Nonperturbative potential model for light and heavy quark-antiquark systems

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A potential model based on quantum-chromodynamics (QCD) considerations is developed. The model attempts to overcome the relativistic limitations associated with earlier models by introducing an effective size for quarks. Application of the model Hamiltonian to both light and heavy mesons yields an accurate description of the mass spectrum, radiative transitions, and annihilation widths for a large number of known mesons. Spin-dependent interactions are treated nonperturbatively using standard diagonalization procedures with an oscillator basis set and relativistic kinematics are adopted throughout. The scaling of the potential parameters is found to be similar to simple QCD predictions. Much of the anomalous behavior for pseudoscalar mesons appears to be resolved both for light and heavy mesons.

I. INTRODUCTION

Much of the attention of particle physics has recently become focused on the quark model.[1] Two new flavors, c (Refs. 2, 3) and b (Ref. 4), have been spectacular additions to hadron spectroscopy. The charmonium model[5-9] and its generalization to "quarkonium"[10] have emerged as the most successful theory thus far. The charmonium interpretation of the narrow states ψ and ψ' with masses in the range 3–4 GeV as charmed-quark–antiquark ($c\bar{c}$) bound states of a simple potential predicted much of the phenomena that was subsequently observed. The discovery[11] of charmed mesons has given further support to this interpretation. Much of the experimental situation for charmonium has been reviewed by Feldman and Perl[3] and the relation to theory has been extensively discussed by Novikov *et al.*[12]

The use of a nonrelativistic model for $c\bar{c}$ and $b\bar{b}$ has been justified on the basis of the large masses involved. Attempts[9] to use such models for light-quark mesons have achieved only qualitative success and in general it has become a lore of particle physics that light-meson dynamics are outside the range of such potential theories. Recent developments associated with accurate mass-difference measurements[13] in the $b\bar{b}$ system also suggest difficulties with the simple modified-Coulomb-potential model[7] based on quantum chromodynamics[14] (QCD). The almost equal energy spacing between ψ and ψ' and Υ and Υ' states suggested[15] a logarithmic potential as the appropriate interquark potential. As an overall logarithmic potential has no justification within QCD and also leads to values of $(v/c)^2$ in the ψ family which are uncomfortably large, more complicated models have been tried.[16] Such models do not, however, apply to light mesons.

The main purpose of the present work is to show that a relatively simple potential model still based on QCD can describe *all* $q\bar{q}$ mesons ranging from the π meson up to the heaviest known states in $b\bar{b}$. For the first time it is possible to calculate the masses of mesons with unequal-mass quark and antiquark (e.g., $u\bar{s}$, $d\bar{c}$, $u\bar{b}$). Moreover, the model appears to be as successful for light mesons as it is for heavy mesons and we do not appear to need logarithmic potentials for $c\bar{c}$ and $b\bar{b}$. Of course, we have had to grapple with relativistic effects which are indeed most important for the lighter-mass mesons but are also noticeable even for charmonium.

The new ingredients to the present model are discussed in Sec. II. The major new feature is that the full Hamiltonian is not solved perturbatively as has been the case in the past. The effects of self-interactions and higher-order terms in the $(v/c)^n$ expansion of the Bethe-Salpeter equation are described in terms of effective sizes (as well as the usual effective masses) for each flavored quark. After averaging the short-range interactions between point fermions, the full Hamiltonian no longer contains divergent terms and a diagonalization procedure using harmonic-oscillator basis states yields an exact solution to the transformed Hamiltonian. Only in this way can a useful solution be found for the light mesons where the effects of strong spin-spin interactions lie outside the scope of perturbation theory. Relativistic kinematics are used throughout and the use of oscillators (Sec. II C) allows this to be included easily. Radial potential form factors for the transformed Hamiltonian are given in the Appendix. The quark-antiquark mass dependence of all potential parameters is discussed in Sec. II B and the smooth dependence expected for a linear confinement potential is verified by the results obtained in Sec. III.

