Constraints on phase-shift analysis from two-variable analyticity

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We discuss the ambiguity in the determination of phase shifts, allowed by knowledge of the modulus in the three physical channels, for an amplitude which satisfies Mandelstam analyticity and has normal thresholds. We show that, under certain (weak) conditions, the ambiguity is of a discrete type, rather than continuous, as one would expect from the situation in one complex variable. The study is independent of the requirement of elastic unitarity and rests mostly on the analyticity properties of the amplitude.

I. INTRODUCTION

The problem of the ambiguity appearing in the determination of the phase of a scattering amplitude from its modulus has attracted considerable attention over the last ten years. There are clear practical reasons for this, since one would like to understand to what extent the baryon and meson spectra are really known at present from the available phase-shift analyses. Very interesting studies of mathematical physics have simultaneously grown out of this research, as one has tried to describe in precise terms the extent of the ambiguities for idealized, error-free data. These studies have intrinsic mathematical beauty, and some of them are a necessary prerequisite for a meaningful discussion of any phase-shift analysis.

In practice, it has become customary to classify solutions of phase-shift analyses according to the distribution of their zeros in the complex cosine plane at fixed energy. This has followed the work of Refs. 1 and 2. The complex position of a zero of the amplitude lying on the physical sheet of the cosine plane can be determined directly from the measured differential cross section (and polarization for πN scattering[2]) up to the sign of the imaginary part (or up to a reflection across the unit circle in πN scattering[2]). This discrete ambiguity is clearly seen in the analytic extension to complex values of $z=\cos\theta$ ($\theta=$c.m. scattering angle) of the measured differential cross section

$$\frac{d\sigma}{d\Omega}(s,z)=A(s,z)A^*(s,z^*)\equiv M_s(z)\,. \qquad (1.1)$$

The function $M_s(z)$ is real analytic in z and has complex-conjugate zeros; we do not know *a priori* how to distribute them among the two factors of Eq. (1.1), one of which is the true amplitude.

Unitarity requirements on A can remove part of this ambiguity,[3] although it has been shown[4] that even elastic unitarity cannot in general resolve it completely.

It is also well known that even if the discrete ambiguity of the zeros in the cut cosine plane is resolved, a continuum ambiguity is still left in the determination of the phase, at energies where inelasticity occurs. The amplitude can indeed be written at fixed energy as

$$A(s,z)=\prod_{i=1}^{N}[z-z_i(s)]O(z)\,, \qquad (1.2)$$

where $z_i(s)$ are N zeros on the physical sheet, and $O(z)$ is a function having the cuts, no zeros in the cut cosine plane, and a known modulus in the physical region. It is easy to construct functions which are free of zeros on the physical sheet at fixed energy and have modulus equal to one on the real axis between the cuts. These functions make up the possible infinite ambiguity which is left in the determination of the phase, after the discrete ambiguity has been disposed of. In practice,[5] $O(z)$ is assumed to produce a slow variation of the phase and one sometimes[5] believes that a good approximation of its effect is obtained by very simple functional forms. The unitarity inequalities restrict the choices for $O(z)$, but almost never completely. It was shown in detail,[6,7] by using refined mathematical techniques, how one can generate a continuum of different solutions, keeping consistency with unitarity and the given data, at fixed energy.

In these studies, the amplitude is regarded as a function solely of the c.m. scattering angle and one can wonder whether additional constraints on the phase appear if its dependence on the energy is also taken into account. In this connection, it has been argued in practical (and nonrigorous) terms in Refs. 8–10 that a partial reduction of the discrete ambiguity can be achieved by applying Weierstrass's preparation theorem to the amplitude $A(s,t)$ at points where two zero trajectories [these are complex functions $t(s)$ which are such that $A(s,t(s))\equiv 0$] come close to each other ("intersect"[8]). The main point is that two zero trajectories $t_{1,2}(s)$ of a function $A(s,t)$, holomorphic in s and t in a certain domain, are allowed (but not obliged) to develop individually singularities at the point s_c where they cross $[t_1(s_c)=t_2(s_c)]$, but only in a correlated manner; namely, so that their singularities cancel each other out when the symmetric combinations $t_1(s)+t_2(s)$ and $t_1(s)t_2(s)$ are built. If it is allowed to regard the amplitude as holomorphic in two variables in the neighborhood of points lying in the physical region (which is not strictly correct), one can check that such a cancellation of singularities does not occur for any resolution of the discrete ambiguity (see Sec. III C); this way one can reduce the number of acceptable solutions.

The reasons for starting the investigations of the present paper were twofold. First, the author wished to understand the procedure of resolving ambiguities by means of zero trajectories in more rigorous terms; does it follow from more general principles that independent reflections of zeros lead to a violation of two-variable analyticity? Secondly, he wished to see whether a resolution of the discrete ambiguity on larger intervals of energy affects the extent of the continuum ambiguity. Let us notice that, on general grounds, this is not an unexpected occurrence. Indeed, knowledge of a zero trajectory $t(s)$ of the amplitude $A(s,t)$ on a certain interval of energies allows its analytic extrapolation to fixed real t, i.e., the determination of those complex points $\bar{s}$, for which $\bar{t}=t(\bar{s})$ is a real number, and $A(\bar{s},\bar{t})=0$. If we know the position of the zeros of the function $A(s,t)$ in the complex s plane at fixed real t, $t<0$, and its modulus on the cuts, $A(s,t)$ is completely determined, and no continuum ambiguity exists. The difficulty is, however, to show that we obtain *all* zeros of $A(s,t)$ lying on the physical sheet of the complex s plane at fixed t, $t<0$, by such an extrapolation.

In the present paper, neither of these questions will be answered in the simple form stated above, which would be interesting for practical purposes. However, the present author thinks that the results which can be proved are sufficient to make it plausible that two-variable analyticity is a rather important restriction on the amplitude, which can limit, in principle drastically, the class of solutions allowed by fixed-energy phase-shift analysis.

To formulate a coherent problem, we shall assume in this paper that the modulus is given in the three physical channels of a reaction which has normal thresholds in all of them, and that the amplitude has Mandelstam analyticity (which does not mean that it satisfies the Mandelstam representation). We then ask whether one can obtain information on the phase if one resolves the discrete ambiguity of the zeros at all energies in all channels, not just at a single value. The conclusion appears to be rather simple and does not seem to have been stated so far; it says essentially that, under some weak conditions to be specified, there is then no continuum ambiguity left, only as an effect of the two-variable structure in the Mandelstam domain. The hypothesis of Mandelstam analyticity plays a crucial role and the result is not true for the axiomatic domain. We discuss this in Sec. II. In these investigations we make essential use of a result of Burkhardt and Martin[11] concerning a general expression for the ratio of two amplitudes which have the same modulus in the three physical channels. The question of the extent to which two-variable analyticity fixes the phase was actually first raised in Ref. 11.

In Sec. III we try to describe, at least in principle, how one can construct the possible discrete ambiguities of the amplitude, if one knows the function giving the modulus in all three channels (Sec. III A). If an upper bound is available for the number of zeros at complex s values in some interval of real t values ($|t|<4m_\pi^2$), then even the extent of the discrete ambiguity is limited with respect to the estimate at fixed $t\geq 4m_\pi^2$ again by considerations based on two-variable analyticity. We shall actually see that, in general, only part of the zeros present in the cosine plane at fixed $t\geq 4m_\pi^2$ can be reflected without violating two-variable analyticity. Further, those reflections of zeros that are allowed cannot be performed independently of each other. This is discussed in Sec. III B. In Sec. III C, we recall the way Weierstrass's preparation theorem can be plausibly used to remove discrete ambiguities.

The author points out that of all the properties of the scattering amplitude, this paper uses only two-variable analyticity, and ignores the constraint of unitarity (except for positivity, in Sec. III B).

II. ON THE CONTINUUM AMBIGUITY

A. Notation and statement of a theorem

Before stating the theorem which is the object of this section, we introduce part of the notation. For the description of the Mandelstam domain, we use the variables

$$\zeta=(4m^2-s)^{1/2}, \quad \theta=(4m^2-t)^{1/2}, \quad \eta=(4m^2-u)^{1/2}, \tag{2.1}$$

where s, t, and u are the usual Mandelstam variables and m is the mass of the pion. The roots in (2.1) have cuts along $(4m^2,\infty)$ and are such that they map the s, t, and u planes minus these cuts onto

$$\mathrm{Re}\zeta<0, \ \mathrm{Re}\theta<0, \ \mathrm{Re}\eta<0, \tag{2.2}$$

respectively. The Mandelstam domain $\mathfrak{M}$ consists of all points $P\in \mathbf{C}^3$ with coordinates (ζ, θ, η) which fulfill the inequalities (2.2) and lie on the manifold, which we call $\mathcal{S}$:

$$\zeta^2+\theta^2+\eta^2=8m^2\,. \tag{2.3}$$

The physical region of the s channel $\mathcal{P}_s$ consists of those points (ζ, θ, η) of the closure $\overline{\mathfrak{M}}$ of the Mandelstam domain, for which ζ is positive imaginary and θ and η are real and less than $-2m$:

$$\mathcal{P}_s=\{(\zeta,\theta,\eta)\,|\,(\zeta,\theta,\eta)\in\overline{\mathfrak{M}},\ \mathrm{Re}\zeta=0,\ \mathrm{Im}\zeta\geq 0,\ \mathrm{Im}\theta=0,\ \mathrm{Re}\theta\leq-2m,\ \mathrm{Im}\eta=0,\ \mathrm{Re}\eta\leq-2m\}. \tag{2.4}$$

Similar definitions hold obviously for the t and u channels and the physical region is $\mathcal{P}=\mathcal{P}_s\cup\mathcal{P}_t\cup\mathcal{P}_u$. For the amplitude, we write either $A(s,t,u)$ or $A(\zeta,\theta,\eta)$. We wish to consider sometimes the properties of $A(s,t,u)$ (or of any other function defined in the Mandelstam domain or on $\mathcal{S}$) as a function of one variable, when s, t, or u is held fixed. We write then, e.g., $A_{s_0}(t)\equiv A(s_0,t,u)$ if we regard A as a function of t at fixed $s=s_0$.

We say that an amplitude $A(s,t,u)$ has Mandelstam analyticity if (i) it is a holomorphic function of two variables in the Mandelstam domain $\mathfrak{M}$, (ii) for $s=s_0+i\epsilon$, $s_0\geq 4m^2$ ($\mathrm{Re}\zeta=0$, $\mathrm{Im}\zeta\geq 0$), the limiting function $A_s(t)$ exists and is holomorphic in the t plane cut along $-\infty<t<-s_0$, $4m^2<t<\infty$, (iii) property (ii) holds when the s channel is replaced by the t and u channels, and $A_s(t)$ by the corresponding limiting functions.

The statement which we wish to prove is the following: *If* A_1 and A_2 are two amplitudes which (a) have Mandelstam analyticity and are of the real type, and (b) have the same modulus in the physical region of the three channels; and are such that (c) for $s=s_0+i\epsilon$, s_0 real, $s_0>4m^2$, the functions of t, $A_{is}(t)$, $i=1,2$ are continuous in the corresponding cut t plane including the cuts and have a phase which is piecewise Hölder continuous along the cuts; (d) there exists an interval I_t of real values of t, $I_t\subset(-4m^2,4m^2)$ so that, for $t\in I_t$, the real analytic functions of s, $A_{it}(s)$, $i=1,2$ are continuous in the cut s plane, including the cuts, and their phases are piecewise Hölder continuous and uniformly bounded with respect to s, when s lies on the cuts; (e) for $t\in I_t$, the functions $A_{it}(s)$, $i=1,2$ are bounded by a polynomial in s, in the whole cut s plane, including the cuts; (f) there exists an interval I_s of real s values, $I_s\subset(-4m^2,4m^2)$, with the same properties (d) and (e) as I_t, with respect to the functions $A_{is}(t)$, $i=1,2$; and (g) the discrete ambiguity of the zeros has been resolved for both amplitudes in the same way in each channel, including the cuts in the s channel, *then* the ratio of the two amplitudes is ±1. The meaning of the phrase "including the cuts in the s channel" in condition (g) of the theorem requires some discussion and is explained under point 5 of the next paragraph.

B. Comments

We next give some comments concerning this statement.

(1) The s channel of condition (c) can be replaced by any other channel. Similarly, any pair of channels can be used in the formulation of conditions (d), (e), and (f), independently of the one appearing in condition (c).

(2) Condition (c) concerning the continuity of the amplitude in the cut t plane, including the cuts, does not imply any restriction on the behavior of the amplitude at infinity in the cut t plane. One needs only the following: Consider a finite point $t_0+i\epsilon$ lying on the cut and the intersection $U_{t_0}(r)$ of an open disk around $t_0+i\epsilon$ of radius r with the open cut t plane (this is an open half disk); we require the amplitudes to be continuous in the closed $\overline{U}_{t_0}$ for all $t_0<\infty$. A similar comment applies to condition (d) concerning the continuity of $A_{it}(s)$.

(3) The phases to which conditions (c) and (d) refer are defined by continuity from thresholds along the cuts, supplemented by small excursions in the complex plane if one meets a zero. This procedure gives a unique result if there is no accumulation of zeros to some point of the cut. We assume this is true. Condition (d) concerning the boundedness of the phase requires in addition that there be only a finite number of zeros on the two cuts. In condition (c), we do not require the phase to stay bounded, as we move to infinity along the cut.

(4) At fixed $s=s_0+i\epsilon$, $s_0>4m^2$, the modulus squared of the functions can be analytically continued to the whole cut t plane by formula (1.1), which we rewrite as

$$M_{is}(t)=A_{is}(t)A^*_{is}(t^*)\,,\quad i=1,2\,. \tag{2.5}$$

The functions $M_{is}(t)$ represent the moduli (squared) of the amplitudes as long as $t=t^*$ in (2.5), that is, on the segment of the real axis lying between the cuts. So, if the moduli of $A_{is}(t)$, $i=1,2$ are equal [condition (b)] in the physical region $\mathcal{P}_s$, they must be equal in the whole t interval $-s<t<4m^2$. Further, since for real t, the functions $A_i(s,t)$ are real analytic in s, it follows that $A_{is^*}(t)=A^*_{is}(t)=A^*_{is}(t^*)$, and we see from (2.5) that $M_{1s^*}(t)=M_{2s^*}(t)$. We conclude that, with the notation of (2.1), the equality of the moduli of the two amplitudes holds in the enlarged region

$$\bar{\mathcal{P}}_s=\{(\zeta,\theta,\eta)\,|\,(\zeta,\theta,\eta)\in\bar{\mathfrak{M}},\ \mathrm{Re}\zeta=0,\ \mathrm{Im}\zeta\neq 0,\ \mathrm{Im}\theta=0,\ \mathrm{Im}\eta=0\}. \tag{2.6}$$

The same is obviously true for the t and u channels. For the purposes of this paper, (2.6) will be referred to as "the physical region of the s channel."

(5) Under condition (c), it is meaningful to talk about a "zero of $A_{1s}(t)$ lying on the cut" of order p, at a point $t_0+i\epsilon$, $t_0\in(4m^2,\infty)\cup(-\infty,-s_0)$ for $s=s_0+i\epsilon$, $s_0>4m^2$ if $\lim_{t\to t_0}A_{1s}(t_0+i\epsilon)/(t-t_0)^p$ is finite, but $\lim_{t\to t_0}A_{1s}(t_0+i\epsilon)/(t-t_0)^{p+1}$ is infinite as t approaches t_0 from above or along the real axis ($p\geq 0$). It is indeed possible to show that if $A_{1s}(t_0+i\epsilon)=0$ and condition (c) holds, then an integer p exists, having the properties stated above (i.e., to any zero we can associate uniquely its order). The proof of this assertion is done in Appendix A. If $A_{1s}(t)$ has, e.g., a simple zero on the cut, then the analytic extension $M_{1s}(t)$ of the modulus function according to (2.5) has also at least a simple zero at $t_0+i\epsilon$ [if condition (c) holds]. The same happens if $A_{1s}^*(t^*)$ has a simple zero at $t_0+i\epsilon$, t_0 on the cut. One is tempted to think that the situation is similar to the one in which t_0 is an interior point of the analyticity domain, and that a simple zero of the function $M_{1s}(t)$ lying on the cut leads to a twofold ambiguity when the amplitude is reconstructed. This is, however, not obvious. Indeed, it is possible *a priori*, for any α, $0<\alpha<1$ to part a simple zero lying on the cut of $M_{1s}(t)$ into two factors $(t-t_0)^\alpha$, $(t-t_0)^{1-\alpha}$, and assign them to $A_{1s}(t), A_{1s}^*(t^*)$, respectively. This can be done without affecting the analyticity properties of $A_{1s}(t)$ or $A_{1s}^*(t^*)$ at fixed energy and would correspond to a continuum ambiguity.

We shall show explicitly in Sec. II M that if two amplitudes obey conditions (a), (b), and (c) of the theorem, then a simple zero of $M_{1s}(t)$ lying on the cut can lead to at most a twofold ambiguity in the reconstruction of $A_{1s}(t)$ at fixed energy. [The ambiguity concerns the behavior of $A_{1s}(t)$ at the zero of $M_{1s}(t)$.] This will justify the formulation of condition (g). For the time being, we shall take condition (g) to mean that the two amplitudes have zeros of the same order at those points of the complex t plane, including the cuts, where $M_{1s}(t)$ $[=M_{2s}(t)]$ in Eq. (2.5) vanishes (for $s=s_0+i\epsilon$, $s_0>4m^2$).

