PHYSICAL REVIEW D VOLUME 21, NUMBER 11 1 JUNE 1980

Problem of fermion generations in grand unified theories

J. Chakrabarti, M. Popović,* and R. N. Mohapatra
Department of Physics, City College, City University of New York, New York, New York 10031
(Received 20 December 1979)

We investigate the problem of incorporating the fermion generations within the framework of a grand unified SU(N) gauge theory. The requirement of no triangle anomaly and an asymptotically free gauge coupling constant appears to limit the number of light-fermion generations to four. Specializing to an SU(8) model, we exhibit a Higgs mechanism that splits the fermions into heavy and light sets, the latter containing only three generations. We argue that the mass of the heavy generation is in the TeV region.

The simplest gauge models that unify weak, electromagnetic, and strong interactions treat the various generations[1] of quarks and leptons in a sequential fashion and do not provide any insight into the question of mixing angles and mass ratios between various generations. Possible ways to make further progress has been suggested in the literature by treating the various fermion generations as representations of some discrete[2] or continuous horizontal symmetry groups.[3] With the recent surge of interest in grand unification, ways to tackle this problem in the context of SU(N) grand unified theories have also been discussed by several authors.[4,5] The basic strategy proposed is to consider an anomaly-free combination[6] of totally antisymmetric representations of SU(N) groups and identify the generation number with the difference of the number of 10 and 10* (or equivalently 5* and 5) dimensional SU(5) representations contained in the above representations. The difference between Refs. 4 and 5 lies in their choice of the anomaly-free set. In Ref. 4, Georgi requires all members of the anomaly-free set (AFS) to belong to different representations of the grand unifying group. The smallest grand unification group of the above type that can yield three generations is found to be SU(11). On the other hand, in Ref. 5, Frampton and Nandi allow repetition of a multiplet in the AFS, which then enables them to work with lower-rank groups such as SU(7) or SU(9), etc.

We feel that since the purpose of employing higher-rank groups is to avoid having to repeat representations, the strategy of working with only those anomaly-free sets where each representation of the grand unifying group appears once ought to be preferred.

We also propose to gauge the "horizontal" degree of freedom[3] that transforms various generations into each other. So, if there are m generations, we propose an SU$(m)_H$ horizontal group. We consider the minimal grand unification symmetry which contains SU(5) $\times$ SU$(m)_H$, i.e., SU$(5+m)$, and we require light-fermion generations to transform as vectors under SU$(m)_H$ group. Let us, therefore, make our grand unification criteria explicit:

(a) Only totally antisymmetric representation of SU(N) are considered[4] for assignment of fermions.

(b) The set of fermion representations that is anomaly-free[6] must not contain any representation more than once.

(c) Self-conjugate representations are barred.

(d) The gauge coupling is required to be asymptotically free.[7]

(e) The light generations of fermions are required to transform as a vector under the horizontal group.

Within this set of criteria, we find that the allowed number of light generation of fermions must be at most four. In the above discussions, we have assumed that the very-low-energy electroweak symmetry group is SU$(2)_L \times U(1)$ with each generation transforming as a weak isodoublet.

To present our discussion, we first give our notation: For the group SU(N), the representation with m antisymmetric indices will be denoted by $[N,m]$. We then find that the only anomaly-free sets of $[N,m]$, satisfying the criteria (a)–(d), are the following[7]:

$$\begin{aligned} &\mathrm{SU}(5):\ [5,1]+[5,3],\\ &\mathrm{SU}(7):\ [7,1]+[7,3]+[7,5],\\ &\mathrm{SU}(8):\ [8,3]+[8,6]+[8,7],\\ &\mathrm{SU}(9):\ [9,2]+[9,5],\\ &\qquad\quad [9,1]+[9,3]+[9,5]+[9,7],\\ &\mathrm{SU}(10):\ [10,3]+[10,6]. \end{aligned} \tag{1}$$