Section III A is devoted to a discussion of the results for the mass spectrum (from 0.1 to 10.6 GeV). The value of the strength parameters A for the Coulomb-type part of the interquark potential is found to be considerably larger than previously used by other workers, whereas the linear potential is somewhat weaker. To further test the model we have calculated $E1$ and $M1$ radiative widths (Sec. III B), and leptonic, two-photon, and gluon annihilation widths (Sec. III C). The results agree remarkably well with the available data and suggest that the wave functions are reasonably accurate at both small and large distances. The anomalous behavior of the pseudoscalar-meson masses which was qualitatively explained[17] by quark-antiquark annihilation into gluons is given a more quantitative basis and yields a reasonably accurate answer to this puzzle, not only for the η-η' problem but also for the η_c mesons in charmonium.

The coupling constant $\alpha_{\rm eff}$ at short distances for strong interactions in discussed in Sec. III C 3 and can be consistently related to the potential coupling constant A only if the number of quark flavors is taken to be eight. A brief discussion of the pion form factor is given in Sec. III D. Some future extensions and applications of the present approach are given in Sec. IV.

II. MODEL CONSIDERATIONS

A. Form of the wave equation

In this section we discuss the new features which we have introduced so that the potential quark models used by previous workers[5-10] can be extended to cover all mesons. A major obstacle presented in previous models is the use of perturbative treatments, particularly for the spin-spin term which was assumed to have the contact form:

$$V^{(0)}_{s_1 s_2} = \tfrac{4}{3}|\alpha_{\rm eff}| \frac{2\pi}{3m_1 m_2} \vec{\sigma}_1 \cdot \vec{\sigma}_2\, \delta(\vec{r}). \tag{1}$$

This interaction arises in a Fermi-Breit approximation to the Bethe-Salpeter[18] equation and yields the dominant energy splitting of pseudoscalar and vector mesons. Such splittings are very large for light mesons (such as the π-ρ situation) and a perturbative treatment is invalid. Unfortunately one cannot solve the above interaction exactly because for pseudoscalar states the attractive three-dimensional δ function leads to a divergent solution.

This apparent paradox is easily resolved when one remembers that the spin-spin term above arises because the approximation uses an expansion only up to $O(v^2/c^2)$. In a "proper" reduction of the Bethe-Salpeter equation which eliminates the small components entirely, an infinite succession of Chraplyvy[19] transformations must be carried out. Clearly such a procedure is not very practical and it is more appropriate to turn to a parametric model for such a transformation.

A second point to be considered arises because the theory of quarkonium ($q\bar{q}$) is based on a non-Abelian field theory such as quantum chromodynamics whereas the Fermi-Breit interaction is deduced for conventional positronium[20] situations. In using a mass parameter for quarks in a Bethe-Salpeter equation we presume such a mass describes a dressed quark which we assume has a surrounding gluonic cloud. The interaction between a dressed quark-antiquark pair will therefore involve gluon exchanges not only between the bare quark-antiquark pair but also between their gluon clouds. Since QCD is not yet a complete theory it would be difficult to proceed without making a simple model of this situation.

The major features of the above reasoning are that dressed and bare quarks will have different interaction strengths and that the dressed quarks have a finite size associated with the gluon cloud and with the series of transformations used to eliminate small components of the fully relativistic wave function. The latter are well known[21] in the case of the Dirac equation to give rise to an effective size for a Dirac particle. Long-range interactions in the Hamiltonian are essentially unaffected by the introduction of an effective size for the particles. On the other hand, short-range terms are modified and lead to a calculable theory.

As an appropriate model we assume all short-range terms (electric or magnetic) for point particles are replaced by conveniently chosen averages, i.e.,

$$\begin{aligned}\langle V \rangle &= V(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\\ &= \int d\vec{r}'\, d\vec{r}''\, U_1^*(\vec{r}-\vec{r}')V(\vec{r}'-\vec{r}'', \vec{\sigma}_1, \vec{\sigma}_2)\\ &\quad \times U_2(\vec{r}''-\vec{r})\,,\end{aligned} \tag{2}$$

where

$$U_i(\vec{s}) = \frac{\beta_i^{\,2}}{4\pi} e^{-\beta_i|\vec{s}|}/|\vec{s}|\,,$$

corresponding to a Yukawa form factor. The latter is both convenient (since all averages are analytic) and somewhat appropriate as it lies between the solutions for a purely Coulomb interaction and a purely linear interaction. We expect β_i^{-1} to be a measure of the finite radius of our quarks (q_i) and to be small relative to the average sizes of hadrons. This is borne out in the actual calculations discussed below where we find $\beta_i^{-1} \lesssim 0.15$ fm for all quarks. Nevertheless, the introduction of a small finite size is essential for the model to have any hope of success.