C. Outline of the proof

The proof of the statement uses in an essential manner a result of Ref. 11, according to which the ratio of two amplitudes which are holomorphic in the axiomatic analyticity domain and have the same modulus in the physical region of the three channels can be represented in the form

$$R(s,t,u)=\frac{A_1(s,t,u)}{A_2(s,t,u)}=\frac{f(s,t,u)-\zeta\theta\eta}{f(s,t,u)+\zeta\theta\eta}, \tag{2.7}$$

where $f(s,t,u)$ is the ratio of two real analytic entire functions of s,t,u. Clearly, $f(s,t,u)$ can be regarded as the ratio of two entire functions of s,t only, because of the linear relation (2.3) between s, t, and u. We denote $f_1(s,t)\equiv f(s,t,u)$. The remarkable quality of Eq. (2.7) is that it contains none of the possible complicated singularities of the amplitude, but has instead just two sheets in each of the variables. Consequently, it is defined at every point of the manifold $\mathcal{S}$, Eq. (2.3), without the restrictions (2.2).

The theorem of Sec. II A rests upon the following property of the ratio R of Eq. (2.7):

(A) If $f_1(s,t)$ is a rational function of s, t and R is such that (i) the function of θ, $R_\zeta(\theta)$, assumes only finite nonzero values in the set

$$\partial_\zeta\mathfrak{M}=\{(\zeta,\theta,\eta)\in\mathcal{S}\,|\,\mathrm{Re}\zeta=0,\ \mathrm{Re}\theta\leq 0,\ \mathrm{Re}\eta\leq 0\}, \tag{2.8}$$

(ii) the function of η, $R_\theta(\eta)$ has neither zeros nor poles in the set

$$\partial_\theta\mathfrak{M}=\{(\zeta,\theta,\eta)\in\mathcal{S}\,|\,\mathrm{Re}\zeta<0,\ \mathrm{Re}\theta=0,\ \mathrm{Re}\eta<0\},$$

and (iii) the function of ζ, $R_\eta(\zeta)$, has neither zeros nor poles in the set

$$\partial_\eta\mathfrak{M}=\{(\zeta,\theta,\eta)\in\mathcal{S}\,|\,\mathrm{Re}\zeta<0,\ \mathrm{Re}\theta<0,\ \mathrm{Re}\eta=0\},$$

then $R=\pm1$. We denote $\partial_1\mathfrak{M}=\partial_\zeta\mathfrak{M}\cup\partial_\theta\mathfrak{M}\cup\partial_\eta\mathfrak{M}$ and notice the slight asymmetry of $\partial_\zeta\mathfrak{M}$ with respect to the other two. The sets $\partial_\zeta\mathfrak{M}$, $\partial_\theta\mathfrak{M}$, and $\partial_\eta\mathfrak{M}$ represent the physical sheets of the cut cosine planes for all energies in the s, t, and u channels, respectively; the cuts themselves are included in $\partial_\zeta\mathfrak{M}$, but not in $\partial_\theta\mathfrak{M}$ and $\partial_\eta\mathfrak{M}$. We prove statement (A) in Sec. II G–II L. To make use of statement (A), we need two preparatory steps (B) and (C):

(B) If A_1 and A_2 obey conditions (a), (b), (d), (e), and (f) of Sec. II A, then $f_1(s,t)\equiv f(s,t,u)$ in (2.7) is a rational function of s and t.

(C) If A_1 and A_2 obey conditions (a), (b), (c), and (g), then the ratio R has neither zeros nor poles in $\partial_1\mathfrak{M}$ (as a function of the variable corresponding to each subset).

The three statements (B), (C), and (A) yield together the announced theorem. We now turn to their explicit proof. We prove statement (B) in Sec. II D and Sec. II E, then statement (C) in Sec. II F, and we turn after this to (A).

D. A preliminary step

To prove statement (B), we need a preliminary step:

(B1) If the amplitude $A_1(s,t)$ obeys conditions (a), (d), and (e), then the function $A_{1t}(s)$ has at most a finite number of zeros in the cut complex s plane, corresponding to $t \in I_t$.

To show this, we use the variable $\nu = s - u$ and define, for $t \in I_t$

$$\Omega_{1t}(\nu) = \exp\left[-\frac{\nu}{\pi}\int_{-\infty}^{-\nu_1}\frac{\phi'_{1t}(\nu')}{\nu'(\nu'-\nu)}d\nu' + \frac{\nu}{\pi}\int_{\nu_1}^{\infty}\frac{\phi''_{1t}(\nu')}{\nu'(\nu'-\nu)}d\nu'\right], \qquad (2.9)$$

where ϕ'_{1t} and ϕ''_{1t} are the phases of $A_{1t}(\nu)$ on the upper sides of the u and s cuts, and $\nu_1 = 4m^2 + t$. By condition (d) of Sec. II A, the integrals converge for all ν in the cut ν plane and have well defined limits at all points on the lips of the cuts (Ref. 13, p. 38). The function $\Omega_{1t}(\nu)$ has clearly no zeros in the open cut ν plane and has the phase of the amplitude $A_{1t}(\nu)$ when ν lies on the cuts. Using the fact that ϕ'_{1t}, $\phi''_{1t}(\nu)$ are bounded for $|\nu'| > \nu_1$, one can even find a lower bound for the magnitude of $\Omega_{1t}(\nu)$ at points lying on a circle of sufficiently large radius $|\nu|$ in the complex plane[14]

$$|\Omega_{1t}(\nu)| > e^{-K\ln(|\nu|/\nu_1|\sin\theta|)}, \qquad (2.10)$$

where we have denoted $\nu \equiv |\nu|\cos\theta + i|\nu|\sin\theta$, and K is a constant depending on the bound on ϕ. The derivation of inequality (2.10) is straightforward, but we give for completeness its details in Appendix B, including the meaning of "sufficiently large $|\nu|$." We define further

$$\tilde{A}_{1t}(\nu) \equiv A_{1t}(\nu)/\Omega_{1t}(\nu), \qquad (2.11)$$

which is a function having the same zeros as $A_{1t}(\nu)$ (except possibly for a finite number of zeros lying on the cut), but is holomorphic (cf. Appendix A) in the whole complex ν plane. We can estimate its magnitude on a circle of radius $|\nu|$, by using condition (e), according to which $A_{1t}(\nu)$ is polynomially bounded in the cut ν plane, for $t \in I_t$: $|A_{1t}(\nu)| < K_1 e^{n\ln(|\nu|/\nu_1)}$ for some $n, K_1 > 0$. We conclude

$$|\tilde{A}_{1t}(\nu)| = |A_{1t}(\nu)/\Omega_{1t}(\nu)| \le e^{K_2\ln(|\nu|/\nu_1|\sin\theta|)} \qquad (2.12)$$

for, e.g, $K_2 > 2(K+n)$ and $|\nu|$ appropriately large. But the principle of the maximum modulus tells us that $|\tilde{A}_{1t}(\nu)|$ is actually a polynomial. Indeed, we write that $|\tilde{A}_{1t}(\nu)|$ is less than a function having no zeros in a circle of radius $2|\nu|$ and modulus $(2|\nu|/\nu_1|\sin\theta|)^{K_2}$ on this circle:

$$|\tilde{A}_{1t}(\nu)| \le \exp\left[\frac{1}{\pi}\int_0^{\pi}\frac{3}{4-4\cos(\theta-\varphi)+1}\ln\left(\frac{|2\nu|}{\nu_1|\sin\varphi|}\right)^{K_2}d\varphi\right] \le D_1|\nu|^{D_2}, \qquad (2.13)$$

where D_1 and D_2 are constants, independent of $|\nu|$. (We have applied Poisson's formula.) Since the bound (2.13) is true for all ν, $\tilde{A}_{1t}(\nu)$ must be a polynomial, and so, have a finite number of zeros. This proves statement (B1).

E. Proof of statement (B)

We can now prove statement (B) of Sec. II C, namely that $f_1(s,t) \equiv f(s,t,u)$ of Eq. (2.7) is rational, under conditions (a), (b), (d), (e), and (f) of Sec. II A. To this end, we choose $t \in I_t$ and compute explicitly the ratio $R \equiv A_1(s,t,u)/A_2(s,t,u)$. We notice that the points lying on the u and s cuts, $s < -t$ and $s > 4m^2$, for $t \in I_t$ also lie in the ("extended," see Sec. II B) physical regions $\bar{\mathcal{P}}_s, \bar{\mathcal{P}}_u$ [see Eq. (2.6)], where the moduli of the two amplitudes are equal, by condition (b) (and Sec. II B). It is then convenient to define, for $t \in I_t$, the function

$$E_t(s) = \exp\left[\frac{1}{2\pi}\int_0^{\pi}\frac{1-z_t^2(s)}{1-2z_t(s)\cos\varphi + z_t^2(s)} \times \ln|A_t(\varphi)|^2 d\varphi\right], \qquad (2.14)$$

where $z_t(s)$ is the mapping from the cut s plane onto the unit disk, so that $\pm i$ corresponds to the points at infinity above and below the cuts:

$$z_t(s) = \frac{(4m^2-u)^{1/2} - (4m^2-s)^{1/2}}{(4m^2-u)^{1/2} + (4m^2-s)^{1/2}}. \qquad (2.15)$$

[Formula (2.14) is not the Poisson formula, but its complexified form (Schwartz-Villat) written for real analytic functions of z_t.] The subscripts 1 and 2 were dropped in (2.14) from $|A_{it}(\varphi)|$, because $|A_{1t}(\varphi)| = |A_{2t}(\varphi)|$, $t \in I_t$. The function $E_t(s)$ has no zeros in the cut s plane (open) and is well defined in $|z_t| < 1$, since on the cuts $|A_t(s)|$ is polynomially bounded, so that the integrand diverges only logarithmically at $\varphi = \pm\pi/2$. Further, it is known (Ref. 15, Theorems 1.2 and 1.3) that, because $|A_t(\varphi)|$ is continuous, one can conclude that, except for those points where $|A_t(\varphi)| = 0$ or is not finite, $|E_t(s)| \to |A_t(s_0 \pm i\epsilon)|$, as $s \to s_0 \pm i\epsilon$, $s_0 > 4m^2$, $s_0 < -t$. If we now define $\mathcal{B}_{it}(z_t(s)) \equiv A_{it}(z_t(s))/E_t(s)$ we see that, for $t \in I_t$,

$$R(s,t,u) = \mathcal{B}_{1t}(z_t(s))/\mathcal{B}_{2t}(z_t(s)). \qquad (2.16)$$

The functions $\mathcal{B}_{it}(z_t(s))$, $i = 1, 2$ have modulus one at all points on the cut, except possibly for a finite number. We have further seen (Sec. II D) that $A_{it}(z_t(s))$, $i=1,2$ have a finite number of zeros N_i in the open cut s plane. If $z_{1i}, z_{2i}, \ldots, z_{N,i}$ are the positions of the zeros of $A_{it}(z_t(s))$, we can define

$$B_{it}(z_t(s)) = \prod_{k=1}^{N_i} \frac{z_t(s) - z_{ki}}{1 - z_{ki}^* z_t(s)} . \tag{2.17}$$

Clearly, $B_{it}(z_t(s))$ have modulus one on the cuts. We define now

$$S_{it}(z_t) = \mathcal{B}_{it}(z_t)/B_{it}(z_t) , \tag{2.18}$$

which are functions real holomorphic and without zeros in the unit disk $|z_t| < 1$, and with modulus one almost everywhere on $|z_t| = 1$. We show in Appendix C that, as a consequence of conditions (d) and (e) of Sec. II A, $S_{it}(z_t) = \pm 1$.

We conclude that for $t \in I_t$,

$$R(s,t,u) = \pm \frac{B_{1t}(z_t)}{B_{2t}(z_t)} = \pm \frac{b_{0,t} + b_{1,t} z_t + b_{2,t} z_t^2 + \ldots + b_{N,t} z_t^N}{b_{0,t} z_t^N + b_{1,t} z_t^{N-2} + b_{2,t} z_t^{N-2} + \ldots + b_{N,t}} , \tag{2.19}$$

where the $b_{i,t}$'s are algebraic combinations of the z_{k1}, z_{k2} of (2.17), and $N = N_1 + N_2$. They are real numbers since B_{1t} and B_{2t} are real analytic functions of z_t. The validity of (2.19) is checked by noticing that the right-hand side is rational, with correct degree, and so constructed that it has modulus one on the unit circle. Using now in (2.19) the expression (2.15) of the mapping $z_t(s)$ we arrive at

$$R(s,t,u) = \frac{\sum_{i=0}^{\bar N} \alpha_i(t) s^{\bar N - i} + \eta\zeta \sum_{i=0}^{\bar N -1} \beta_i(t) s^{\bar N - i - 1}}{\sum_{i=0}^{\bar N} \alpha_i(t) s^{\bar N - i} - \eta\zeta \sum_{i=0}^{\bar N -1} \beta_i(t) s^{\bar N - i - 1}} , \tag{2.20}$$

where $\bar N = [\frac{1}{2}(N+1)]$ and we have eliminated terms in η^2 by means of condition (2.3); $\alpha_i(t)$ and $\beta_i(t)$ are real algebraic combinations of $b_{i,t}$ in (2.19) and of possible terms containing t, coming from (2.3). If $N = N_1 + N_2$ is odd, one needs to multiply both numerator and denominator by a factor ζ (or η) in order to come to (2.20).

Clearly, the numbers N_1, N_2 of zeros depend on t. However, there exists a set I_t' of values of t, dense in some interval contained in I_t, such that, for all $t \in I_t'$, the number of zeros in the complex s plane belonging to A_{1t}, A_{2t} is bounded from above by a certain number $N_{\max}$ (see Appendix D). We conclude that Eq. (2.20) with $\bar N$ replaced by $\bar N_{\max} = N_{\max}$ is valid for all $t \in I_t'$ (with some of the $\alpha_i(t), \beta_i(t)$ possibly equal to zero). We now compare (2.20) with the general formula for $R(s,t,u)$ given by (2.7) and obtain this way, at fixed $t \in I_t'$ an expression for $f_1(s,t) \equiv f(s,t,u)$:

$$f_1(s,t) = \frac{\sum_{i=0}^{\bar N_{\max}} \alpha_i(t) s^{\bar N_{\max} - i}}{\sum_{i=0}^{\bar N_{\max}-1} \beta_i'(t) s^{\bar N_{\max} - i - 1}} , \tag{2.21}$$

with $\beta_i'(t) = \beta_i(t)/\theta$. Let us now fix (2.21) at $2\bar N_{\max} + 1$ points $s_1, s_2, \ldots s_{2\bar N_{\max}+1}$ of the complex s plane and obtain this way, at all $t \in I_t'$, $2\bar N_{\max} + 1$ homogeneous equations which must be satisfied by the coefficients $\alpha_0, \alpha_1, \ldots \alpha_{\bar N_{\max}}, \beta_0, \ldots \beta_{\bar N_{\max}-1}$. We show now that we can choose the solutions of this set at each $t \in I_t'$ in such a way that, when regarded as a function of t, they are analytic in t in some domain. To see this, we notice first that the rank of the matrix of the set of equations is at least $\bar N_{\max} + 1$. Indeed, the Vandermonde determinant constructed from $1, s_k, \ldots, s_k^{\bar N_{\max}}$, $k = 1, 2, \ldots, \bar N_{\max} + 1$ does not vanish. Also, the rank is certainly less than $2\bar N_{\max} + 1$, since there exists a nontrivial solution to the equations. Let then ρ be the rank of the matrix, as far as identical vanishing in I_t' is concerned (notice that all minors of the matrix are ratios of entire functions of t) and let $\{\alpha_i, \beta_j'\}_{i=i_1,\ldots,i_{\rho'}; j=i_{\rho'+1},\ldots,i_\rho}$ be the unknowns corresponding to the nonvanishing minor. A particular solution $\{\alpha_k^0, \beta_l^0\}_{k=1,\bar N_{\max}+1}^{l=1,\bar N_{\max}}$ of the set of equations is obtained[16] by setting the unknowns outside the set $\{i_1, \ldots i_\rho\}$ equal to zero, except for one of them, which we set, e.g., equal to one. The solutions obtained in this way are rational combinations of a certain subset of the $\{f(s_k,t)\}_{k=1}^{2\bar N_{\max}+1}$, i.e., are ratios of entire functions of t. Further, the rational function of s constructed with $\alpha_k^0(t), \beta_l^0(t)$ coincides, at each $t \in I_t'$, with the right-hand side of (2.21) at $2\bar N_{\max} + 1$ values of s, and so in the whole complex s plane. We conclude that we can replace $\alpha_i(t), \beta_j'(t)$ by $\alpha_i^0(t), \beta_j^0(t)$ in (2.21) and obtain for any complex s and for $t \in I_t'$ an identity in t. Since both sides are ratios of entire functions of t, this can be extended to the whole complex t plane and so becomes an identity for all s and t. By multiplying all coefficients $\alpha_i^0(t), \beta_j^0(t)$ by an appropriate entire function of t, we preserve the identity (2.21) and find new coefficients, denoted again by $\alpha_i(t), \beta_j'(t)$, which are now entire functions of t themselves.

Clearly, we can repeat the reasoning of this paragraph for $s \in I_s$, of condition (f), Sec. II A, and obtain another form for the function $f_1(s,t)$, namely,

$$f_1(s,t) = \frac{\sum_{i=0}^{\bar N_{\max}'} \gamma_i(s) t^{\bar N_{\max}' - i}}{\sum_{i=0}^{\bar N_{\max}'-1} \delta_i'(s) t^{\bar N_{\max}' - i - 1}} , \tag{2.22}$$

where $\gamma_i(s)$, $\delta_i'(s)$ can be chosen, as before, to be entire functions of s, and $\overline{N}'_{\max}$ is analogous to $\overline{N}_{\max}$ in (2.21).

To sum up, we have shown that the function $f_1(s,t)$ has the property that, on one hand, it is a rational function of s, with coefficients that are entire (and real analytic) in t, and on the other hand, it is a rational function of t with coefficients that are entire (and real analytic) in s. We now show that this implies it is a rational function of s and t.