We emphasize that this result is true for *all N*, provided we avoid self-conjugate representations and the AFS with two representations which are complex conjugates of each other. We note that this excludes the SU(11) group of Ref. 4 from our considerations. To establish the connection between the number of generations and the grand unifying symmetry group, we display the SU(5) content for each set displayed in Eq. (1) in the notation $[N,m]=\sum C_{m_i}\{m_i\}$ where C_{m_i} is the number of m_i-dimensional representations of SU(5) in $[N,m]$ under SU($N-5$):

$$\text{(a) SU(7):}\quad \begin{aligned}[7,1]&=2\{1\}+\{5\},\\ [7,3]&=\{5\}+2\{10\}+\{10^*\},\\ [7,5]&=\{10^*\}+2\{5^*\}+\{1\};\end{aligned} \tag{2}$$

$$\text{(b) SU(8):}\quad \begin{aligned}[8,3]&=\{1\}+3^*\{5\}+3\{10\}+\{10^*\},\\ [8,6]&=3\{1\}+3^*\{5^*\}+\{10^*\},\\ [8,7]&=3^*\{1\}+\{5^*\};\end{aligned} \tag{3}$$

$$\text{(c) SU(9):}\quad \begin{aligned}[9,2]&=6\{1\}+4\{5\}+\{10\},\\ [9,5]&=\{5\}+4\{10\}+6\{10^*\}+4\{5^*\}+\{1\},\end{aligned} \tag{4}$$

also a similar decomposition for the other set;

$$\text{(d) SU(10):}\quad \begin{aligned}[10,3]&=10\{1\}+10\{5\}+5\{10\}+\{10^*\},\\ [10,6]&=\{5\}+5\{10\}+10\{10^*\}\\ &\quad+10\{5^*\}+5\{1\}.\end{aligned} \tag{5}$$

From Eqs. (2)–(5), we see that the SU(10) case does not satisfy our criterion for light generations belonging to the fundamental representation of SU(5). From this, we conclude that the maximum number of generations is four.[8]

To demonstrate our procedure, we describe the SU(8) model of Eq. (3). The fermion multiplets are denoted by $(\psi_{\alpha\beta\gamma})_L$, $(\psi_{\alpha\beta\gamma\sigma\lambda\tau})_L$, $(\bar{\psi}^{\alpha})_R$. According to our prescription for identifying light fermions, we see that they are given by $(\psi_{abA})_L$ and $(\bar{\psi}^{aA})_R$, where $a,b,c,\ldots$ stand for SU(5) indices $1,\ldots,5$, $A,B,\ldots$ stand for the horizontal SU(3) indices 6, 7, 8, and $\alpha,\beta,\ldots$ stand for SU(8) indices. The heavier fermions are then identified as $(\psi_{ABC})_L$, $(\psi_{aAB})_L$, $(\psi_{abc})_L$, $(\psi^{AB})_L$, $(\psi^{ab})_L$, $(\psi^{a})_L$, and $(\psi^{A})_L$. We will now proceed to show that it is possible to choose Higgs multiplets and vacuum expectation values that yield the above mass hierarchy for the fermions. We first note that at the level of heavy-fermion masses, the horizontal symmetry $SU(3)_H$ is already broken. To simplify discussions, we give explicit particle labels. Our notation will be as follows: Lower-case symbols with subscripts p ($p=1,2,3$) such as $(u_p^i, d_p^i, e_p, \nu_p)$ will stand for light fermions of pth generation; i stands for color index; capital-letter symbols with subscripts such as (U_p^i, D_p^i, E_p, N_p) will stand for heavier fermions (superscript c indicates charge conjugate of the state):