The model Hamiltonian we have chosen to solve is of the form ($\hbar = c = 1$)

$$H_{12} = (m_1^2 + p^2)^{1/2} + (m_2^2 + p^2)^{1/2} + \langle V_{SR} \rangle + V_{LR} + \langle V_A \rangle , \quad (3)$$

in which we neglect electromagnetic interactions and we use relativistic kinematics as represented by the first two terms. The potential-energy terms have three contributions. The first is the "electric" and "magnetic" short-range (SR) interaction presumably associated with one-gluon exchange. To reduce the number of parameters involved we have neglected some interactions, e.g., a term involving $\vec{L} \cdot (\vec{\sigma}_1 - \vec{\sigma}_2)$ becomes significant for $L \neq 0$ and $m_1 \gg m_2$. Such states are not calculated in the present analysis. Using $\vec{s} = \vec{r}' - \vec{r}''$, we have

$$V_{SR}(\vec{s}, \vec{\sigma}_1, \vec{\sigma}_2) = -\frac{4\lambda}{3} |\alpha_{eff}| \left\{ \frac{1}{s} - \frac{2\pi}{3} C_{SS} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta(\vec{s}) - \frac{3}{4} C_{LS} \vec{L} \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) \frac{1}{s^3} - \frac{3}{4} C_T \left[(\vec{\sigma}_1 \cdot \vec{s})(\vec{\sigma}_2 \cdot \vec{s}) - \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{3} s^2 \right] \frac{1}{s^5} \right\} \quad (4)$$

in which $\vec{L}$ is the orbital angular momentum operator $\vec{s} \times \vec{p}$. The averaged values for each component (central, spin-spin, spin-orbit and tensor) are given in the Appendix. As expected for finite-size particles the divergences in V_{SR} for $s \to 0$ are all removed and the resulting interactions are all well-behaved smooth functions of r. The multiplying factors λ, C_{SS}, C_{LS}, and C_T are discussed below.

The second term in Eq. (3) is chosen here to be a linear potential with no magnetic terms, i.e.,

$$V_{LR} = \frac{4\lambda}{3} |\alpha_{eff}| \frac{r}{r_0^2} . \quad (5)$$

This choice is consistent with the idea proposed by lattice gauge theory[22] that magnetic terms should be negligible relative to the electric term associated with confinement. We assume the strength of the linear term is dependent upon the running coupling constant α_{eff} in the *same* way as the short-range central potential. Such a dependence would be automatic if the short-range term were smoothly joined to the long-range term using an intermediate-range potential such as $\ln(r/r_0)$. We avoid using an intermediate-range interaction here since it raises further complications for the magnetic terms. The parameter r_0 determines the relative strength of the linear potential and is assumed to be a constant for all $q\bar{q}$ pairs independent of their mass or flavor.

The last term $\langle V_A \rangle$ in Eq. (3) was first considered qualitatively by De Rújula *et al.*[17] and attempts to account for the fact that 0^- mesons and 1^- mesons can annihilate into at least two gluons and three gluons, respectively. Such annihilations are also used to describe the hadronic decay widths of the hidden-charm (ψ) mesons. For point particles (1 and 2) the leading graphs are replaced by convenient parametric forms

$$V_A(0^-) = 4\pi K_2 \lambda^2 |\alpha_{eff}(1) \alpha_{eff}(2)| \delta(\vec{s}) \int d\vec{s}' \delta(\vec{s}') , \quad (6)$$

$$V_A(1^-) = 4\pi K_3 \lambda^3 |\alpha_{eff}^{3/2}(1) \alpha_{eff}^{3/2}(2)| \delta(\vec{s}) \int d\vec{s}' \delta(\vec{s}') , \quad (7)$$

which lead to a calculable theory when the δ functions are replaced by their averaged values as in the spin-spin case. As emphasized by De Rújula *et al.*,[17] the above mechanism yields zero for isovector states and important energy shifts and mixings for the isoscalar S-wave mesons. By assuming a simple mass dependence for the annihilation matrix elements given below we are able to eliminate the nonorthogonality difficulties encountered in the earlier work. Furthermore, we are able to extend the previous two-state theory (e.g., η-η' and ω-ϕ) to include all states which can couple together, i.e., excited radial excitations of each $q\bar{q}$ pair plus the mixing with several $c\bar{c}$ states.