To this end, we choose again $2\overline{N}'_{\max}+1$ points $t_1, t_2, \ldots, t_{2\overline{N}'_{\max}+1}$ and compute $f_1(s,t_1), \ldots, f_1(s, t_{2N'_{\max}+1})$ according to formula (2.21). These are all rational functions of s. We obtain thus a set of equations for $\gamma_0(s), \ldots, \gamma_{\overline{N}'_{\max}}, \delta_0'(s), \ldots, \delta'_{\overline{N}'_{\max}-1}(s)$ which, by the same reasoning as before, admits of a solution in terms of rational functions of s. We conclude then that $f_1(s,t)$ is actually a rational function of s. This has completed the proof of statement (B).

We see we can now write

$$f_1(s,t)=h(s,t)/g(s,t)\,, \tag{2.23}$$

with h, g polynomials in s, t with real coefficients and

$$R(s,t,u)=\frac{h(s,t)-\zeta\theta\eta g(s,t)}{h(s,t)+\zeta\theta\eta g(s,t)}\equiv\frac{R_N(\zeta,\theta,\eta)}{R_D(\zeta,\theta,\eta)}\,, \tag{2.24}$$

where R_N and R_D are obviously the numerator and denominator of R. Reasonings similar to those of this paragraph can be found in Ref. 12, p. 275.

F. Proof of statement (C)

We now prove statement (C) of Sec. II C.

According to condition (g) the amplitudes A_1 and A_2 are such that, e.g., at fixed $u=u_0+i\epsilon$, $u_0>4m^2$ the zeros of the functions of t, $A_{1u}(t)$ and $A_{2u}(t)$ lying in the open cut t (cosine) plane coincide with each other and have the same order. Since the amplitudes are real analytic, $A_i(u,t)=A_i^*(u^*,t^*)$, $i=1,2$ their zeros in the complex t plane must coincide and have the same order also for $u=u_0-i\epsilon$. We conclude that their ratio $R_u(t)=A_{1u}(t)/A_{2u}(t)$ has no zeros and poles in the complex cut t plane, for $u=u_0\pm i\epsilon$, $u_0>4m^2$. For $u_0=4m^2$, we see in (2.19) that $R(-t,t,4m^2)=\pm1$ ($z_t=-1$ at $u_0=4m^2$), $t\in I_t$, and so for all t. So, the set of points at which zeros and poles are excluded in this way, for all $u_0\geq 4m^2$, makes up the set $\partial_\eta\mathfrak{M}$ of Sec. II C. Clearly, $R(s,t,u)$ has by the same argument no zeros and poles in $\partial_\theta\mathfrak{M}$, and in that part of $\partial_\zeta\mathfrak{M}$ which is obtained by removing the equal sign in the two inequalities in (2.8).

We now show that condition (c) concerning the continuity of the amplitude on the cuts prevents zeros of $R_\zeta(\theta)\equiv R_s(t)$ from appearing in the whole $\partial_\zeta\mathfrak{M}$ of (2.8), provided the zeros on the cut of A_1 and A_2 are of the same order (Sec. II B). To this end, let us notice, also for further purposes, that at fixed ζ, $\zeta\neq\pm2m\sqrt{2}$, the function $R_\zeta(\theta)$ has two sheets, which can be uniformized by means of the variable

$$\omega=\theta+i\eta\equiv\theta+i(8m^2-\zeta^2-\theta^2)^{1/2}\,. \tag{2.25}$$

The only singularities of the function $R_\zeta(\omega)$ are then a finite number of poles in ω. Indeed, one has $\theta=[\omega+(8m^2-\zeta^2)/\omega]/2$, $\eta=[\omega-(8m^2-\zeta^2)/\omega]/2i$ which, when introduced in (2.24) yield a rational function of ω (at fixed ζ). Let us then consider a zero of order p in both $A_{1\zeta}(\theta)$ and $A_{2\zeta}(\theta)$, lying at $\omega_0=\omega(\theta_0)$, on the image of the cuts through the mapping (2.25), and compute

$$\lim_{\omega\to\omega_0}\frac{R_\zeta(\omega)}{\omega-\omega_0}\equiv\lim_{\omega\to\omega_0}\frac{A_{1\zeta}(t)}{(\omega-\omega_0)^{p+1}}\,\frac{(\omega-\omega_0)^p}{A_{2\zeta}(t)}\,. \tag{2.26}$$

We can assume $\lim_{t\to t_0}[(\omega-\omega_0)/(t-t_0)]$ is finite and nonzero. Indeed, the only points where this does not happen are $\theta=0$ and $\eta=0$. But $R=\pm1$ there, by (2.19) and so, the fact that R is finite and nonzero is trivially true. Then, according to condition (g) and Sec. II B, $\lim_{\omega\to\omega_0}A_{1\zeta}(t)/(\omega-\omega_0)^{p+1}$ is infinite, whereas $\lim_{\omega\to\omega_0}A_{2\zeta}(t)/(\omega-\omega_0)^p$ is finite (or zero). Consequently, a constant $L>0$ exists, with the property $|(\omega-\omega_0)^p/A_{2\zeta}(t)|>L$ for ω close to ω_0. We see that the limit in (2.26) is infinite. Since $R_\zeta(\omega)$ is rational, we conclude that it cannot have a zero at ω_0, but rather must be finite there or have a pole.

We compute then

$$\lim_{\omega\to\omega_0}1/(\omega-\omega_0)R_\zeta(\omega)$$
$$\equiv\lim_{\theta\to\theta_0}[A_{2\zeta}(t)/(\omega-\omega_0)^{p+1}][(\omega-\omega_0)^p/A_{1\zeta}(t),$$

which is again infinite, and so conclude that $R_\zeta(\omega)$ is finite at ω_0. We have shown this way that $R_\zeta(\theta)$ is finite and nonzero in the whole $\partial_\zeta\mathfrak{M}$ [Eq. (2.8)]. We shall state that $R_\zeta(\theta)$" has neither zeros nor poles in $\partial_\zeta\mathfrak{M}$", meaning that this is true at all points of $\partial_\zeta\mathfrak{M}$, except for the branch point $\eta=0$, where $R=\pm1$.

G. Comment on the proof of (A)

We are now in a position to prove statement (A) of Sec. II B, namely that $R(s,t,u)=\pm1$ if it is of the form (2.24), and has no zeros and poles in $\partial_\zeta\mathfrak{M}$, $\partial_\theta\mathfrak{M}$, and $\partial_\eta\mathfrak{M}$ as a function of the corresponding variable. The main part of the argument will be to show that $R(s,t,u)$ has under these conditions neither zeros nor poles in the interior of the Mandelstam analyticity domain [Eqs. (2.2) and (2.3)].

To render this result plausible, we recall Hartogs's theorem (Ref. 12, p. 206), according to which a function that is holomorphic in two variables in the two complex dimensional neighborhood of any point of the boundary of a bounded domain (in $\mathbf{C}^2$) is holomorphic everywhere inside the domain. Since the complex cosine planes at fixed $s=s_0 \pm i\epsilon$, $s_0 \geq 4m^2$, $t=t_0 \pm i\epsilon$, $t_0 \geq 4m^2$, $u=u_0 \pm i\epsilon$, $u_0 \geq 4m^2$ (s_0, t_0, u_0 real) make up the boundary $\partial\mathfrak{M}$ of the Mandelstam domain, we expect that the absence of zeros or poles of $R(s,t,u)$ on the boundary somehow propagates to the interior of the analyticity domain.

The main difficulty one faces in using this theorem is to prove that the absence of zeros and poles of R as a function of one variable in $\partial_\zeta\mathfrak{M}$, $\partial_\theta\mathfrak{M}$, and $\partial_\eta\mathfrak{M}$ implies the absence of zeros and poles in a two (complex) dimensional neighborhood of a point on the boundary of $\mathfrak{M}$. This can be done, but it is complicated to study the analyticity and absence of zeros of $R(s,t,u)$ in the neighborhood of the points at infinity. The author believes he can circumvent this with an argument which exploits directly the polynomial character of h, g [in Eq. (2.24)].

We are then rewarded with the (esoteric) finding that, under conditions (d), (e), and (f) of Sec. II A, it is not even necessary to exclude zeros and poles in all of $\partial\mathfrak{M}$, which includes the closure of $\partial_\theta\mathfrak{M}$, $\partial_\eta\mathfrak{M}$ (equality signs in their definitions), as would be required by Hartogs's theorem, but only on $\partial_\zeta'\mathfrak{M} \cup \partial_\theta\mathfrak{M} \cup \partial_\eta\mathfrak{M}$, where

$$\partial_\zeta'\mathfrak{M} = \{(\zeta,\theta,\eta) \in \mathcal{S} \mid a < |\mathrm{Im}\zeta| < b,\ \mathrm{Re}\zeta = 0,\ \mathrm{Re}\theta \leq 0,\ \mathrm{Re}\eta \leq 0\} \subset \partial_\zeta\mathfrak{M}$$

for some $a, b > 0$.

H. A lemma

We study first the zeros of $R(s,t,u)$. They are given by those zeros of $R_N(\zeta,\theta,\eta)$, Eq. (2.24) which are not canceled by zeros of the denominator $R_D(\zeta,\theta,\eta)$. To simplify the discussion, we note the following statement:

(A1) The function R does not vanish identically on $\mathcal{S}$ along any complex "line" $\zeta=\zeta_0$, $\theta=\theta_0$, or $\eta=\eta_0$ for any value of $(\zeta_0,\theta_0,\eta_0)$.

We first recall that $R(s,t,u)=\pm 1$ along $\zeta=0$, $\theta=0$, or $\eta=0$ so that we can assume $\zeta_0 \neq 0$, $\eta_0 \neq 0$, $\theta_0 \neq 0$. Let us suppose then that, for a certain complex ζ_0, we had $R_N(\zeta_0,\theta,\eta) \equiv 0$ for all (θ,η) so that $(\zeta_0,\theta,\eta) \in \mathcal{S}$. We choose then an interval $I \subset (-2m\sqrt{2}, 2m\sqrt{2})$ of real values of θ and consider $\eta(\zeta_0,\theta)=(8m^2-\zeta_0{}^2-\theta^2)^{1/2}$ (any determination of the root) which is holomorphic in θ in a complex neighborhood of I. The function $\eta(\zeta,\theta)$ is real for ζ real and $\theta \in I$, $-(8m^2-\theta^2)^{1/2} < \zeta < (8m^2-\theta^2)^{1/2}$ and so $\eta(\zeta_0^*,\theta)=\eta^*(\zeta_0,\theta)$, for $\theta \in I$. [At fixed real θ, ζ_0^* lies on the same sheet as ζ_0, if the cuts of $\eta(\zeta,\theta)$ go from $\pm(8m^2-\theta^2)^{1/2}$ to infinity]. But for $\theta \in I$, $R_N^*(\zeta_0,\theta,\eta)=R_N(\zeta_0^*,\theta,\eta^*)$, by the real analyticity of R_N. Therefore, the holomorphic function of θ, $R_N(\zeta_0^*,\theta,\eta(\zeta_0^*,\theta))$ vanishes on I and so everywhere. So, $R_N(\zeta_0^*,\theta,\eta) \equiv 0$ for all (θ,η) so that $(\zeta_0^*,\theta,\eta) \in \mathcal{S}$.

Consider now an interval I' of values θ along $\mathrm{Re}\theta=0$. We can solve for $\eta=\eta(\zeta_0,\theta)$ in the neighborhood of I'. We write $\theta=i|\theta|$ for $\theta \in I'$; for $-(8m^2+|\theta|^2)^{1/2} < \zeta < (8m^2+|\theta|^2)^{1/2}$, $\theta \in I'$, $\eta(\zeta,\theta)$ is real, and so $\eta(\zeta_0^*,\theta)=\eta^*(\zeta_0,\theta)$. From the definition of R_N, Eq. (2.24), we see that

$$R_N^*(\zeta_0,\theta,\eta)=h(s_0^*,t)+i\zeta_0^*|\theta|\eta^* g(s_0^*,t)=0, \quad \theta \in I'. \tag{2.27}$$

We have denoted $s_0=4m^2-\zeta_0{}^2$. We combine this with

$$R_N(\zeta_0^*,\theta,\eta)=h(s_0^*,t)-i\zeta_0^*|\theta|\eta^*(\zeta_0,\theta)g(s_0^*,t)=0, \quad \theta \in I'. \tag{2.28}$$

We conclude that $h(s_0^*,t) \equiv 0$ for an interval of t values, $\mathrm{Im}t=0$, $t>4m^2$. Also, since $\zeta_0 \neq 0$, $g(s_0^*,t) \equiv 0$ there. Since h and g are polynomials, the only possibility is that both h and g contain a certain number of factors of $(s-s_0^*)$. Let p_0 be the number of common factors of this type of h, g. They contain the same number of common factors of $(s-s_0)$ by real analyticity. We define then new polynomials

$$h_1(s,t)=\frac{h(s,t)}{(s-s_0)^{p_0}(s-s_0^*)^{p_0}}, \qquad g_1(s,t)=\frac{g(s,t)}{(s-s_0)^{p_0}(s-s_0^*)^{p_0}}, \tag{2.29}$$

and with them new R_N', R_D', according to (2.24), by replacing h, g with h_1, g_1. Clearly, $R_N'/R_D'=R$. At least one of h_1 and g_1 no longer vanishes identically for $s=s_0$. We can repeat now the reasoning of this paragraph, with h, g replaced by h_1, g_1, and reach a contradiction if we assume that the set of zeros of R (included in that of R_N) contains the line $\zeta=\zeta_0$, on $\mathcal{S}$. This concludes the proof. [If $\zeta_0 \neq 0$ and real, we combine Eq. (2.28) with its complex conjugate and reach the same conclusion.]

From now on, we shall assume replacements like that of (2.29) have been performed, so that, for no $\zeta_0 \neq 0$ is $R_N(\zeta_0,\theta,\eta) \equiv 0$ (similarly, for θ_0, η_0).

I. Description of the zeros of $R(s,t,u)$

To proceed, we need some definitions. A polynomial $\pi(\zeta,\theta)$ is said to be irreducible (over the complex numbers) if there are no two other polynomials $\pi_1(\zeta,\theta)$ and $\pi_2(\zeta,\theta)$ so that $\pi=\pi_1\pi_2$. It is easy to show the following statements concerning irreducible polynomials:

(i) Two irreducible polynomials coincide up to a constant factor if they have a common root $\theta=\theta(\zeta)$ [or $\zeta=\zeta(\theta)$] on no matter how small a continuum in ζ (or θ).

(ii) Any polynomial $P(\zeta,\theta)$ has a unique decomposition (up to order and constants) in irreducible polynomials. These statements are proved in Appendix E.

Given a polynomial $P(\zeta,\theta)$, we call $Z(P)$ the set of points $(\zeta_0,\theta_0)\in\mathbb{C}^2$ which are such that $P(\zeta_0,\theta_0)=0$ and refer to it as the set of zeros of $P(\zeta,\theta)$. We call further Z_N the set of points of $\mathcal{S}$, Eq. (2.3), where $R_N(\zeta,\theta,\eta)$, Eq. (2.24), vanishes. It is convenient to consider the projections $Z_N^\eta, Z_N^\zeta, Z_N^\theta$ on the planes $\eta=0$, $\zeta=0$, $\theta=0$. Obviously, the coordinates in $\mathbb{C}^3$ of any point of Z_N are completely known if we know two of its projections.

We can now show the following:

(A2) The set Z_N^η is a finite union of sets $Z(P^{(i)})$, where $P^{(i)}(\zeta,\theta)$, $i=1,2,\ldots$ are irreducible polynomials of ζ and θ.

To prove this, we notice that the set Z_N^η is identical to the set of points (ζ,θ) where the function

$$P(\zeta,\theta)=R_N(\zeta,\theta,\eta_{\rm I})R_N(\zeta,\theta,\eta_{\rm II}) \qquad (2.30)$$

vanishes. In (2.30) the indices I and II denote the two possible values of η, at fixed ζ,θ, according to (2.3). The function $P(\zeta,\theta)$ is a polynomial in ζ,θ; in our special case it is even a polynomial in s,t, as one verifies from the definition (2.24) (notice $\eta_{\rm II}=-\eta_{\rm I}$).

$$P(\zeta,\theta)\equiv P(s,t)=h^2(s,t)-(4m^2-s)(4m^2-t)(4m^2-u)g^2(s,t)\,. \qquad (2.31)$$

This polynomial admits in general of a decomposition in irreducible factors, when it is expressed as a function of ζ,θ. We call the sets of zeros of these polynomials $Z(P^{(k)})$ $k=1,2,\ldots$. So, we see that $Z_N^\eta=\cup_k Z(P^{(k)})$, and the result is proved.

Clearly, statements similar to (A2) are true also for Z_N^θ and Z_N^ζ. The families of irreducible polynomials whose sets of zeros exhaust Z_N^η, Z_N^ζ, and Z_N^θ are denoted by F_N^η, F_N^ζ, and F_N^θ, respectively. We count each distinct irreducible polynomial only once, and we omit from these families the possible polynomials: ζ, θ, η, $\theta^2+\eta^2-8m^2$, $\zeta^2+\eta^2-8m^2$, and $\zeta^2+\theta^2-8m^2$, since they do not correspond to zeros of R, according to (A1). Polynomials of F_N^η, F_N^θ, and F_N^ζ are denoted by P^η, P^θ, and P^ζ, respectively.

J. Comments on statements (A1) and (A2)

Now follow some comments concerning statements (A1) and (A2).

(a) It follows from (A1) that all polynomials of the families F_N^η, F_N^ζ, and F_N^θ depend effectively on both their variables. Indeed, if it were not so, their zeros would be lines $\zeta=$const, $\theta=$const, and $\eta=$const, where R_N would vanish, contrary to our assumption, following (A1). As a consequence, except for a finite number of points of the (ζ,θ) plane, we can always solve the equation $P^\eta(\zeta,\theta)=0$, both with respect to ζ and θ (see Appendix E). We shall make use of this observation in the next paragraph.