$$\begin{aligned}
[8,3]:&\quad (\psi_{ABC})_L\equiv N_L,\\
&\quad (\psi_{aAB})_L\equiv (D_{pL}^{i}, E_{pL}^{+}, E_L^0),\\
&\quad (\psi_{abA})_L\equiv (u_{pL}^{i}, u_{pL}^{ci}, d_{pL}^{i}, e_{pL}^{+}),\\
&\quad (\psi_{abc})_L\equiv (U_{1L}^{i}, U_{1L}^{ci}, D_{1L}^{ci}, E_{1L}^{-}),\\
[8,6]:&\quad (\psi^{aA})_L\equiv (d_{pL}^{ci}, e_{pL}^{-}, \nu_{pL}),\\
&\quad (\psi^{AB})_L\equiv M_{pL}^0,\\
&\quad (\psi^{ab})_L\equiv (U_{2L}^{i}, U_{2L}^{ci}, D_{2L}^{ci}, E_{2L}^{-}),\\
[8,7]:&\quad (\psi^{A})_L\equiv M_{pL}^0,\\
&\quad (\psi^{a})\equiv (D_{3L}^{ci}, E_{3L}^{-}, N_{3L}).
\end{aligned} \tag{6}$$

In making the above identification of heavy-particle states, we have used the fact that there is negligible mixing between heavy- and light-particle states resulting from the Higgs mechanism displayed below. We first give the Higgs multiplets, Yukawa couplings, and vacuum expectation values (VEV's) that contribute to the heavy-particle masses. We choose the following Higgs multiplets: $\Phi^{\alpha\beta}$, $\Phi^{\alpha\beta\gamma\delta}$, $\Phi^{\alpha}_{\beta\gamma}$, $\Phi^{\alpha\pi\eta}_{\beta\gamma\lambda\rho\sigma}$, where we assume antisymmetry in both super- and subindices and tracelessness (e.g., $\sum_\gamma \phi^{\gamma}_{\alpha\gamma}=0$). The heavy fermions in Eq. (6) acquire their masses and mixings from the following gauge-invariant couplings:

$$\begin{aligned}
&\psi^{T\alpha\beta\gamma\lambda\sigma}C^{-1}\psi_{\pi\eta\phi}\Phi^{\pi\eta\phi}_{\alpha\beta\gamma\lambda\sigma},\quad \psi^{T\alpha}C^{-1}\psi_{\alpha\beta\gamma}\Phi^{\beta\gamma},\\
&\psi^{T\alpha\beta}C^{-1}\psi_{\alpha\beta\sigma\lambda\mu\nu},\Phi^{\sigma\lambda\mu\nu},\quad \psi^{T\alpha\beta}C^{-1}\psi_{\mu\alpha\nu}\Phi^{\mu\nu}_{\beta},
\end{aligned} \tag{7}$$

where ψ's with superscripts are obtained by applying the totally antisymmetric ϵ symbol for the SU(8) group. The desired pattern of heavy-particle masses ensue on setting

$$\begin{aligned}
&\langle \Phi^{ab5}_{abABC}\rangle\neq 0,\quad \langle\Phi^{AB}\rangle\neq 0,\quad \langle\Phi^{5ABC}\rangle\neq 0,\\
&\langle\Phi^{AB}_{5}\rangle\neq 0,\quad \langle\Phi^{abc}_{abc5A}\rangle\neq 0.
\end{aligned} \tag{8}$$

It is clear that since three of the above VEV's break $SU(2)_L\times U(1)$ symmetry, they contribute to the W-boson mass, These VEV's are therefore constrained to be of order $m_W/g\approx 300$ GeV. If we therefore choose the coupling constants associated with the invariants in Eq. (7) to be of order 2–3, then the heavy generation acquires its largest mass, of the order of TeV's. The important point that we stress here is that none of the couplings in Eq. (7) upon using Eq. (8) contribute to the light-fermion masses or light-heavy mixings. All the other Yukawa coupling constants are chosen smaller, compared to those in Eq. (7), since they involve light to heavy mixings as well as light-light mixings.