B. Mass dependence of parameters

So far the model involves several coupling parameters which, because of the flavor independence of gluons, would be expected to be flavor independent. However, as pointed out by several authors[12] from QCD considerations these effective coupling constants should have a mass dependence, e.g.,

$$\alpha_{eff}(M^2) = \{ [\alpha_{eff}(M_0^2)]^{-1} + C \ln(M^2/M_0^2) \}^{-1} , \quad (8)$$

with $C = (11 - \frac{2}{3} N_f / 4\pi$ being directly related to the number of flavored quarks N_f. The above is expected to hold for small values of α_{eff}. Since we use $|\alpha_{eff}|$ in the range $0.15 \leq |\alpha_{eff}| \leq 0.45$ in the calculations below we adopt the above form for the running coupling constant α_{eff}. The characteristic mass M_c where $\alpha_{eff}(M_c^2) = 0$, is a parameter and in the present calculations we assume M_c depends upon the spin $\vec{S}=\vec{S}_1+\vec{S}_2$. For each quark-antiquark pair we replace the mass M by the sum of their individual quark masses (m_1+m_2). The parameter λ was introduced to allow for the gluon-cloud component of the dressed pair and we allow it to depend upon the total spin S. The overall coupling constant $A_S = \lambda_S \alpha_{\rm eff}$ is therefore assumed to satisfy the relation

$$A_S(m_1+m_2)=\left[A_S^{-1}(m_1'+m_2')+b_S \ln\left(\frac{m_1+m_2}{m_1'+m_2'}\right)^2\right]^{-1}, \tag{9}$$

with $b_S=(11-\frac{2}{3}N_F)/4\pi\lambda_S$. Clearly b_0, b_1 plus A_0 and A_1 for a given pseudoscalar and vector meson determine our coupling constants for all our quark pairs. The dependence of $A(m_1+m_2)$ on spin is necessary to fit all the data accurately, however, as indicated below for any meson multiplet the values of A_0 and A_1 do not differ by more than 15%.

On obtaining the mass dependence of the quark size we at first assumed β_i would be given by

$$\beta_i \sim m_i ,$$

corresponding to a Compton wavelength for the size parameter. However, in applying the model we soon realized that the above mass variation of β_i was too rapid and in the view of QCD not even appropriate. For a particle developing self-energy in a linear confining field the size of the gluon cloud should be scaling according to the results for a linear potential. The appropriate scaling for inverse lengths involves[15]

$$\beta_i=\beta[A(m_1+m_2)m_i]^{1/3}, \tag{10}$$

where β is a single mass-independent parameter. It is this scaling law which is used in all our calculations.

The C coefficients of the magnetic sums above also have a definite mass dependence which differs from the relation

$$C(m_1,m_2)\sim \frac{1}{m_1 m_2}$$

used for C_{SS} and C_T in atomic physics. An appropriate choice appears to be obtained by replacing the static-free particle propagators m_i^{-1} by effective momenta p_i^{-1} in a linear potential. In this case we estimate $(p_i^2)^{1/2}$ to scale as its expectation value in a Schrödinger approach:

$$\langle|p_i^2|\rangle^{1/2}\sim[m_i\langle|V_{\rm LR}|\rangle]^{1/2}$$

or

$$\langle|p_1^2|\rangle^{-1/2}\langle|p_2^2|\rangle^{-1/2}\sim[m_1^{-1}m_2^{-1}A_S(m_1+m_2)]^{1/3}.$$

Consequently we assume all the C coefficients above satisfy

$$C(m_1,m_2)=C\left[\frac{A(m_1+m_2)}{m_1 m_2}\right]^{1/3}, \tag{11}$$

where the strength C may be a different value for each term (SS, LS, or T) but is now mass independent.