(b) Consider a polynomial $P_1^\eta(\zeta,\theta)$. Since $P_1^\eta\neq\zeta^2+\theta^2-8m^2$, we can find points $(\bar\zeta,\bar\theta)$ of $Z(P_1^\eta)$ which are such that the functions $\eta(\zeta,\theta)=\pm(8m^2-\zeta^2-\theta^2)^{1/2}$ are holomorphic in the neighborhood $U_{\bar\zeta}\times U_{\bar\theta}$ of $(\bar\zeta,\bar\theta)$. We further assume $(\bar\zeta,\bar\theta)$ is such that we can solve in its neighborhood the equation $P_1^\eta(\zeta,\theta)=0$ and get in $U_{\bar\theta}$ the holomorphic functions of θ: $\bar\eta_{1,2}(\theta)=\pm[8m^2-\zeta^2(\theta)-\theta^2]^{1/2}$. Clearly, the points $[\theta,\bar\eta_i(\theta)]$ ($i=1$ or 2 or both) belong to Z_N^ζ (for $\theta\in U_{\bar\theta}$). They must be contained in the set of zeros $Z(P_1^\zeta)$ of some irreducible polynomial $P_1^\zeta(\theta,\eta)$, according to (A2). We associate this way to each polynomial P_1^η at least one polynomial $P_1^\zeta(\theta,\eta)$ with the following property: to any branch $\zeta=\zeta(\theta)$ of $P_1^\eta(\zeta,\theta)=0$, there exists a branch $\eta=\eta(\theta)$ of $P_1^\zeta(\theta,\eta)=0$ so that

$$\zeta^2(\theta)+\theta^2+\eta^2(\theta)\equiv 8m^2\,. \qquad (2.32)$$

Since all polynomials in F_N^η are by construction distinct, we can find at most two such polynomials in F_N^ζ (corresponding to $\bar\eta_1$ and $\bar\eta_2$) and we are sure to find at least one. Since F_N^ζ and F_N^η are finite, we can list the set of all "compatible" pairs of polynomials (P^η,P^ζ). The coordinates of the sets of zeros of these polynomials completely describe Z_N in $\mathbb{C}^3$ [if the branches are appropriately paired, to satisfy (2.32)]. We refer to the set of points (ζ,θ,η) of Z_N, obtained by equating to zero the members of the pair (P^η,P^ζ) and matching the branches correctly, according to (2.32), as the "zero trajectory described by (P^η,P^ζ)."

The projection of the zero trajectory on the plane $\theta=0$ is described by an irreducible polynomial P^θ, which we can add, for symmetry, to the description of the trajectory. It is known up to a constant, if P^η and P^ζ are known.

(c) Consider a point $(\zeta,\theta,\eta)\in\mathcal{S}$, so that $R_N(\zeta,\theta,\eta)=0$ and so that we can solve Eq. (2.3) with respect to η, say. Consider then the function $\bar R_N(\zeta,\theta)=R_N(\zeta,\theta,\eta(\zeta,\theta))$. According to (A2) we can find in F_N^η irreducible polynomials $P_i^\eta(\zeta,\theta)$ so that $\bar R_N(\zeta,\theta)/\prod_{i=1}^m(P_i^\eta(\zeta,\theta))^{k_i}$ is holomorphic and nonvanishing in a neighborhood $U_\zeta \times U_\theta$ of (ζ, θ). This is intuitively obvious; one can justify this completely by means of Weierstrass's preparation theorem applied to $\overline{R}_N$ and P_i^η (see Appendix F). The index k_i is called the multiplicity of $P_i^\eta(\zeta, \theta)$.

It is easy to convince oneself that (a) the notion of multiplicity is independent of the point (ζ, θ), and (b) that the polynomial $P_i^\theta(\zeta, \eta)$ which describes, together with $P_i^\eta(\zeta, \theta)$, a zero trajectory, has the same multiplicity as $P_i^\eta(\zeta, \theta)$. This latter statement allows one to talk about the multiplicity of a trajectory. For completeness, we prove this in Appendix F.

K. Statement (A3) and proof

In this paragraph, we wish to prove the following statement:

(A3) If R is of the form (2.24) and has no zeros in the set $\partial_1\mathfrak{M} = \partial_\zeta\mathfrak{M} \cup \partial_\theta\mathfrak{M} \cup \partial_\eta\mathfrak{M}$ as a function of the corresponding variable, then R has no zeros in the whole Mandelstam domain [Eq. (2.2) and (2.3)].

We call $Z_N^{\mathfrak{M}}$ the set of trajectories of Z_N which contains points of $\mathfrak{M}$. If $Z_N^{\mathfrak{M}}$ is void, the proof is finished. If it is not void, we prove the following statement:

(A3.1) Let $(\overline{P}_0^\eta, \overline{P}_0^\theta)$ be a zero trajectory of $Z_N^{\mathfrak{M}}$. If R, Eq. (2.24), has no zeros in $\partial_1\mathfrak{M}$, as a function of the variable corresponding to each subset, then $(\overline{P}_0^\eta, \overline{P}_0^\theta)$ is also a zero trajectory of the denominator R_D of R. Moreover, the trajectories have the same multiplicity in R_N and R_D.

Proof: Since $(\overline{P}_0^\eta, \overline{P}_0^\theta) \in Z_N^{\mathfrak{M}}$, there exists a point $(\overline{\zeta}_0, \overline{\theta}_0, \overline{\eta}_0) \in \mathfrak{M}$ so that $\overline{P}_0^\eta(\overline{\zeta}_0, \overline{\theta}_0) = 0$, $\overline{P}_0^\theta(\overline{\zeta}_0, \overline{\eta}_0) = 0$. Recall $\mathrm{Re}\overline{\zeta}_0 < 0$, $\mathrm{Re}\overline{\theta}_0 < 0$, $\mathrm{Re}\overline{\eta}_0 < 0$. It follows $\overline{P}_0^\eta \in F_N^\eta$, $\overline{P}_0^\theta \in F_N^\theta$. We assume for simplicity that the multiplicity of the trajectory is one. It is very easy to allow for higher multiplicities and we show this at the end of the proof. Because $\mathfrak{M}$ is open, and the zero trajectory does not consist of isolated points, we can even choose $(\overline{\zeta}_0, \overline{\theta}_0, \overline{\eta}_0)$ so that we can solve in its neighborhood, in a regular way, the equations $\overline{P}_0^\eta(\zeta, \theta) = 0$, $\overline{P}_0^\theta(\zeta, \eta) = 0$ with respect to all variables [cf. Sec. II J, comment (a)]. Let $\theta(\zeta)$, $\eta(\zeta)$ be the two solutions of these equations, holomorphic in a neighborhood $U_{\overline{\zeta}_0}$ of $\overline{\zeta}_0$.

We join $\overline{\zeta}_0$ by a curve $\mathfrak{C}$ to a point $\overline{\zeta}$ lying on the boundary $\mathrm{Re}\zeta = 0$, such that (a) $\mathrm{Im}\overline{\zeta} \neq 0$, and (b) the sign of $\mathrm{Im}\overline{\zeta}$ is the same as that of $\mathrm{Im}\overline{\zeta}_0$ (if $\mathrm{Im}\overline{\zeta}_0 = 0$, the sign of $\mathrm{Im}\overline{\zeta}$ is irrelevant). The curve $\mathfrak{C}$ is subjected to the conditions that (i) it lie completely in $\mathrm{Re}\zeta < 0$; (ii) $\mathrm{Im}\zeta$ does not change sign along it; (iii) it avoids the finite number of branching points of the functions $\theta(\zeta)$, $\eta(\zeta)$ and the images of the branching points of their inverses; (iv) it avoids those points ζ corresponding by (2.3) to points (θ, η) for which $\overline{P}_0^\zeta(\theta, \eta) = 0$ cannot be solved with respect to θ or η ($\overline{P}_0^\zeta$ is uniquely determined by $\overline{P}_0^\theta$ and $\overline{P}_0^\eta$); (v) it avoids those points of the ζ plane where the coefficient of the highest power of θ and η in $\overline{P}_0^\eta$ and $\overline{P}_0^\theta$, respectively, vanishes; these coefficients are polynomials in ζ and at those points the functions $\theta(\zeta)$ and $\eta(\zeta)$ become infinite. There are only a finite number of points which we must avoid, and so $\mathfrak{C}$ can be constructed.

We follow $\theta(\zeta)$ analytically along this curve, as well as $\eta(\zeta)$. In this process, we come to a point ζ_1 where ζ, θ, or η (or pairs of them) reaches the line $\mathrm{Re}\zeta = 0$, $\mathrm{Re}\theta = 0$, or $\mathrm{Re}\eta = 0$.

(a) We discuss first the case when, e.g., we reach the line $\mathrm{Re}\theta = 0$ at $\theta = \theta_1$ and at that point $\mathrm{Re}\zeta_1 < 0$, $\mathrm{Re}\eta_1 \equiv \mathrm{Re}\eta(\zeta_1) < 0$. We conclude that we have generated, by analytic continuation, a zero of $R_N(\zeta, \theta, \eta)$ on the physical sheet of the cosine plane, for a value θ, lying on $\mathrm{Re}\theta = 0$ (i.e., at a point of $\partial_\theta\mathfrak{M}$). It is here that Mandelstam analyticity plays a crucial role. This is not true for the axiomatic domain, since we are not sure that we can choose the curve $\mathfrak{C}$ in such a way that the analytic continuation of $\theta(\zeta)$ and $\eta(\zeta)$ starting from an interior zero leads to zeros lying in the Lehmann ellipse, as we reach the lines $\mathrm{Re}\theta = 0$, $\mathrm{Re}\eta = 0$, or $\mathrm{Re}\zeta = 0$.

Now, a zero of $R(s, t, u)$ on the physical sheet of the cosine plane (in $\partial_\theta\mathfrak{M}$) is forbidden by assumption, and the only way to accommodate the zero of $R_N(\zeta, \theta, \eta)$ which we have obtained is to assume that $R_D(\zeta, \theta, \eta)$ has a zero at the same position. Because the zero trajectory is of multiplicity one, the zero of R_N on $\partial_\theta\mathfrak{M}$ must be simple and so must be the zero of $R_D(\zeta, \theta, \eta)$.

Let then $\overline{\overline{P}}_0^\eta(\zeta, \theta)$ and $\overline{\overline{P}}_0^\zeta(\theta, \eta)$ be the irreducible polynomials which are responsible for the zero of $R_D(\zeta, \theta, \eta)$. By changing the curve $\mathfrak{C}$ slightly, we see there is a whole neighborhood of values of θ around θ_1 for which a zero of $\overline{P}_0^\eta(\zeta, \theta)$ is the same as a zero of $\overline{\overline{P}}_0^\eta(\zeta, \theta)$. The same is true for $\overline{\overline{P}}_0^\zeta$. Because $\overline{P}_0^\eta, \overline{\overline{P}}_0^\eta, \overline{P}_0^\zeta, \overline{\overline{P}}_0^\zeta$ are irreducible, we conclude, by (i) that $\overline{P}_0^\eta \equiv \overline{\overline{P}}_0^\eta$ and $\overline{P}_0^\zeta \equiv \overline{\overline{P}}_0^\zeta$ up to constants. So, statement (A3.1) is proven for case (a).

(b) We now assume that a pair of $(\zeta, \theta(\zeta))$ or $(\zeta, \eta(\zeta))$ values reach simultaneously the lines $\mathrm{Re}\zeta = 0$ and $\mathrm{Re}\theta = 0$ at ζ_1 and θ_1 (or the lines $\mathrm{Re}\zeta = 0$, $\mathrm{Re}\eta = 0$). Let us notice that it is impossible to have $\mathrm{Re}\zeta < 0$, $\mathrm{Re}\theta = 0$, and $\mathrm{Re}\eta = 0$ along $\mathfrak{C}$; the latter two equalities and (2.3) imply $\mathrm{Im}\zeta = 0$, which is outside $\mathfrak{C}$.

We can now modify the curve $\mathfrak{C}$, so that it joins $\overline{\zeta}_0$ with points ζ_1^i in an interval $(i\lambda, i\mu)$ with λ, μ real around ζ_1, and perform the same continuation process as before, starting from $(\overline{\zeta}_0, \overline{\theta}_0, \overline{\eta}_0)$. It could happen that in this way we come across situations similar to case (a), that is, we generate zeros in the interior of the cut cosine plane, corresponding to $\operatorname{Re}\zeta=0$, $\operatorname{Re}\eta=0$, or $\operatorname{Re}\theta=0$. Then (A3.1) is again proved.

It can happen, however, that $\operatorname{Re}\theta(\zeta)=0$ as long as $\operatorname{Re}\zeta=0$. Then we recall that R has no zeros even on $\operatorname{Re}\zeta=0$ and $\operatorname{Re}\theta=0$ [cf. the definition of $\partial_\zeta\mathfrak{M}$, Eq. (2.8)]. We can then repeat the reasoning of case (a), concerning the necessary coincidence of zeros of R_N and R_D, and reach the same conclusion.

Let us notice that it is impossible to have simultaneously $\operatorname{Re}\zeta=0$, $\operatorname{Re}\theta=0$, and $\operatorname{Re}\eta=0$, by Eq. (2.3), so that case (b) exhausts all possibilities.

If $(\overline{P}_0^\eta, \overline{P}_0^\theta)$ had a multiplicity m higher than one, we would have obtained, by continuing along $\mathfrak{C}$, a zero of order m for $R_N(\zeta,\theta,\eta(\zeta,\theta))$ at $\theta=\theta_1$ on $\operatorname{Re}\theta=0$. We conclude that R_D must have a zero of the same order in an interval $(i\lambda, i\mu)$ around θ_1. This is possible only if the trajectory of R_D has the same multiplicity m. This concludes the proof of (A3.1).

The following statement is now obvious, using the definition of multiplicity of a trajectory, of Sec. II J:

(A3.2) If the sets $Z_N^{\mathfrak{M}}$; $Z_D^{\mathfrak{M}}$ of zeros of R_N, R_D, lying inside the Mandelstam domain, contain the same zero trajectories, with the same multiplicities, then $R=R_N/R_D$ has no zeros in the Mandelstam domain.

If we repeat the reasonings of (A3) for the function $1/R$ we conclude:

(A4) If R is of the form (2.24) and has no poles in the set $\partial_\zeta\mathfrak{M}\cup\partial_\theta\mathfrak{M}\cup\partial_\eta\mathfrak{M}$ (as a function of one variable) then it has no poles in $\mathfrak{M}$.

L. Conclusion of proof of (A)

We can now bring the proof of (A) to an end and so prove the theorem of Sec. II A completely. It is true that:

(A5) If $R(s,t,u)$ is of the form (2.24) and has neither zeros nor poles in $\mathfrak{M}$, then $R=\pm1$.

Proof: Consider a real value of $t\in I_t$. The function $R_t(s)$ has neither zeros nor poles in the corresponding open cut s plane. Indeed, all the points belonging to this set also belong to $\mathfrak{M}$. On the other hand, we have seen in Sec. II E that, by means of the mapping $z_t(s)$, Eq. (2.14), we can write $R_t(s)=\pm B_{1t}(z_t(s))/B_{2t}(z_t(s))$ [cf. Eq. (2.19)], with $B_{it}(z_t)$, $i=1,2$ given by Eq. (2.17). All the zeros and poles of $R_t(s)$ in the cut complex s plane are given by the noncoincident zeros of $B_{it}(z_t)$, $i=1,2$. Since $R_t(s)$ has no such zeros and poles, it follows that $B_{1t}(z_t)=B_{2t}(z_t)$. So, $R_t(s)=\pm1$ for all $t\in I_t$. We conclude then, by analytic continuation, that

$$R(s,t,u)\equiv\pm1\,. \tag{2.33}$$

The proof of the theorem of Sec. II A is finished.

M. Comments on the theorem of Sec. II A

In this subsection, we comment on the theorem that has been proved. We define, also for the next chapter, the class C of amplitudes as those complex functions of two variables which obey conditions (a), (c), (d), (e), and (f) of the theorem of Sec. II A. The statement of Sec. II A says then that, within C, the determination of the phase from the modulus is equivalent to the resolution of the discrete ambiguity at all energies in all channels.

It might not yet be clear why we talk about discrete ambiguities for zeros lying on the cut. The following statement, which is a simple consequence of (2.24), answers this question.

If the modulus $M_{1s}(t)$ in Eq. (2.5) has a simple zero on the cut (see Sec. II B) at a point $t_0+i\epsilon$, and A_1, A_2 are two amplitudes of class C having the same modulus in the physical region of the three channels, then only the following situations can occur at fixed energy: (α) $\lim_{t\to t_0+i\epsilon}A_{1s}(t)/A_{2s}(t)=\text{const}\neq0$; ($\beta$) $t_0+i\epsilon$ is a first-order zero of A_{2s}, $A_{2s}/A_{1s}=\text{const}\times(t-t_0)$, const $\neq0$ and $t_0-i\epsilon$ is a first-order zero of A_{1s}, $A_{1s}/A_{2s}=\text{const}\times(t-t_0)$, const $\neq0$; (γ) situation (β) with A_1, A_2, interchanged (t_0 real).

