrt{2}\,\hat{k}_{b'}(\theta\theta)_{a'c'} & 2(\omega\theta)_{ab} & 2(\theta\omega)_{ab} \end{pmatrix} , \tag{A2d}$$

$$P_{ij}^{\psi\phi}(2^+) = \begin{pmatrix} (\theta\theta\omega)_{cab} + (\theta\theta\omega)_{acb} - \tfrac{2}{3}\theta_{cb}\omega_{aa'}\theta_{c'b'} & \sqrt{2}\,\hat{k}_b[(\theta\theta)_{ca} - \tfrac{1}{3}\theta_{ca}\theta_{a'b'}] \\ \sqrt{2}\,\hat{k}_{b'}[(\theta\theta)_{c'a'} - \tfrac{1}{3}\theta_{c'a'}\theta_{ab}] & (\theta\theta)_{ab} - \tfrac{1}{3}\theta_{ab}\theta_{a'b'} \end{pmatrix} , \tag{A2e}$$

$$P_{ij}^{\psi\phi}(0^+) = \begin{pmatrix} \tfrac{2}{3}\theta_{cb}\omega_{aa'}\theta_{c'b'} & \dfrac{\sqrt{2}}{3}\hat{k}_b\theta_{ca}\theta_{a'b'} & \sqrt{2/3}\,\hat{k}_{b'}\theta_{cb}\omega_{aa'} \\ \dfrac{\sqrt{2}}{3}\hat{k}_{b'}\theta_{c'a'}\theta_{ab} & \tfrac{1}{3}\theta_{ab}\theta_{a'b'} & \dfrac{1}{\sqrt{3}}\theta_{ab}\omega_{a'b'} \\ \sqrt{2/3}\,\hat{k}_b\theta_{c'b'}\omega_{a'a} & \dfrac{1}{\sqrt{3}}\theta_{a'b'}\omega_{ab} & \omega_{ab}\omega_{a'b'} \end{pmatrix} . \tag{A2f}$$

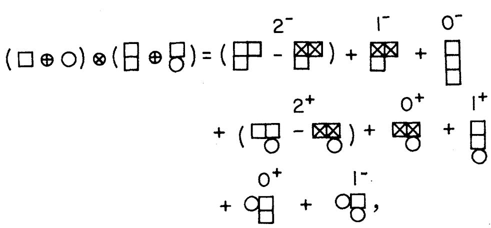
$P^{\omega\omega}$, $P^{\omega\chi}$, and $P^{\chi\chi}$ are to be antisymmetrized in (a, b) and (a′, b′), $P^{\omega\varphi}$ is to be symmetrized in (a′b′) and antisymmetrized in (a, b), $p^{\varphi\varphi}$ is to be symmetrized in both (a, b) and (a′b′).

Consider first the diagonal projection operators $P_{ii}^{\phi\phi}(J^P)_{\alpha\beta}$. Their derivation amounts to addition of angular momenta. Since θ_{ab} projects out the $J^P = 1^-$ part and ω_{ab} projects out the $J^P = 0^+$ part of a Lorentz vector index b, the field h_{ab} decomposes as follows[17]:

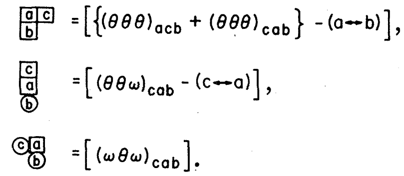

(A3)

where □ represents θ_{ab}, ○ represents ω_{ab}, and double crosses denote the "trace," in the sense that, say, the trace of $(\theta\theta)_{ab}$ is $\frac{1}{3}\theta_{ab}\theta_{a'b'}$. Thus, from Eq. (A3), after fixing the overall normalization factor, one obtains the $\varphi\varphi$ and $\chi\chi$ operators.

On the other hand, the field ω_{cab} decomposes as follows:


(A4)

where, for example,


(A5)

Thus, from the diagrams given in Eq. (A5), by first symmetrizing the row indices and then antisymmetrizing the column indices *of the boxes*, all $P_{ii}^{\omega\omega}(J^P)_{cab}^{c'a'b'}$ can be constructed. From these the transition operators were obtained, using (16a). [It is understood that all the expressions are to be symmetrized as mentioned below (A2f).]

In Ref. 5, Neville has constructed, in a somewhat different way, a set of projection operators excluding the ones acting on χ_{ab}, since he works in the gauge $e_{a\mu} = e_{\mu a}$. With our complete set of projection operators one can work in any gauge. Our $P_{ij}^{\psi\phi}(1^\mp)_{\alpha\beta}$, $i,j \leq 2$ are not the same as those of Neville, but rather they are related by an orthogonal transformation as follows:

$$P_{ij}^{\psi\phi}(1^\mp)_{\alpha\beta}\big|_{\text{Neville}} = A_{ik} P_{kl}^{\psi\phi}(1^\mp)_{\alpha\beta}\big|_{\text{ours}} A^T{}_{lj}\,, \qquad (A6)$$

where

$$A_{ij} = \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \frac{1}{\sqrt{3}}\,, \quad AA^T = 1\,.$$

In general, spin projection operators are unique up to orthogonal transformations, as we may easily verify for the two-dimensional case.

A complete check of these spin projection operators involves an examination of many relations. Although many of these relations are trivial, there are less trivial ones and for that matter we have used the algebraic computer program SCHOONSCHIP to check all of these relations.

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