The remaining mass dependence of the parameters involves the annihilation factors K_2,K_3. Since by dispersion theory level shifts Δ are related to level widths Γ and since the major mass dependence of annihilation widths involves[12] (see below) a simple M^{-2} variation we expect

$$\Delta \sim M^{-2} \tag{12}$$

as the approximate scaling law. Forming matrix elements of V_A in the basis which are eigenfunctions Ψ_α of the remaining Hamiltonian yields the results ($m_1=m_2=m_\alpha$ for state α)

$$\langle\alpha'|V_A(0^-)|\alpha\rangle=K_2\frac{A_0(2m_\alpha)A_0(2m_{\alpha'})}{E_\alpha E_{\alpha'}}\times\langle R_\alpha(0)\rangle\langle R_{\alpha'}(0)\rangle, \tag{13}$$

$$\langle\alpha'|V_A(1^-)|\alpha\rangle=K_3\frac{A_1^{3/2}(2m_\alpha)A_1^{3/2}(2m_{\alpha'})}{E_\alpha E_{\alpha'}}\times\langle R_\alpha(0)\rangle\langle R_{\alpha'}(0)\rangle, \tag{14}$$

in which K_2, K_3 are mass independent and $\langle R_\alpha(0)\rangle$ is the value of the radial solution $R_\alpha(r)$ in state α averaged around its value at $r=0$. The energies E_α are given by $2m_\alpha$ for ground states and $2m_\alpha+\Delta E_\alpha$ for excited states. The additional energy ΔE_α is just the excitation energy of the state α relative to its ground state. A more exact approach would iterate by successively using the calculated masses of the mixed systems until convergence is achieved. In the calculations below we ignore the interative procedure since a reasonable fit can be achieved without it.

The potential introduced here involves eleven parameters; four of them (A_0,A_1,b_0,b_1) represent the weakly spin-dependent central short-range term and three (C_{SS},C_{LS},C_T) of them describe the short-range magnetic terms. The remaining parameters are r_0 for the linear confinement, β for the averaging constant, and K_2, K_3 for the annihilation potential. Except for the weak spin dependence of A and b and the new averaging procedure via the parameter β the above model introduces no new parameters over previous potential models. The improved quality of the fits to data (Sec. III) over the entire meson spectrum suggests that the addition of a small number of parameters is a worthwhile exercise. Of course, we also use four quark masses

$m_n = m_u = m_d$, m_s, m_c, and m_b as fitting parameters but this is common to all potential models.

C. Method of solution

Solutions to the Hamiltonian [Eq. (3)] can be written as

$$\psi_{JM} = \sum_L [\chi_S \otimes u_L(r) Y_L(\hat{r})]^J_M , \quad (15)$$

where χ_S is the spinor for two fermions coupled to spin S, $Y_L(\hat{r})$ is a spherical harmonic of rank L in the quark-antiquark relative coordiniate, and $u_L(r)$ is the radial wave function associated with $Y_L(\hat{r})$. This state is coupled to total angular momentum J with projection M. Because the Hamiltonian contains tensor terms, two different values of the orbital angular momentum L will occur for some values of J. This leads to coupled equations for the radial wave functions $u_L(r)$. Straightforward numerical integration of these coupled equations is in general a difficult method of finding solutions. However, such a Hamiltonian can be easily solved as a matrix problem where the radial solution is expanded in a convenient basis.

The equation we wish to solve is

$$H_{12}\psi^n_{JM} = E_n \psi^n_{JM} . \quad (16)$$

We write the solution as

$$\psi^n_{JM} = \sum_L |SLJM\rangle u_{nL}(r) , \quad (17)$$

where $|SLJM\rangle$ contains only spin and angular parts of the wave function. We expand the radial solution by introducing the transformation in which for brevity the dependence on SLJ, etc., is suppressed:

$$u_{nL}(r) = \sum_\lambda u_{\lambda L}(r) T_{\lambda n} . \quad (18)$$

Using this in Eq. (16) along with orthogonality of the states ψ^n_{JM} we arrive at

$$\sum_{LL'} \sum_{\lambda\lambda'} T_{\lambda' n'} \langle \lambda' L' | H_{L'L} | \lambda L \rangle T_{\lambda n} = E_n \delta_{nn'} , \quad (19)$$

where we have defined

$$H_{L'L} = (SL'JM | H | SLJM) . \quad (20)$$

The round parentheses signify integration over spin and angular variables only.