To prove this, we notice first that $A_{1s}(t)$ cannot have a second-order zero at $t_0+i\epsilon$. Indeed,

$$\lim_{t\to t_0+i\epsilon} M_{1s}/(t-t_0)^2=\lim_{t\to t_0+i\epsilon}A_{1s}(t)A_{1s}^*(t^*)/(t-t_0)^2 =\infty\,,$$

by assumption, and $\lim_{t\to t_0+i\epsilon}A_{1s}^*(t^*)$ is finite. So, $\lim_{t\to t_0+i\epsilon}A_{1s}(t)/(t-t_0)^2$ cannot be finite. The same is true for A_{2s}. A_{1s} cannot have a second-order zero at $t_0-i\epsilon$, either, and the same holds for A_{2s}. We now define $R=A_1/A_2$ and recall that R is given by (2.24). We show that $R_s(t)$ cannot have a second-order pole at $t_0+i\epsilon$. Indeed, assume it had one, $R_s(t)=\overline{R}_s(t)/(t-t_0)^2$, with $\overline{R}_s(t_0+i\epsilon)\neq0$. But, on one hand,

$$\lim_{t\to t_0+i\epsilon}A_{1s}(t)=\lim_{t\to t_0+i\epsilon}[A_{2s}(t)R_s(t)]$$

is finite, and on the other hand,

$$\lim_{t\to t_0+i\epsilon}A_{2s}(t)R_s(t)=\lim_{t\to t_0+i\epsilon}A_{2s}(t)\overline{R}_s(t)/(t-t_0)^2=\infty\,,$$

since $A_{2s}(t)$ cannot have a second-order zero at $t_0+i\epsilon$. So, we have reached a contradiction, and $R_s(t)$ cannot have a pole of order higher than one at $t_0+i\epsilon$. The same is true if we replace $t_0+i\epsilon$ by $t_0-i\epsilon$. Since we also know that $R_s(t_0\pm i\epsilon)=1/R_s^*(t_0\mp i\epsilon)$, for $s=s_0+i\epsilon$, $s_0>4m^2$ it follows that R_s cannot have a double zero at $t_0\pm i\epsilon$. So, let us assume that $R_s(t)$ has a simple pole at $t_0+i\epsilon$.

Then,

$$\lim_{t\to t_0+i\epsilon} A_{1s}(t) = \lim_{t\to t_0+i\epsilon} A_{2s}(t)R_s(t) = \text{finite}$$

implies

$$\lim_{t\to t_0+i\epsilon} A_{2s}(t)/(t-t_0) = \text{finite}$$

so that A_{2s} has a simple zero at $t_0+i\epsilon$. Clearly, $A_{2s}(t)/A_{1s}(t) = \text{const}\times(t-t_0)$, $\text{const}\neq 0$. Since R has a zero at $t_0-i\epsilon$, $1/R$ has a pole. Interchanging $A_{1s}(t)$ and $A_{2s}(t)$, we conclude that A_{1s} has a simple zero at $t_0-i\epsilon$, and $A_{1s}(t)/A_{2s}(t) = \text{const}\times(t-t_0)$, $\text{const}\neq 0$. So, situation (β) occurs. One verifies that, if $R_s(t)$ has a zero at $t_0+i\epsilon$, situation (γ) occurs. Finally, if $R_s(t)$ is finite at t_0, and $R_s(t_0+i\epsilon)\neq 0$, situation (α) occurs, and this way the statement is proved.

The three possibilities enumerated in this statement also occur if $M_{1s}(t)$ has a zero lying in the interior of the open cut t plane. This might justify the formulation "discrete ambiguity of zeros lying on the cut" of Sec. II A.

N. Role of conditions (c), (d), and (e)

In this subsection, we discuss the role played by conditions (c), (d), and (e) of Sec. II A. We show, namely, that they cannot be completely abandoned, and the result be preserved.

If conditions (d) and (e) concerning the finiteness of the phase on the cut and the polynomial boundedness of the amplitude, in some interval of real s and t values $[\subset(-4m^2, 4m^2)]$, are removed, a simple example of an ambiguity which could appear is

$$\mathfrak{A}(s,t,u) = \exp\{[(4m^2-s)(4m^2-t)(4m^2-u)]^{1/2} \times f(s,t,u)\}, \qquad (2.34)$$

where $f(s,t,u)$ is a real entire function of s, t, u. The function $\mathfrak{A}(s,t,u)$ has modulus one at all points of the physical region, no zeros and poles in the whole Mandelstam domain, and is not of the form (2.24).

(b) The condition concerning the Hölder continuity of the phase in (d) can certainly be relaxed to functions $\phi_{1t}(\nu)$ [Eq. (2.9)] for which the limits of the integrals in (2.9) exist almost everywhere as we approach the cuts of the ν plane. This is, however, an uninteresting direction of refinement.

(c) It is possible to restrict the validity of condition (c) (concerning the continuity of the amplitude and of its phase on the cuts of the complex t plane) to an interval of values of s in the physical region ($s=s_0+i\epsilon$, $s_0>4m^2$). Indeed, we can choose the curve $\mathfrak{C}$ in (A3.1) so that the point $\bar{\zeta}$ (its end on $\text{Re}\zeta=0$) lies in the interval of values of s we choose. This remark justifies the statement at the end of Sec. II G. However, a device must exist to forbid R to develop freely poles and zeros on the cuts of the cosine planes. These belong to the boundary of the Mandelstam domain, and we expect zeros and poles that are present there to propagate also inside the domain. The fact that f in (2.7) is a rational function has allowed us to reduce this interdiction to an interval of values of s only.

A simple example which shows the effect of poles and zeros that are freely allowed on the boundary has been given to the author by Professor A. Martin:

$$\mathfrak{A}_\lambda(s,t,u) = \frac{\lambda+[(4m^2-s)(4m^2-t)(4m^2-u)]^{1/2}}{\lambda-[(4m^2-s)(4m^2-t)(4m^2-u)]^{1/2}}, \qquad (2.35)$$

with λ a real number. These functions have modulus one in the physical region and, for all physical s, have a zero and a pole on each of the t and u cuts. The poles of $\mathfrak{A}_\lambda$ stay outside the Mandelstam domain for $\lambda>0$.

III. ON THE DISCRETE AMBIGUITY

A. Irreducible ambiguities

We have seen in the previous section that two amplitudes of class C (defined in Sec. II M), having the same modulus in the three physical channels can be different only if the discrete ambiguity of the zeros has been resolved in different ways in some interval of energies, in some channel. (Strictly speaking, we can state so far that, if $R\neq\pm 1$, there must exist a value of s, t, or u in the physical region of some channel, for which the discrete ambiguity has been resolved in different ways. The fact that there must be a whole interval is shown at the end of this subsection, but is intuitively obvious.) We go from one possible resolution of the discrete ambiguity in the amplitude A_1 to another resolution in A_2, by means of the ratio $R(s,t,u)=A_1/A_2$ given in Eq. (2.24). For fixed, physical values of one variable (i.e., $\text{Re}\zeta=0$, $\text{Re}\theta=0$, or $\text{Re}\eta=0$), the positions of the zeros of the ratio R (regarded as a function of the other variable) are complex conjugate to those of its poles. The poles of the denominator of R cancel some complex zeros of the amplitude A_2 in some interval of energies, and the numerator of R creates new zeros in the complex-conjugate positions, so that in $A_1=RA_2$, the discrete ambiguity of the zeros is resolved in a different way from A_2.

Clearly, the complex positions of the zeros that are "reflected" by $R(s,t,u)$ are determined (up to a reflection) by the modulus of the amplitude. We expect, therefore, that, given the modulus of the amplitudes of class C in the physical region, we should also be able to find those functions R which constitute the possible ambiguities in the determination of the phase. The problem we would like to solve now is then: Knowing the complex zeros of $R(s,t,u)$ which lie in the sets $\partial_\zeta \mathfrak{M}$, $\partial_\theta \mathfrak{M}$, and $\partial_\eta \mathfrak{M}$ (Sec. II B), where R is regarded as a function of one variable (appropriate to each set), how can one determine R? So far, we only know that, if there are no such zeros and poles, $R=\pm 1$ (theorem of previous section).

In this subsection, we shall give a construction which, in principle at least, solves this problem explicitly. To this end, we introduce some definitions; we call functions like $R(s,t,u)$, which can be written in the form (2.24), "ambiguities." An ambiguity R is said to be irreducible if there are no two other ambiguities R_1, R_2 so that $R=R_1R_2$. We shall prove now the following statements:

(i) An irreducible ambiguity is determined (up to a sign) by one of its zeros (or poles) as a function of one variable on an interval of "physical" values of ζ, θ, or η, no matter how small. (These are such that $\mathrm{Re}\zeta=0$, $\mathrm{Re}\theta=0$, $\mathrm{Re}\eta=0$.)
(ii) Any ambiguity R can be uniquely written as a product of irreducible ambiguities (up to a sign).

To prove (i), we take R irreducible, write as in (2.24) $R=R_N/R_D$, and assume that there exists an interval $I=(i\lambda, i\mu)$, λ, μ real, of values of ζ, where we know functions $\theta_0(\zeta)$ and $\eta_0(\zeta)$ so that

$$R_N(\zeta, \theta_0(\zeta), \eta_0(\zeta)) \equiv 0\,, \quad \text{for } \zeta \in I\,. \tag{3.1}$$

We next construct families $Z_{0,N}, Z_{0,D}$ of zero trajectories (see Sec. II J) which are necessarily contained in the sets Z_N, Z_D of zeros of R_N, R_D, respectively, if (3.1) holds. We first construct the irreducible polynomials $P^\eta_{0,N}(\zeta,\theta)$, $P^\theta_{0,N}(\zeta,\eta)$ whose sets of zeros coincide with the analytic continuation of $\theta_0(\zeta)$, $\eta_0(\zeta)$ (see Appendix E for this construction). Since the lines $\zeta=\text{const}$, $\theta=\text{const}$, or $\eta=\text{const}$ are not among the zeros of R [cf. (A1), Sec. II H], we can solve arbitrarily the equations $P^\eta_{0,N}=0$, $P^\theta_{0,N}=0$ with respect to ζ, θ, or η (except for a finite number of points). It is convenient, for symmetry, to add to the description of the trajectory the irreducible polynomial $P^\zeta_{0,N}(\theta,\eta)$, which is completely determined by the first two, and whose set of zeros is the projection of the trajectory on $\theta=0$ (cf. Sec. II H). Since $|R(\zeta,\theta,\eta)| =1$ for $\zeta\in I$, θ,η real, one concludes that the denominator R_D must vanish along the zero trajectory described by $P^\eta_{0,D}(\zeta,\theta)\equiv P^{*\eta}_{0,N}(-\zeta^*,\theta^*)$, $P^\theta_{0,D} \equiv P^{*\theta}_{0,N}(-\zeta^*,\eta^*)$ [to which we add the corresponding $P^\zeta_{0,D}(\theta,\eta)$].

Since R is real analytic, if the trajectory $[P^\eta_{0,N}(\zeta,\theta), P^\theta_{0,N}(\zeta,\eta), P^\zeta_{0,N}(\theta,\eta)]\in Z_{0,N}$, then the trajectory $[P^{*\eta}_{0,N}(\zeta^*,\theta^*), P^{*\theta}_{0,N}(\zeta^*,\eta^*), P^{*\zeta}_{0,N}(\theta^*,\eta^*)]$ must also belong to $Z_{0,N}$, if it is not identical to the former. An analogous statement is true for $Z_{0,D}$.

Further, since $|R(\zeta,\theta,\eta)|=1$ for $\mathrm{Re}\theta=0$, ζ,η real, it follows that $Z_{0,N}$ must contain the zero trajectories described by $P^\eta_{1,N}(\zeta,\theta)=P^{*\eta}_{0,D}(\zeta^*,-\theta^*)$, $P^\zeta_{1,N}(\theta,\eta)=P^{*\zeta}_{0,D}(-\theta^*,\zeta^*)$. It could be that $P^\eta_{1,N} \equiv P^\eta_{0,N}$ and $P^\zeta_{1,N}\equiv P^\zeta_{0,N}$. If this is not the case, we must add this new trajectory to $Z_{0,N}$. We continue this process at physical values of η ($\mathrm{Re}\eta=0$) and add again new trajectories to $Z_{0,N}$, if we find that those already contained in it are not complex conjugate to those of the denominator. We must also check the consistency of $Z_{0,N}, Z_{0,D}$ with real analyticity and possibly add new trajectories to restore consistency. We then go back to the s channel, etc.

This inclusion of new trajectories in $Z_{0,N}, Z_{0,D}$ must stop after a finite number of steps (i.e., we find consistency of the trajectories of $Z_{0,N}, Z_{0,D}$ with real analyticity and the unit modulus of R in the physical region). Indeed, the set Z_N of zeros of R_N contains only a finite number of zero trajectories, by (2.24).

Clearly, the set of zeros described by $Z_{0,N}$ is such that $Z_{0,N}\subset Z_N$. We now construct the function

$$\tilde{R}_0(s,t,u)=\frac{\prod_{i=1}^{N_{\text{tot}}}[\theta_i(\zeta)+i\eta_i(\zeta)-\theta-i\eta]}{\prod_{i=1}^{N_{\text{tot}}}[\theta_i^*(-\zeta^*)+i\eta_i^*(-\zeta^*)-\theta-i\eta]}\,, \tag{3.2}$$

where the product is taken over all zero trajectories of $Z_{0,N}$ and their branches, which are paired according to (2.32). We claim that (a) $\tilde{R}_0$ is uniform and has no other singularities on $\mathcal{S}$ but zeros and poles, (b) it has the reality property on $\mathcal{S}$, (c) it has no other zeros on $\mathcal{S}$ but those of $Z_{0,N}$, (d) its zeros exhaust the trajectories of $Z_{0,N}$, (e) it has modulus one in the physical region of all channels. These claims are justified in Appendix G. The claims (c) and (d) are plausible if one recalls from Sec. II F that the variable $\omega=\theta+i\eta$ makes uniform the two sheets of the θ plane at fixed ζ, $\zeta\neq\pm 2m\sqrt{2}$. The correctness of the claims (a), (b), and (e) is, however, not obvious, at least to the author. He was unable to find anything but a rather lengthy justification, given in detail in Appendix G.

Now, if $\tilde{R}_0$ has the properties (a), (b), and (e) claimed above, it can be written in the form (2.7). This follows from Ref. 11. Since we show in Appendix G that $\tilde{R}_0$ is actually a rational function of ζ,θ,η (in $\mathbf{C}^3$), one concludes that it can even assume the form (2.24). We then see that $R_N(s,t,u)/\tilde{R}_{0,N}(s,t,u)$ has no poles on $\mathcal{S}$, so that

$\tilde{R}_{0,N}$ is actually a divisor of R_N. The ratio $R_1 = R/\tilde{R}_0$ is again real analytic, has modulus one in the physical region, and has no other singularities on $\mathcal{S}$ but poles. Consequently, it is of the form (2.24). We seem to contradict, this way, our assumption that $\tilde{R}$ is irreducible, since we have shown that $R = \tilde{R}_0 R_1$, where both $\tilde{R}_0$ and R_1 are ambiguities. The only way out is that $R_1 = \pm 1$, so that $R = \tilde{R}_0$ (up to a sign). This completes the proof of (i), since $\tilde{R}_0$ is determined starting from $\theta_0(\zeta)$, $\eta_0(\zeta)$ on some interval $(i\lambda, i\mu)$ of values of ζ.

The proof of (ii) seems to be very simple. Namely, we exhaust gradually the sets of zeros Z_N of $R_N(\zeta, \theta, \eta)$ by means of "minimal" families of zero trajectories $Z_{0,N}, Z_{1,N}, \ldots,$ according to the procedure of the previous proof. We construct then the corresponding $\tilde{R}_0, \tilde{R}_1, \tilde{R}_2, \ldots$ by (3.2). Clearly, $R = \tilde{R}_0 \tilde{R}_1 \ldots \tilde{R}_p$ up to a sign. The decomposition is unique, since the partition $Z_N = \cup_i Z_{i,N}$ is unique.

To make use of these statements, let us notice that, if $R \neq \pm 1$, there must exist a whole interval of physical values of ζ (or θ, or η) so that R has zeros in the corresponding cut cosine planes (including the cuts for physical values of ζ). This follows from the proof of the statement of Sec. II A. Indeed, if there had been only a finite number of isolated values of ζ (or θ, or η) for which R had zeros in the corresponding cosine planes, we could have still carried the proof through by suitably choosing the end $\bar{\zeta}$ of the curve $\mathcal{C}$ in (A3.1) to avoid them and get $R = \pm 1$. So there must be an infinite number of such values ζ. But we have only a finite number of irreducible polynomials at our disposal, to exhaust the zeros of R [cf. Eq. (2.31)], and so there must exist one which vanishes on an infinite set $(\zeta_i, \theta(\zeta_i))$, with $\mathrm{Re}\theta(\zeta_i) \leq 0$. Then, however, R vanishes on the whole set of zeros of this polynomial. The latter cannot contain an infinite set of points $(\zeta_i, \theta(\zeta_i))$, with $\mathrm{Re}\theta(\zeta_i) \leq 0$, without the existence of an interval of values ζ for which $\mathrm{Re}\theta(\zeta) \leq 0$ is true.

With this observation, the possible ambiguities R are obtained by simply inspecting all the analytic continuations of the zero trajectories determined by the modulus in the cut cosine plane at fixed energies, in all channels. Those which give rise to ambiguities are such that their analytic continuation is possible and the result consistent with the zero trajectories of an irreducible ambiguity [as described in the proof of statement (i)]. We obtain this way, at least in principle, a set (maybe void) of possible irreducible ambiguities of the type (3.2).

The situation we face is analogous to what would happen if we tried to determine a polynomial amplitude $A_s(z)$ at fixed energy, from data on its modulus. Then only the discrete ambiguity is present; its extent is fixed by the measured modulus and, given a solution, we can obtain another one by multiplying with $(z - z_0^*)/(z - z_0)$, where z_0 is one of the zeros.

In our case, the role of the zeros is played by the irreducible ambiguities, and the assumptions concerning the amplitude are less artificial. Knowing one amplitude of class C compatible with the given modulus, we can obtain a whole set of such amplitudes by multiplication (or division) with products of irreducible ambiguities like (3.2).

Let us also notice that it is by no means clear that a given modulus distribution, which is consistent with an amplitude of class C, allows ambiguities at all. Indeed, we expect in general a zero trajectory $t(s)$ of the amplitude $[A(s, t(s)) \equiv 0]$ to have a branch point at an inelastic threshold. Such trajectories cannot lead, in the limit of exact data on the modulus, to discrete ambiguities. This is so because the function $R(s, t, u)$ of Eq. (2.7) does not contain any inelastic branch points, and so, cannot produce reflections of zero trajectories which do contain one.