The light-particle sector is completed by adjoining the following extra Higgs multiplets:

$\phi^{\alpha}_{\beta\gamma\lambda}$ with $\langle\Phi^{A}_{5BC}\rangle \neq 0$ and $\Phi^{\alpha\beta}_{\gamma}$ with $\langle\Phi^{5A}_{B}\rangle \neq 0$. The latter contributes to the mass matrix for down quarks whereas the former contributes to the up-quark masses.

Finally, we envision that the grand unification symmetry SU$(5+m)$ (for m generations) will break down to SU(5) $\times$ SU$(m)_H$ at some mass scale higher than 10^{15} GeV by a Higgs multiplet of type Φ^{α}_{β}. Then, as we go below the mass scale of 10^{15} GeV, SU(5) symmetry breaks down to SU$(3)_C \times$ SU$(2)_L \times U(1)$. The next important mass scale is around 10^2 TeV, where the horizontal symmetry SU$(m)_H$ is completely broken down. The heavy-fermion mass (~TeV) constitute the next level in the mass hierarchy.

In summary, we have presented a new approach to SU(N) grand unification based on economy and the requirement of asymptotic freedom for gauge couplings, which embeds a horizontal gauge symmetry. This enables us to limit theoretically the number of light-fermion generations to four and leads a set of new fermion with masses in the TeV range.

This work was supported in part by a National Science Foundation Grant No. PHY-78-24888 and a CUNY-PSC-BHE Research Award Grant No. RF13096.

---

*On leave from Laboratory of Theoretical Physics, "Boris Kidric," Institute for Nuclear Sciences, Beograd, Yugoslavia.

[1]For a review, see H. Harari, Phys. Rep. 42C, 235 (1978).

[2]A. De Rújula, H. Georgi, and S. L. Glashow, Ann. Phys. (N.Y.) 109, 258 (1977); F. Wilczek and A. Zee, Phys. Lett. 70B, 436 (1977); S. Weinberg, Trans. N.Y. Acad. Sci. 38 (1977); H. Fritzsch, Phys. Lett. 73B, 317 (1978); R. N. Mohapatra and G. Senjanović, *ibid.* 73B, 176 (1978); G. Branco, *ibid.* 76B, 70 (1978); D. Wyler, Phys. Rev. D 19, 330 (1979); A. Ebrahim, Phys. Lett. 76B, 605 (1978); T. Hagiwara, T. Kitazoe, G. Mainland, and K. Tanaka, *ibid.* 76B, 602 (1978); M. deCrombruggie, *ibid.* 80B, 365 (1979); H. Harari, H. Haut, and J. Weyers, *ibid.* 78B, 459 (1978); G. Segrè, A. Weldon, and J. Weyers, *ibid.* 83B, 351 (1979); R. Barbieri, R. Gaatto, and F. Strocci, *ibid.* 74B, 344 (1978). For a review, see R. Gatto, Invited talk at the Einstein Symposium at Jerusalem, 1979 (unpublished).

[3]R. N. Mohapatra, Phys. Rev. D 9, 3461 (1974); R. N. Mohapatra, J. C. Pati, and L. Wolfenstein, *ibid.* 11, 3319 (1975); F. Wilczek and A. Zee, Phys. Rev. Lett. 42, 421 (1979); C. L. Ong, Phys. Rev. D 19, 2738 (1979); A. Davidson, M. Koca, and K. C. Wali, *ibid.* 20, 1195 (1979); J. Chakrabarti, *ibid.* 20, 2411 (1979).

[4]H. Georgi, Nucl. Phys. B156, 126 (1979).

[5]P. Frampton and S. Nandi, Phys. Rev. Lett. 43, 1460 (1979).

[6]S. Okubo, Phys. Rev. D 16, 3528 (1977); J. Banks and H. Georgi, *ibid.* 14, 1159 (1976).

[7]We have not included the effect of Higgs boson in the calculation of the β function, since we hope, in some future theory, they will arise dynamically as composites of the basic fermion fields.

[8]We note that for the case of SO(N) grand unification, asymptotic-freedom constraints on the grand unifying coupling constant allow a much larger number of light generations.