Consequently, a small number of zero trajectories will in general be such that their continuations generate irreducible factors like (3.2). It is not obvious that there are any. If there are none, the set of ambiguities is void and there is just one amplitude, compatible with the given modulus.

B. Use of information on number of zeros

So far, we have used in this study only the analyticity in two variables of the amplitude and its reality property. We now study briefly the effect of the statements we have proved before, if we include information on the number of zeros p_0 that lie in the complex s plane, when t is in the interval $0 < t < 4m^2$. If the imaginary part is known to be positive in the interval $0 < t < 4m^2$, then one knows that $p_0 \leq 2$.

By comparison with Eq. (2.20), it is easy to see that, if R in Eq. (2.24) is the ratio of two amplitudes, each having at most p_0 zeros in $0 < t < 4m^2$, then the degree of $h(s, t)$ with respect to s must be at most p_0, whereas the degree of $g(s, t)$ with respect to s is at most $p_0 - 1$. Now, an ambiguity $R(s, t, u)$ having this property cannot contain more than $2p_0$ irreducible ambiguities. This is so because any irreducible ambiguity must have at least a zero or a pole on the physical sheet of the complex s plane, at these values of t. Indeed, if it had none, it would be ± 1 by the reasoning of Sec. II. With this observation, it is convenient to define the degree of an ambiguity with respect to s as D = number of zeros plus number of poles of R on the physical sheet of the s plane, for $t\in(-4m^2, 4m^2)$.

Let us now study the situation $p_0=1$. One can easily show the following: If the amplitude admits of an irreducible ambiguity of degree 2, then it cannot admit of any other one (of any degree). Indeed, assume the contrary, and let the two ambiguities be R, R', where degree $(R)=2$. This means that A, $A_1=AR$, and $A_2=AR'$ are all amplitudes of class C, with $p_0\leq 1$. Then $R_0=A_1/A_2=R/R'$ has, in general, degree greater than 2, if R is irreducible. The only possibility is $R'=\pm1$, and so, $A_1\equiv A_2$ (up to sign).

If the amplitude admits of no irreducible ambiguity of degree 2, it can nevertheless admit of many irreducible ambiguities R_i of degree 1 with respect to s. Indeed, the ratio of any two members of the family $\{AR_i\}$ is R_i/R_j which has degree 2 with respect to s. However, for no other R' can $AR'R_i$ again be an admissible amplitude, in this case $(R'\neq R_i^{-1})$. It is possible that $AR_i'R_0$ be admissible for a whole class of ambiguities R_i' of degree 1 and a fixed R_0 of degree 1 too. Then AR_i' is not admissible, but AR_0 is so.

If $p_0=2$ and if the amplitude admits of an irreducible ambiguity of degree 4, then this is the only ambiguity, by a reasoning similar to that for $p_0=1$. If it admits of an irreducible ambiguity of degree 3, it can still admit of many ambiguities of degree 1, etc.

According to the usual point of view, if there are q zeros at a fixed physical value of t, in the corresponding cut cosine plane, there exists a corresponding 2^q-fold discrete ambiguity for the phase. If the modulus is known in all three channels, the preceding discussion shows that this number is much too large for low values of p_0; many of the solutions obtained by reflecting the zeros will violate two-variable analyticity, the conditions defining the class C or the bound for p_0.

If $p_0=1$, we have seen that we can reflect at most those zeros whose analytic extrapolation gives rise to ambiguities of degree one or two, with respect to s. All the others are fixed. The ambiguities of degree one in s have at most one zero on the physical sheet of the cosine plane at fixed t. So, at most, we are able to reflect either one zero at a time, and keep the others in their original positions, or reflect a pair and then keep fixed one of its members. If the zeros of the pair are correlated, then there is just one ambiguity. The view of this process is more complicated in the s channel. There may be many zeros lying in the cosine plane, at physical values of s, which are correlated to each other and can be reflected only as a whole.

For $p_0=2$, the situation is more complicated, but the discrete ambiguity at fixed energy is again less than 2^q, for $q>4$.

C. Detecting "wrong" resolutions

It follows from the preceding subsection that we cannot arbitrarily reflect zero trajectories of the amplitude without endangering two-variable analyticity, among others of its properties. In this subsection, we shall describe a simple and nonrigorous way of detecting practically "wrong" resolutions of the discrete ambiguity, which lead to violations of two-variable analyticity.

It follows directly from Weierstrass's preparation theorem that, if a function $A(s,t)$ is holomorphic in a complex domain $D=D_s\times D_t$ of $\mathbb{C}^2$, and for each s in D_s, $A(s,t)$ has n zeros lying in D_t (D_t finite), then $A(s,t)$ can be represented in D as

$$\begin{aligned}A(s,t)&=[t^n+A_1(s)t^{n-1}+\cdots+a_n(s)]\Omega(s,t)\\&\equiv W(s,t)\Omega(s,t),\end{aligned}\tag{3.3}$$

where the functions $a_1(s),\ldots,a_n(s)$ are holomorphic in D_s, and $\Omega(s,t)$ is nonvanishing in $D_s\times D_t$.

We assume we can apply (3.3) even if D contains part of the physical region of the s channel. This means that we can continue the amplitude on the second sheet in a certain neighborhood of the physical region. This is the same assumption as in current searches for resonances.

The number of zero trajectories of the amplitude that lie near the physical region and can be detected from data increases with increasing energy, and so we take D_s to contain a limited energy interval of the order 1 GeV/c. It is of interest to consider a domain $D_s\times D_t$ which contains two trajectories of the amplitude lying in the neighborhood of the physical region, approaching each other and then moving apart in t, as the energy increases ("intersect"[8]).

It has been argued in Ref. 8 in detail that the behavior of the trajectories in such a region can be understood if $W(s,t)$ in Eq. (3.3) is near a reducible case, i.e., $W(s,t)=(t-t_1^0(s))(t-t_2^0(s))+\beta(s,t)$ with $t_1^0(s)$ and $t_2^0(s)$ analytic functions of s in D_s, and $\beta(s,t)$ a "small" function. Each zero trajectory has then two branching points in the neighborhood of the physical region [the distance between the two branching points is, roughly speaking, proportional to the magnitude of the perturbation $\beta(s,t)$]. The effect of these singularities is an "oscillation" of the real and imaginary parts of the functions describing the zero trajectories, at energies in the "intersection" region.[8]

These oscillations are clearly seen in the "experimentally" determined zero trajectories. But, according to (3.3), the singularities of $t_1(s)$ and

$t_2(s)$ must cancel each other out when one builds $t_1(s)+t_2(s)=a_1(s)$, $t_1(s)t_2(s)=a_2(s)$. So, crudely speaking, $t_1(s)$ and $t_2(s)$ must oscillate in opposite directions in such an intersection region.

Now, such a cancellation will no longer occur if we replace $t_2(s)$ by $t_2^*(s)$, which would correspond to another choice for the resolution of the discrete ambiguity. This one can check by explicitly solving the second-order equation in t obtained by setting the (pseudo) polynomial $W(s,t)$ in (3.3) equal to zero. The oscillations of the zero trajectories will reinforce each other in the neighborhood of the singularity instead of destroying themselves. The same happens if we let t_2 remain unchanged but replace $t_1(s)$ by $t_1^*(s^*)$. We shall not be able, however, to eliminate the situation when both $t_1(s)$ and $t_2(s)$ are reflected with respect to their true position, since $t_1^*(s)+t_2^*(s)$ and $t_1^*(s)\,t_2^*(s)$ can be analytically extended to a domain of the same size as that where $t_1(s)+t_2(s)$ and $t_1(s)t_2(s)$ are analytic (but reflected with respect to the real axis). Continuity arguments might help in discarding this situation. In Refs. 9 and 10 applications to practical situations in πN and πK scattering are described.

Clearly, to establish the presence of a singularity of the $a_i(s)$, $i=1,2$, one uses analyticity tests of the type described in Ref. 17. These produce results which have only a limited amount of certainty, when the errors of the data are finite. The coherence of smoothness in both $t_1(s)+t_2(s)$, and $t_1(s)t_2(s)$ for the correct resolution of the discrete ambiguity adds weight, however, to the analyticity test.

IV. CONCLUSIONS

The main result of this paper is that, for a large class of amplitudes, the ambiguity which appears if one tries to construct the phase from the available modulus in all three physical channels is of a discrete type. This means that, if we know an amplitude (of class C, see Sec. II M), we cannot obtain another one from it, having the same modulus in the three physical channels, by an "infinitesimal displacement" [analogous to a small change of the function $O(z)$ in (1.2)], but rather by the reflection of a zero trajectory on some interval of energies in some channel. This result came to the author as a surprise, since it differs sensibly from what one knows from the study of ambiguities in one variable (fixed energy or fixed momentum transfer).

We could also see that the pattern of zero trajectories determined by the modulus of the amplitude enjoys a certain rigidity, and does not allow in general for reflections, as is commonly assumed. Analyticity in two variables or positivity is violated by most of these reflections.

There are a number of mathematical questions which are still unsolved. The first one is the complete description of ambiguities in the Mandelstam domain. One would, namely, like to know the extent of the ambiguity appearing if one relaxes conditions (c), (d), (e), and (f) of Sec. II A. What is the general form of ambiguities having the properties of (2.35)? Is it true that Eq. (2.34) times such factors gives the whole ambiguity?

A rather strange feature of the theorem of Sec. II A is that it depends essentially on the supposed analyticity of the amplitude in the Mandelstam domain, rather than in the axiomatic domain. We could see in the proof (Sec. II K) where this difference actually comes in, but the present author is unable to state a deeper reason why this is the case.

From a practical point of view, it would be interesting to understand the ambiguity allowed by two-variable analyticity, knowledge of the modulus in only two channels, and positivity. This question has been studied in Ref. 11. However, at first sight it seems that the answer given there—namely, the dependence of the ratio of two amplitudes satisfying these conditions on a certain number of arbitrary functions—does not take into account the fact that the modulus itself determines to some extent the set of zeros lying in the complex s plane at fixed real t. It is not clear to the present author whether a refinement is possible; he hopes, however, to return to some of these questions in the future.

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In the course of this work, I had the opportunity of having a discussion with Professor A. Martin, who indicated to me a number of useful references (in particular, Ref. 14), helped me realize the importance of the continuity condition on the cuts of the cosine plane in Sec. II A [by means of example (2.35)], as well as remove an incorrect statement in an earlier version of this paper, by placing it under doubt. I wish to express my gratitude to him. The author gratefully acknowledges the financial support of the Deutsche Forschungsgemeinschaft for the period during which this research was carried out.

APPENDIX A: THE ORDER OF A ZERO LYING ON THE CUT

Consider a point $t_0+i\epsilon$, t_0 real, on the cut of the complex t plane, for $s=s_0+i\epsilon$, $s_0>4m^2$. We surround it by a disk of radius r and consider the intersection $U_{t_0}(r)$ of this disk with the cut t plane. We show that if $A_s(t)$ is holomorphic in $U_{t_0}(r)$, continuous in $\overline{U}_{t_0}(r)$, has a piecewise Hölder continuous phase $\phi_s(t)$ on (t_0-r, t_0+r) and $A_s(t_0+i\epsilon)=0$, then there exists an integer p, $p\geq 0$ so that

$$\lim_{t\to t_0+i\epsilon} A_s(t)/(t-t_0)=\text{finite},$$

$$\lim_{t\to t_0+i\epsilon} A_s(t)/(t-t_0)^{p+1}=\infty .$$

To this end, we define

$$\Omega_s(t)=\exp\left[\frac{1}{\pi}\int_{-r}^{r}\phi_s(t')/(t'-t)dt'\right]$$

and

$$\hat{A}_s(t)=A_s(t)/\Omega_s(t) .$$

Assume t_0 is a point of discontinuity of $\phi_s(t')$. (If it is not, the proof is simpler and is a special case of the following.) Assume the magnitude of the discontinuity $\phi(t_{0+})-\phi(t_{0-})$ is $\pi\beta\geq 0$. It is well known that $\Omega_s(t)\propto(t-t_0)^{\beta}$ in the neighborhood of t_0 (Ref. 13, p. 73) (for all β). We can also find $r', r''<r$, so that for all $r'>r''>0$, $\Omega_s(t)$ is continuous on (t_0+r'', t_0+r'); it is actually continuous (in two real variables) and nonvanishing in the closed intersection $\overline{U}_{r'r''}$ of a disk $|t-t_0-\frac{1}{2}(r'+r'')|<\frac{1}{2}(r'-r'')$ with the cut t plane (Ref. 13, p. 38). Since the phase of $\Omega_s(t)$ is equal to the phase of $A_s(t)$, it follows that the function $\hat{A}_s(t)$ is holomorphic and continuous in $\overline{U}_{r'r''}$, real on (t_0+r'', t_0+r'), and so can be extended analytically to Im$t<0$. We can actually conclude that $\hat{A}_s(t)$ is analytic and uniform in a disk of radius $r'''<r'$ around t_0, except for an isolated singularity at t_0. This singularity can be at most a pole, since the function

$$\hat{A}_s(t)(t-t_0)^{\beta+1}\equiv A_s(t)(t-t_0)^{\beta+1}/\Omega_s(t)$$

is real along Im$t=0$, is holomorphic and continuous in $\overline{U}_{t_0}(r''')$ [for definition, compare with $U_{t_0}(r')$], and so, can be extended by reflection to all $|t-t_0|<r'''$ where it is holomorphic. Let p_0 be the number of poles of $\hat{A}_s(t)$; the variation of the phase of $\hat{A}_s(t)$ as we go from $t_0-\epsilon$ to $t_0+\epsilon$ is $-p_0\pi$. But if $\phi_s(t')$ is the phase of the amplitude, as defined in Sec. II B, the only possibility is $p_0=0$. So, $\hat{A}_s(t)$ is holomorphic at t_0. We can then choose $p=[\beta]$ and prove our statement. Further, it is easy to check that $\beta<0$ is inconsistent with the continuity of A on the cuts, and the definition of the phase of Sec. II A. So, our assertion is completely proved.

APPENDIX B: THE INEQUALITY (2.10)

Inequality (2.10) can be read off Eq. (6, Sec. 2) of Ref. 14 as a special case. The present author thinks its derivation is straightforward, but somewhat lengthy and so he chose to give the main steps below.

The modulus of the two integrals in Eq. (2.9), which we together denote by I, can be estimated as follows:

$$|I|\leq\left|\frac{\nu}{\pi}\int_{\nu_1}^{\infty}\frac{\phi_t''(\nu')d\nu'}{\nu'(\nu'-\nu)}\right|+\left|\frac{\nu}{\pi}\int_{\nu_1}^{\infty}\frac{\phi_t'(\nu')d\nu'}{\nu'(\nu'+\nu)}\right|$$
$$\leq\frac{|\nu|}{\pi}C_0\left(\left|\int_{\nu_1}^{\infty}\frac{d\nu'}{|\nu'||\nu'-\nu|}\right|+\left|\int_{\nu_1}^{\infty}\frac{d\nu'}{|\nu'||\nu'+\nu|}\right|\right)\equiv I_1+I_2 . \tag{B1}$$

Here C_0 is a common bound for the phases $|\phi_{1t}'(\nu')|$ and $|\phi_{1t}''(\nu')|$, which exist according to condition (d), Sec. II A. Consider first,

$$|I_1|\leq\frac{|\nu|}{\pi}C_0\int_{\nu_1}^{\infty}\frac{d\nu'}{|\nu'||\nu'-\nu|}\leq\frac{|\nu|}{\pi}C_0\sqrt{2}\int_{\nu_1}^{\infty}\frac{d\nu'}{|\nu'|(|\nu'-\text{Re}\nu|+|\text{Im}\nu|)}, \tag{B2}$$

where we have used $|\nu'-\nu|\geq 1/\sqrt{2}(|\nu'-\text{Re}\nu|+|\text{Im}\nu|)$. Assume first Re$\nu>\nu_1$. We perform explicitly the last integration and get

$$|I_1|\leq\frac{C_0\sqrt{2}}{\pi}\left\{\frac{1}{\cos\theta+|\sin\theta|}\ln\left[\frac{\cos\theta}{\cos\theta_1}\,\frac{\cos\theta+|\sin\theta|-\cos\theta_1}{|\sin\theta|}\right]+\frac{1}{\cos\theta-|\sin\theta|}\ln\frac{\cos\theta}{|\sin\theta|}\right\}\equiv I_1'+I_1'', \tag{B3}$$

where we have denoted $\nu=|\nu|\cos\theta+i|\nu|\sin\theta$ and $\cos\theta_1=\nu_1/|\nu|$. Clearly, if Re$\nu>\nu_1$, $\cos\theta>\cos\theta_1$. For I_1' we use $\cos\theta+|\sin\theta|\geq 1$, and the fact that the argument of the logarithm is always greater than 1 and less than $2/(\cos\theta_1|\sin\theta|)\equiv 2|\nu|/(\nu_1|\sin\theta|)$. So

$$|I_1'|\leq\frac{C_0\sqrt{2}}{\pi}\ln\left(\frac{2}{\nu_1}|\nu|/|\sin\theta|\right).$$

For I_1'', we notice that the function

$$f(\theta)\equiv|\ln\cos\theta-\ln|\sin\theta||(|\ln\cos\theta+\ln|\sin\theta||\times|\cos\theta-|\sin\theta||)$$

is continuous in $0 \leq \cos\theta \leq 1$. Let its maximum there be $A \geq 1$. We conclude

$$|I_1''| \leq (C_0 A\sqrt{2}/\pi)\,|\ln\cos\theta + \ln|\sin\theta|| \leq (C_0 A\sqrt{2}/\pi)\,|\ln\cos\theta_1 + \ln|\sin\theta|| = (C_0 A\sqrt{2}/\pi)\,\ln[|\nu|/(\nu_1|\sin\theta|)].$$

So we see

$$|I_1| \leq (C_0 A\sqrt{2}/\pi)\ln(2|\nu|/(\nu_1|\sin\theta|)) \leq (2C_0 A\sqrt{2}/\pi)\ln(|\nu|/(\nu_1|\sin\theta|)) \text{ for } |\nu| \geq 2\nu_1. \tag{B4}$$

If $\mathrm{Re}\nu < \nu_1$, then I_1 is bounded by a constant B, so that we can choose $|\nu|$ so large that the majorization (B4) holds for all $|\nu|$. The same majorization holds for I_2 since (B4) is invariant at a change $\nu \to -\nu$. So, we obtain (2.10), by noticing that $|e^I| > e^{-|I|}$.

APPENDIX C: PROOF THAT $S_{it}(z_t) \equiv \pm 1$ (SEC. II E)

We have seen that $S_{it}(z_t)$ is such that it has modulus one on the unit circle of the z_t plane [obtained by (2.15) from the cut s plane] except possibly for a finite number of points, and is real holomorphic and without zeros in $|z_t| < 1$. We can consequently define unambiguously $L_{it}(z_t) \equiv \ln S_{it}(z_t)$ in $|z_t| < 1$; then $\mathrm{Re}L_{it}(z_t) = 0$ for $|z_t| = 1$, except for a finite number of points. To show that $\mathrm{Re}L_{it}(z_t) \equiv 0$ for $|z_t| < 1$, it is enough to show that, e.g., $\oint |L_{it}(\gamma e^{i\theta})|d\theta$ is bounded for $0 < \gamma < 1$. If this is the case, then, by Theorem 3.1 of Ref. 15, p. 34, $\oint |\mathrm{Re}L_{it}(e^{i\theta})|d\theta$ exists, and $\mathrm{Re}L_{it}(\gamma e^{i\theta})$ can be obtained from $\mathrm{Re}L_{it}(e^{i\theta})$ by means of a Poisson integral. Since $\mathrm{Re}L_{it}(e^{i\theta})$ vanishes except for a finite number of points, it follows that $\mathrm{Re}L_{it}(\gamma e^{i\theta}) \equiv 0$ for all $0 < \gamma < 1$. We conclude $L_{it}(\gamma e^{i\theta}) =$ an imaginary constant. The real analyticity of S_{it} fixes this number to be 0 or πi, and so $S_{it} = \pm 1$.

To estimate $|L_{it}(\gamma e^{i\theta})|$, we use the fact that, as a consequence of conditions (d) and (e) of Sec. II A, the amplitude $A_{it}(s)$ can be written as a polynomial $\tilde{A}_{it}(s)$ [Eq. (2.11)] times the function $\Omega_t(\nu)$, defined by (2.9). This has been shown in Sec. II D (in the proof of statement B.1). But

$$\begin{aligned}|L_{it}(\gamma e^{i\theta})| &\equiv |\ln[\tilde{A}_{it}(s)\Omega_{it}(s)]/[B_{it}(s)E_t(s)]| \\ &\leq |\ln[\tilde{A}_{it}(s)/B_{it}(s)]| \\ &\quad + |\ln\Omega_{it}(s)| + |\ln E_t(s)|,\end{aligned} \tag{C1}$$

where $B_{it}(s)$ and $E_t(s)$ are defined in (2.17) and (2.14), respectively. The function $\tilde{A}_{it}(s)/B_{it}(s)$ has no zeros in the cut s plane, is obviously polynomially bounded, and so, its logarithm has only integrable singularities. We conclude that the integral of the first term in (C1) is bounded. The third term is the exponent of the function defined in (2.14). Since $\oint|\ln|A_t(\theta)||^2 d\theta$ exists, it follows that the integral of the modulus squared of $\ln E_t(s)$ is also bounded (theorem 4.1, Ref. 15, p. 54). We now consider the second term, which is the modulus of the integral I in the exponent of (2.9). Using the mapping (2.15) one obtains after some calculation,

$$\begin{aligned} I &= \frac{z_t}{2\pi}\oint \frac{\phi_t(z_t')(z_t'^2-1)}{z_t'(z_t - z_t')(1 - z_t z_t')}dz_t' \\ &= -\frac{z_t}{\pi}\oint \frac{\phi_t(z_t')}{z_t'}\left(\frac{1}{z_t - z_t'} + \frac{z_t'}{1 - z_t z_t'}\right)dz_t' \\ &\equiv \bar{I}_1 + \bar{I}_2 . \end{aligned} \tag{C2}$$

In (C2) we have extended the notation $\phi_t(z_t')$ to mean ϕ_t', ϕ_t'' in (2.9), and $-\phi_t'$, $-\phi_t''$ on that part of the circle corresponding to the lower lips of the cut. Now, in (C2) the integral $\bar{I}_1$ is a Cauchy integral of a function $\tilde{\phi} \equiv \phi_t(z_t')/z_t'$, which, among others, belongs to $L^2(0, 2\pi)$, since it is bounded. This function has a decomposition $\tilde{\phi} \equiv \tilde{\phi}_+ + \tilde{\phi}_-$ in positive and negative frequencies, so that both $\tilde{\phi}_+$ and $\tilde{\phi}_- \in L^2(0, 2\pi)$. Consequently, $\bar{I}_1$ represents a function with a bounded squared modulus in $|z_t| < 1$. The same can be argued about $\bar{I}_2$ if the change of variables $z_t'' = 1/z_t'$ is performed.

The fact that the bounds for the integrals of $|\ln E_t(s)|^2$ and $|\tilde{\phi}_+(z_1)|^2$ are uniform in γ is checked by simply resorting to the Taylor expansions of the functions $\ln E_t(z_1)$ and $\tilde{\phi}_+(z_1)$, and using the fact that the boundary values are in $L^2(0, 2\pi)$. This completes the proof.

APPENDIX D: COMMENT ON SEC. II D

To show that there exists a set I_t', dense in some interval of I_t, such that for $t \in I_t'$, the number of zeros of $A_{1t}(s)$, $A_{2t}(s)$ in the complex s plane is bounded by a certain number $N_{\max}$, we construct first the sets M_n of t values, contained in I_t, for which the numbers of zeros of A_{1t}, A_{2t} in the complex s plane are both less than n. We see that $M_n \subset M_{n+1}$ and that $\cup_{n=1}^{\infty} M_n = I_t$. The following lemma (see Ref. 12, p. 230) proves the assertion of Sec. II D:

Let S be an interval of the real axis and M_i a sequence of subsets of S with properties (a) $M_i \subset M_{i+1}$ for all i, and (b) No set M_i is dense in some open subset of S. Then the set $M = \cup_{i=1}^{\infty} M_i$ can contain no open subset of S.

The proof is done in Ref. 12, p. 230. This lemma proves the statement of Sec. II D, since in our case $M \equiv I_t$ is an interval, and so we must conclude that there exists an $n = N_{\max}$, so that $M_{N_{\max}}$ is dense in some open subset of I_t.

APPENDIX E: IRREDUCIBLE POLYNOMIALS

In this Appendix, we prove statements (i) and (ii) of Sec. II I concerning irreducible polynomials. The reasonings that follow are similar to those of Ref. 12 (see p. 108). The present author was unable to find in Ref. 12 a statement with the precise content of (i) and (ii) so he proves them here below.

(i) We first show that two irreducible polynomials $P_1(\zeta,\theta)$ and $P_2(\zeta,\theta)$ coincide up to a constant if they have a common root $\theta=\theta(\zeta)$ [or $\zeta=\zeta(\theta)$] on no matter how small a continuum in ζ (or θ).

Assume first that the common root is such that on some one (real) dimensional continuum I in ζ, $\theta(\zeta)=\theta_0=$constant. It follows that $P_{1,2}(\zeta,\theta_0)=0$ for $\zeta\in I$, and so everywhere. This means $P_{1,2}(\zeta,\theta)=\overline{P}_{1,2}(\zeta,\theta)(\theta-\theta_0)^p$, with $\overline{P}_{1,2}(\zeta,\theta)$ polynomials in ζ, θ and $p>0$, integer. This contradicts the irreducibility of $P_{1,2}$ unless $P_{1,2}\equiv$ constant, and $p=1$. This proves statement (i) in this simple case [and also in the similar one, $\zeta(\theta)=\zeta_0$ = const].

If $\theta(\zeta)$ is not a constant, consider the set of all its analytic continuations, along all possible ways in the ζ plane. We obtain n values of $\theta(\zeta)$ at ζ : $\theta_1(\zeta),\ldots,\theta_n(\zeta)$, where n is less than (or equal to) the degree m of $P_1(\zeta,\theta)$ with respect to θ. We construct then the symmetric combinations $\theta_1(\zeta)+\theta_2(\zeta)+\cdots+\theta_n(\zeta)$, $\theta_1(\zeta)\theta_2(\zeta)+\cdots+\theta_{n-1}(\zeta)\theta_n(\zeta),\ldots,$ and $\theta_1(\zeta)\theta_2(\zeta)\ldots\theta_n(\zeta)$. Let $a_0(\zeta)$ be the polynomial which is the coefficient of the highest power in θ of $P_1(\zeta,\theta)$. The m roots of $P_1(\zeta,\theta)$ are all finite except at the points where $a_0(\zeta)$ vanishes. There are only a finite number of such points. Except for them, we can check that the symmetric combinations above are uniform and holomorphic functions at all points of the ζ plane. They are uniform because all branching points of $\theta_i(\zeta)$ fall out when one builds the symmetric combinations; further, they are bounded at those points where the individual $\theta_i(\zeta)$ have branching points and they are uniform and holomorphic in their neighborhood; so they can be extended by holomorphy to these points too.

To study those points where $a_0(\zeta)$ vanishes, we make the replacement $\theta'=1/\theta$ and notice that the set of zeros of $P_1(\zeta,\theta)$ is transformed to that of $\theta'^m P_1(\zeta,1/\theta')$. Assume first that $a_n(\zeta)$, the coefficient of the free term in θ of $P_1(\zeta,\theta)$, does not vanish at the same point as $a_0(\zeta)$. Then the m roots $1/\theta_i(\zeta)$ stay finite in the neighborhood of the points where $a_0(\zeta)=0$. We construct the symmetric combinations of $1/\theta_i(\zeta)$ and conclude as before that they are holomorphic and uniform functions in the neighborhood of these points. It follows that the symmetric combinations of $\theta_i(\zeta)$ are meromorphic functions of ζ at these points themselves.

If, by chance, $a_0(\zeta)$ and $a_n(\zeta)$ vanish at once at some point, we can find a constant C, so that, with the change of variables $\theta_1=\theta+C$, the free coefficient $a_n'(\zeta)$ of $P_1(\zeta,\theta_1)$ no longer has this property. This we can always do, unless $P_1(\zeta,\theta)\equiv 0$. From the meromorphy with respect to ζ of the symmetric combinations in θ_i+C we conclude the meromorphy of the symmetric combinations in θ_i.

So, we have proven the meromorphy with respect to ζ of the symmetric combinations of the θ_i's at all finite points of the ζ plane. We make now the change of variables $\zeta'=1/\zeta$ and show similarly that at $\zeta'=0$, the symmetric combinations are again meromorphic. It is essential at this point that $P_1(\zeta,\theta)$ is a polynomial in ζ. A function which is meromorphic at all points of the complex plane, including infinity, is rational.

We next construct a polynomial in θ of degree n, $\overline{P}_0(\zeta,\theta)$, so that the coefficients of the powers of θ are the symmetric combinations of the n roots $\theta_i(\zeta)$; these coefficients are rational functions of ζ, and $\overline{P}_0(\zeta,\theta)$ vanishes only at the points $(\zeta,\theta_i(\zeta))$, $i=1,2,\ldots n$, and nowhere else. By multiplying $\overline{P}_0(\zeta,\theta)$ with a polynomial $\overline{a}_0(\zeta)$—the common denominator of the coefficients, we obtain a polynomial in two variables $P_0(\zeta,\theta)$. The degree of this polynomial with respect to θ is $n\leq m$, and the set of its zeros is contained in that of $P_1(\zeta,\theta)$. Then the ratio $P_1(\zeta,\theta)/P_0(\zeta,\theta)$ has no singularities in the whole ζ, θ plane. This follows from Weierstrass's preparation theorem [see Appendix F, statement (iii)]. But, for all ζ, P_1/P_0 is bounded by const$\times\theta^{m-n}$, as $\theta\to\infty$. Therefore P_1/P_0 is a polynomial in θ. We call it $\overline{P}_1(\zeta,\theta)$. The coefficients of this polynomial are rational functions of ζ, as one sees by direct computation from those of P_1 and P_0. But $\overline{P}_1(\zeta,\theta)$ is holomorphic in ζ, θ in the whole ζ, θ plane; it follows that it is a polynomial in ζ. This way, we have decomposed the irreducible polynomial P_1 in a product $P_1=\overline{P}_1P_0$ of two polynomials, one of which (P_0) has degree n bigger than zero, by construction. The only possibility is therefore that $n=m$, and $\overline{P}_1=$ constant. We have proved this way that knowledge of one zero trajectory $\theta_1=\theta_1(\zeta)$ on a small continuum in ζ determines the polynomial $P_1(\zeta,\theta)$ up to a constant, as stated in (i). Statement (ii) of Sec. II H is obtained by simply parting the set of zeros of $P(\zeta,\theta)$ into sets of zeros of irreducible polynomials. Then $P=C\prod_{i=1}^{n}P_i$ where P_i is irreducible and determined by the zeros up to a constant.

Let us notice that the equation $P(\zeta,\theta)=0$, where P is irreducible, $\partial P/\partial\zeta\neq 0$, and $\partial P/\partial\theta\neq 0$, can always be solved with respect to ζ or θ, except for a finite number of points (ζ, θ) in $\mathbb{C}^2$. Indeed, if it were not so, one would conclude, for instance, that P_1 and $\partial P_1/\partial\theta$ vanish together on an infinite set of points (ζ, θ), and that they are either identical, or that the degree with respect to θ of $\partial P_1/\partial\theta$ is bigger than that of P_1. Both situations are impossible, however.

APPENDIX F: WEIERSTRASS'S PREPARATION THEOREM

Weierstrass's preparation theorem is a fundamental statement of the theory of functions of several complex variables. For two complex variables (and not for more than two) it reads: If $A(\zeta, \theta)$ is holomorphic in two variables in a domain D of $\mathbb{C}^2$, and $A(\zeta_0, \theta_0)=0$, then there exists a neighborhood $U_{\zeta_0}\times U_{\theta_0}$ of (ζ_0, θ_0) where A can be represented as

$$A(\zeta, \theta)=(\zeta-\zeta_0)^p[(\theta-\theta_0)^m+\alpha_1(\zeta)(\theta-\theta_0)^{m-1}+\cdots+\alpha_m(\zeta)]\omega(\zeta, \theta), \qquad \text{(F1)}$$

where p and m are positive integers (possibly zero), $\alpha_i(\zeta)$ are m functions holomorphic in ζ in U_{ζ_0}, and such that $\alpha_i(\zeta_0)=0$ and $\omega(\zeta, \theta)$ is a holomorphic nonvanishing function of (ζ, θ) in $U_{\zeta_0}\times U_{\theta_0}$.

A proof of this theorem is found in Ref. 12, p. 89. The expression in brackets is called a Weierstrass pseudopolynomial (with peak at ζ_0); we denote it by $W_m(\theta-\theta_0; \zeta-\zeta_0)$. Geometrically, (F1) means that, for a function of two variables, holomorphic in $U_{\zeta_0}\times U_{\theta_0}$, one can associate to its set of zeros in $U_{\zeta_0}\times U_{\theta_0}$ a Riemann manifold with finitely many sheets. The point ζ_0 is a branching point of order m for the functions $\theta(\zeta)$ defined by $W_m(\theta-\theta_0; \zeta-\zeta_0)=0$.

We next quote from Ref. 12 a number of statements and definitions related to Weierstrass pseudopolynomials (WP). They are intuitively obvious and they serve making rigorous some statements of the text (in Sec. II J and Appendix E). Their proof is given in Ref. 12:

(i) A WP (ζ_0):$W(\theta-\theta_0; \zeta-\zeta_0)$ is said to be irreducible if there are no two other WP (ζ_0), W', W'' so that $W=W'W''$ in $U_{\zeta_0}\times U_{\theta_0}$ (p. 98 and Theorem 2, p. 104); (ii) A WP (ζ_0) admits of a unique decomposition (up to order) in irreducible factors (theorems 1 and 2, p. 104); (iii) If a WP (ζ_0), W' is irreducible and vanishes at all points of a neighborhood of (ζ_0, θ_0) where another WP (ζ_0), W'' vanishes, then W''/W' is also a WP (ζ_0) (Theorem 3, p. 105).

Clearly, a proper Riemann surface can be attached only to irreducible WP's. If W_1 is an irreducible WP (ζ_0) of degree m, then $\partial W_1/\partial\theta$ is also a WP (ζ_0) of degree $m-1$. It is true that (iv) if W_1 is irreducible, there are only a finite number of points in $U_{\zeta_0}\times U_{\theta_0}$, where W_1, $\partial W_1/\partial\theta$ simultaneously vanish (theorems 1 and 2, p. 107, 108); (v) an irreducible WP (ζ_0) is completely determined if one of its roots $\theta_1(\zeta)$ is known on any open subset of U_{ζ_0} (p. 108, Sec. 10).

We next prove for completeness some obvious statements about the multiplicity of a zero trajectory.

(a) The notion is independent of point: Indeed, assume

$$R(\zeta, \theta, \eta(\zeta, \theta))\Big/\prod_{i=1}^{n}[P_i(\zeta, \theta)]^{k_i}\neq 0$$

and is holomorphic in a neighborhood $U_{\zeta_0}\times U_{\theta_0}$ of ζ_0, θ_0, where $R(\zeta_0, \theta_0, \eta(\zeta_0, \theta_0))=0$. According to (iv) above, there exists an open set $U'_\zeta\subset U_{\zeta_0}$ so that all the roots of $P_i(\zeta, \theta)$ in θ are distinct. Consider one of these roots $\theta_e(\zeta)$, $\zeta\in U'_\zeta$ and a neighborhood of it U'_θ, so that, if $\zeta\in U'_\zeta$, $\theta_e(\zeta)\in U'_\theta$, and no other root of another P_i is in U'_θ. Then in $U'_\zeta\times U'_\theta$,

$$R(\zeta, \theta, \eta(\zeta, \theta))=[\theta-\theta_e(\zeta)]^{k_i}\omega(\zeta, \theta),$$

where $\omega(\zeta, \theta)$ is $\neq 0$ and holomorphic. So,

$$(\partial^p R/\partial\theta^p)(\zeta, \theta_e(\zeta), \eta(\zeta, \theta_e(\zeta)))\equiv 0$$

in U'_ζ for $p\leq k_i-1$. This is an identity which must hold for all analytic continuations of $\theta_e(\zeta)$, and so, proves the statement.

(b) Two polynomials P^η, P^ζ which describe the same trajectory have the same multiplicity. It is sufficient to establish this in the neighborhood of some point (ζ, θ) where all roots in θ of $P^\eta(\zeta, \theta)$ are distinct, and we can solve (2.3) for $\eta(\zeta, \theta)$ and $\theta(\zeta, \eta)$. So, assume in the neighborhood $U_\zeta\times U_\theta$ of the point $(\zeta, \theta=\theta_e(\zeta))$, one has

$$R(\zeta, \theta, \eta(\zeta, \theta))=[\theta-\theta_e(\zeta)]^{k_i}\omega(\zeta, \theta),$$

with $\omega(\zeta, \theta)$ holomorphic and $\neq 0$. Let U_η be the image of $U_\zeta\times U_\theta$ through $\eta=(8m^2-\zeta^2-\theta^2)^{1/2}$. Then in $U_\zeta\times U_\eta$ the function $\theta(\zeta, \eta)-\theta_e(\zeta)$ vanishes only at the points $(\zeta, \eta_e(\zeta))$, where $\eta_e(\zeta)=[8m^2-\zeta^2-\theta_e^2(\zeta)]^{1/2}$ is the root of $P^\theta(\zeta, \eta)$ associated to $\theta_e(\zeta)$ by (2.32). By Weierstrass's preparation theorem, we can write $\theta(\zeta, \eta)-\theta_e(\zeta)=(\eta-\eta_e(\zeta))\times\omega_1(\zeta, \eta)$. So,

$$\begin{aligned}R(\zeta, \theta(\zeta, \eta), \eta)/(\eta-\eta_e(\zeta))^{k_i}&\equiv R(\zeta, \theta, \eta(\zeta, \theta))\,\omega_1^{k_i}(\zeta, \eta)/(\theta(\zeta, \eta)-\theta_e(\zeta))^{k_i}\\&=\omega(\zeta, \theta(\zeta, \eta))\omega_2^{k_i}(\zeta, \eta),\end{aligned}$$

holomorph and free of zeros in $U_\zeta\times U_\theta$. This proves the statement.

APPENDIX G: THE PROPERTIES OF $R_0(s,t,u)$

(i) We describe first in more detail the way $\tilde{R}_0(s, t, u)$ is constructed. We call $F^\eta_{0,N}$ the family of irreducible polynomials $P^\eta_i(\zeta, \theta)$ describing the zero trajectories of $Z_{0,N}$ (and similarly $F^{\zeta}_{0,N}, F^{\theta}_{0,N}$). Let $\theta_1(\zeta)$ be a root of $P^{\eta}_1(\zeta,\theta)=0$, and let $\eta_1(\zeta)$ be one root of $P^{\theta}_1(\zeta,\eta)$ associated to it by (2.32). In principle there could be two such roots. We consider one of them and continue analytically the pair of functions $(\theta_1(\zeta),\eta_1(\zeta))$ along all possible ways in the ζ plane. It could be that, for each point ζ, we obtain this way (a) n different values $\theta_i(\zeta)$ of θ and n different values $\eta_i(\zeta)$ of η, in one-to-one correspondence to each other; (b) n different values of θ and $2n$ different values of η, or (c) $2n$ different values of θ and n different values of η. In the last two cases, to each root $\theta_i(\zeta)$ [or $\eta_i(\zeta)$] there correspond two roots $\eta_i(\zeta)$ [or $\theta_i(\zeta)$] satisfying (2.32). It is easy to see that there are no other possibilities. In case (a), the pair $(P^{\eta}_1, P^{\theta}_1)$ contributes n different factors to the product (3.2), whereas in cases (b) and (c), it brings $2n$ factors.

Situations (a), (b), or (c) occur at all points ζ except for a finite number at which the equations $P^{\eta}_1(\zeta,\theta)=0, P^{\theta}_2(\zeta,\eta)=0$ do not have simple roots in θ or η. We call $\Pi^{\eta}_N(\zeta,\theta)$ $\Pi^{\zeta}_N(\theta,\eta)$, and $\Pi^{\theta}_N(\zeta,\eta)$ the products of the irreducible polynomials contained in $F^{\eta}_{0,N}$, $F^{\zeta}_{0,N}$, and $F^{\theta}_{0,N}$, respectively. They are real analytic functions of their arguments.

(ii) We now show that the numerator $\tilde{R}_{0,N}(\zeta,\theta,\eta)$ of $\tilde{R}_0$ is a meromorphic function in the Mandelstam domain. We shall actually show that it is meromorphic in all of $\mathbb{C}^3$. Meromorphy in $\mathbb{C}^3$ with respect to θ and η is obvious. Further, $\tilde{R}_{0,N}$ is holomorphic in ζ at all points where the trajectory functions stay finite and do not cross each other. At points where they do cross, the symmetry of (3.2) under permutation of the factors ensures the existence of a neighborhood where $\tilde{R}_{0,N}$ is uniform and holomorphic. If the expressions $\theta_i(\zeta)+i\eta_i(\zeta)$ stay finite in the neighborhood of such points, the symmetric combinations $\sum_i[\theta_i(\zeta)+i\eta_i(\zeta)]$, and $\sum_{i>j}[\theta_i(\zeta)+i\eta_1(\zeta)][\theta_j(\zeta)+i\eta_j(\zeta)]$, etc., have a unique extension by holomorphy to the points themselves.

If $\theta_i(\zeta)+i\eta_i(\zeta)$ becomes unbounded in the neighborhood of a point $\bar{\zeta}$, one can check that there exists an integer m, so that both $\theta_i(\zeta)(\zeta-\bar{\zeta})^m$ and $\eta_i(\zeta)(\zeta-\bar{\zeta})^m$ stay bounded in the neighborhood of $\bar{\zeta}$. This follows from the fact that the coefficients of the powers of θ,η in $\Pi^{\eta}(\zeta,\theta)$ and $\Pi^{\theta}(\zeta,\eta)$ are analytic functions of ζ. We conclude that the symmetric combinations of $\theta_i(\zeta)+i\eta_i(\zeta)$ behave meromorphically at all finite ζ points.

By performing the change of variable $\zeta'=1/\zeta$, we see that the zeros of $\tilde{R}_{N,0}$ around $\zeta=\infty$ are also described by polynomials $\zeta'^{m_\eta}\Pi^{\eta}(1/\zeta',\theta)$ and $\zeta'^{m_\theta}\Pi^{\theta}(1/\zeta',\eta)$ (m_η, m_θ degrees of Π^{η},Π^{θ}). Applying the same reasoning as above, we conclude that the symmetric combinations behave meromorphically also at $\zeta=\infty$, and so we conclude that $\tilde{R}_{0,N}$ (and $\tilde{R}_0$) is a rational function of ζ,θ,η in all of $\mathbb{C}^3$.

(iii) We next show that the number of zero trajectories N_{tot}, Eq. (3.2), must be even as a consequence of the real analyticity of $\Pi^{\eta}(\zeta,\theta)$, $\Pi^{\theta}(\zeta,\eta)$, and $\Pi^{\zeta}(\theta,\eta)$. This is clear if to each trajectory $\theta=\theta(\zeta)$ of an irreducible factor $P^{\eta}_1(\zeta,\theta)$ of Π^{η}, there correspond two trajectories $\eta_i=\eta_i(\zeta)$ of $P^{\theta}_1(\zeta,\eta)$, or conversely. One can check that the product of all irreducible polynomials P^{η}_1 with this property is real analytic. The same is true for P^{θ}_1. We divide these products out of Π^{η}, Π^{θ}, and obtain again real analytic polynomials $\Pi^{\eta}_1, \Pi^{\theta}_1$, so that to each root $\theta(\zeta)$ of Π^{η}_1 there corresponds only one root $\eta(\zeta)$ of Π^{θ}_1. We notice then that, at $\zeta=2m\sqrt{2}$, the trajectories must fulfill one of the two conditions $\theta(2m\sqrt{2})\pm i\eta(2m\sqrt{2})=0$. If $\eta(2m\sqrt{2})$ is real, then $\theta(2m\sqrt{2})$ is imaginary. But then the real analyticity of $\Pi^{\eta}_1(\zeta,\theta)$ enforces the existence of a second root $-\theta(2m\sqrt{2})$, which corresponds to a root of $\Pi^{\zeta}_1(\eta,2m\sqrt{2})$ at $-\eta(2m\sqrt{2})$, or to a second root at $\eta(2m\sqrt{2})$. In both cases, the polynomials must have an even number of roots. A similar reasoning holds if η and θ are complex at $\zeta=2m\sqrt{2}$. So, $N_{\text{tot}}=2p_N$, p_N an integer.

(iv) We now show that $\tilde{R}$ is a real analytic function on $\mathcal{S}$ defined by Eq. (2.3). We shall prove that the function $\tilde{R}_0(\zeta,\theta,\eta(\zeta,\theta))$ is real when ζ and θ are real. More precisely, we shall see that the phases of $\tilde{R}_{0,N}$, and $\tilde{R}_{0,D}$ are the same for these values of ζ and θ. Consider then ζ_0 real, $0<\zeta_0<2m\sqrt{2}$; the zeros of the many-valued function of θ, $\tilde{R}_{0,N}(\zeta_0,\theta,\eta(\zeta_0,\theta))$ are by construction either real or build complex-conjugate pairs. Since we have seen that there exists an even number of trajectories, the real ones must also fall in pairs. If the zeros are real, the corresponding $\theta(\zeta_0)+i\eta(\zeta_0)$ values lie on the circle of radius $(8m^2-\zeta_0{}^2)^{1/2}$ in the ω plane. Otherwise, the corresponding ω_1,ω_2 are reflected with respect to this circle: $\omega_1=(8m^2-\zeta_0^2)/\omega_2^*$. We consider first a pair of real zeros, lying on the circle above at angles θ_1 and θ_2. We write

$$\begin{aligned}(8m^2-\zeta_0{}^2)&(e^{i\theta}-e^{i\theta_1})(e^{i\theta}-e^{i\theta_2})\\ &=2\omega(8m^2-\zeta_0{}^2)^{1/2}e^{i(\theta_1+\theta_2)/2}\\ &\quad\times\{\cos[\tfrac{1}{2}(\theta-\theta_1-\theta_2)]-\cos[\tfrac{1}{2}(\theta_1-\theta_2)]\},\end{aligned}\tag{G1}$$

for ω on the circle of radius $(8m^2-\zeta_0{}^2)^{1/2}$. We see that each pair of factors $(\omega-\omega_1)(\omega-\omega_2)$ divided by ω gives rise to a function which has a constant phase on the circle. The same is true for complex-conjugate zeros. We conclude that the function $\tilde{R}_{N,0}(\zeta,\theta,\eta)/\omega^{p_N}$ has a constant phase on the circle of radius $(8m^2-\zeta_0{}^2)^{1/2}$ (modulo π), and equal (modulo π) to half the phase of the product of all the roots $\prod_{i=1}^{N}\omega_i(\zeta)$.

We next show that this product has a phase which is independent of ζ. We have seen that this product must be a rational function. Since the ω_i's can be grouped in pairs of values, either both of modulus $(8m^2 - \zeta_0^2)^{1/2}$, or reflected with respect to the circle, one concludes that, for ζ_0 real, the product has a modulus equal to $(8m^2 - \zeta_0^2)^{p_N}$. But the function $\prod_i \omega_i(\zeta)$ can have zeros only at $\zeta = \pm 2m\sqrt{2}$. Indeed, it is only there that $\theta(\zeta) + i\eta(\zeta)$ can vanish, according to (2.3). We conclude that $\prod_{i=1}^N \omega_i(\zeta) = C_N(8m^2 - \zeta^2)^{p_N}$, where C_N is a complex phase $C_N = e^{i\psi_N}$. With the previous reasoning, $\tilde{R}_{N,0}/\omega^{p_N}$ has for real ζ_0, $0 < \zeta_0 < 2m\sqrt{2}$, and real θ and η [ω on the circle of radius $(8m^2 - \zeta_0^2)^{1/2}$] a constant phase ψ_N independent of ζ_0. The same reasoning is true for $\tilde{R}_{D,0}/\omega^{p_N}$. We now show that $\psi_N = \psi_D$. To see this, note that the product of the roots of the denominator is, for ζ_0 real,

$$\prod_{i=1}^{N} [\theta_i^*(-\zeta_0) + i\eta_i^*(-\zeta_0)] = \prod_{i=1}^{N} [\theta_i(-\zeta_0) + i\eta_i(-\zeta_0)]$$

because of real analyticity. The right-hand side of this equality is even in ζ_0 [$= e^{i\psi_N}(8m^2 - \zeta_0^2)^{p_N}$], and has just been computed. So, $\psi_N = \psi_D$ and the phase of $\tilde{R}_{N,0}$ is equal to the phase of $\tilde{R}_{D,0}$, modulo π. We conclude $\tilde{R}_{D,0}$ is real analytic on $\mathcal{S}$.

(v) Clearly, for all $\zeta \neq \pm 2m\sqrt{2}$, the variable $\theta + i\eta$ makes uniform the two sheets of the θ plane, so that $\tilde{R}_0$ has the same zeros described by Π^η, Π^θ, for $\zeta \neq \pm 2m\sqrt{2}$. It could have supplementary zeros at $\zeta = \pm 2m\sqrt{2}$. At $\zeta = \pm 2m\sqrt{2}$, $\mathcal{S}$ falls into two factors $\theta = \pm i\eta$. For $\theta = i\eta$, we see directly from (3.2) that $\tilde{R}_0$ does not vanish identically; it vanishes only at isolated points corresponding to those roots $\theta_i(2m\sqrt{2})$ and $\eta_i(2m\sqrt{2})$ of the polynomials $\Pi^\theta(2m\sqrt{2}, \theta)$ and $\Pi^\eta(2m\sqrt{2}, \eta)$, which are such that $\theta_i(2m\sqrt{2}) - i\eta_i(2m\sqrt{2}) = 0$. For $\theta = -i\eta$, both numerator and denominator vanish identically; indeed, we have seen in (iv) that $\prod_i \omega_i(\zeta) = C(8m^2 - \zeta^2)^p$, so that $\prod_i \omega_i(\zeta) \equiv 0$ at $\zeta = \pm 2m\sqrt{2}$. But we know that $\tilde{R}_0$ is a real rational function on $\mathcal{S}$. Consequently, $\tilde{R}_0^*(2m\sqrt{2}, \theta, i\theta) = \tilde{R}_0(2m\sqrt{2}, \theta^*, -i\theta^*)$, so that if $\tilde{R}_0$ vanishes at a point (θ, η) lying on $\theta + i\eta = 0$, it will also vanish at the point (θ^*, η^*) which lies on $\theta - i\eta = 0$. But we have seen that, because of the reality of the polynomials Π^η and Π^θ, if there exists a pair of roots satisfying $\theta(2m\sqrt{2}) - i\eta(2m\sqrt{2}) = 0$, there must also exist a pair obeying $\theta(2m\sqrt{2}) + i\eta(2m\sqrt{2}) = 0$. So, $\tilde{R}_0$ vanishes at $\zeta = \pm 2m\sqrt{2}$ only at the roots of these polynomials and nowhere else.

(vi) We can now show that $\tilde{R}_0$ has modulus one in all three channels. This is by construction guaranteed for the s channel. Since $\tilde{R}_0$ has no other zeros but those generated by Π^η, Π^θ, and Π^ζ, it follows that in the physical region of the t and u channels, the zeros of the denominator $\tilde{R}_D$ are complex conjugate to those of the numerator $\tilde{R}_N$. We conclude that the modulus of $\tilde{R}_0(\zeta, \theta, \eta)$ for θ imaginary, ζ, η real and negative on $\mathcal{S}$, is a function of θ only. But,

$$|\tilde{R}_0(\zeta, \theta, \eta)|^2 = \tilde{R}_0(\zeta, \theta, \eta)\tilde{R}_0(\zeta, \theta^*, \eta)$$
$$= \tilde{R}_0(\zeta, \theta, \eta)\tilde{R}_0(\zeta, -\theta, \eta) = f(\theta^2) \qquad \text{(G2)}$$

for ζ, η real, θ imaginary, $(\zeta, \theta, \eta) \in \mathcal{S}$. This is then an identity at all points of $\mathcal{S}$. We evaluate now the modulus of both sides of the last equality of (G2) for ζ on $\mathrm{Re}\,\zeta = 0$, $(\zeta, \theta, \eta) \in \mathcal{S}$. By construction, $|\tilde{R}_0(\zeta, \theta, \eta(\zeta, \theta))| = 1$ for θ, η real and negative. At fixed ζ, we can extend this last equality also to positive values of θ [$\eta(\theta)$ is even, $-(8m^2 - \zeta^2)^{1/2} < \theta < (8m^2 - \zeta^2)^{1/2}$], and so, $|\tilde{R}_0(\zeta, -\theta, \eta)| = 1$ too. Consequently, $f(\theta^2) = 1$, as announced. A similar reasoning can be done in the u channel.

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