In general this expansion is not carried out over the complete basis set. Once the choice of basis has been made, the number of states is increased until the solution is stable. We thus need only find the transformation $T_{\lambda n}$ which diagonalizes the finite matrix $\langle \lambda' L' | H_{L'L} | \lambda L \rangle$.

We have chosen to use radial harmonic-oscillator wave functions[23] as the basis $u_{\lambda L}(r)$. Oscillators are well known and easily generated. In addition, momentum-space oscillator wave functions are simply related to the radial space wave functions. This allows trival evaluation of momentum-dependent terms in the Hamiltonian. With proper choice of the oscillator-size parameter, eigenvalues of the Hamiltonian considered can be obtained to ~1% of the exact solution using as few as 8 states for each $u_L(r)$. If 18 states are used the solution is found to agree with the exact solution to at least three figures. All of the results quoted below were calculated with a basis set of 18 harmonic-oscillator states.

Because the interaction due to gluon annihilation $\langle V_A \rangle$ acts on only $J^P = 0^-$ and 1^- states, and because the form of the interaction requires some knowledge of the eigenvalues, we chose to prediagonalize all of the Hamiltonian except $\langle V_A \rangle$ and then use these solutions as the basis for diagonalizing $\langle V_A \rangle$.

III. RESULTS

A. Eigenvalues and eigenfunctions

Eigenvalues of quark-antiquark systems calculated with the Hamiltonian [Eq. (3)] are tabulated and compared with data in Table I. Eigenvalues and quark-antiquark admixtures are given for pseudoscalar mesons in Table II. The parameters of these calculations are given in Table III.

The parameters for $S=1$ states were determined by considering the $I^G(J^P)C = 0^-(1^-)-$ states of the $c\bar{c}$ and $b\bar{b}$ systems and the $I^G(J^P)C = 0^+(J^+)+$ P-wave states in $c\bar{c}$. These states, especially those in $c\bar{c}$, are well known and have been the target of several previous potential-model considerations of the meson spectrum. As will be discussed below, the averaging parameter β was determined by requiring that the known width for annihilation of the ψ state into e^+e^- be reproduced. The strength of the linear confining potential was chosen mostly by the $2S$-$1P$ splitting in $c\bar{c}$. The Coulomb-type term and the strength of the spin-spin interaction were adjusted to give the observed $1S$-$2S$ energy gap in both $c\bar{c}$ and $b\bar{b}$. Tensor and spin-orbit strengths were adjusted to fit the P-wave fine structure in $c\bar{c}$ and the $\psi''(3772)$ state.

Only the masses of normal and strange quarks and the gluon-annihilation coupling parameter K_3 in $I^G(J^P)C = 0^-(1^-)-$ mesons remain as adjustable parameters of $S=1$ mesons. The effect of the annihilation term $\langle V_A \rangle$ on vector states of $c\bar{c}$ and $b\bar{b}$ was found to be negligible. This is mostly due to the large masses and small α_{eff} relative to the normal and strange quarks. These three pa-

TABLE I. Calculated eigenvalues and assignments to physical mesons. Unless otherwise specified, data are from Ref. 36. Assignments enclosed in square brackets are felt to be somewhat questionable.

$I^G(J^P)C$	$^{2S+1}L_J$	$q\bar{q}$ content	E (MeV)	Assigned meson (E (MeV))	Γ_{total} (MeV)
$1^-(0^-)+$	$^1S_0$	$n\bar{n}$	132	$\pi^\pm(139.57)$ $\pi^0(134.96)$	
$\frac{1}{2}(0^-)$	$^1S_0$	$n\bar{s}$	490	$K^+(493.71)$ $K^0(497.70)$	
$0^+(0^-)+$	$^1S_0$	$n\bar{n}\oplus s\bar{s}\oplus c\bar{c}$	565	$\eta(548.8+0.6)$	
$1^+(1^-)-$	$^3S_1$	$n\bar{n}$	755	$\rho(773\pm2)$	152 ± 3
$0^-(1^-)-$	$^3S_1$	$n\bar{n}\oplus s\bar{s}$	787	$\omega(782.7\pm0.3)$	10.0 ± 0.4
$\frac{1}{2}(1^-)$	$^3S_1$	$n\bar{s}$	889	$K^*(892.2\pm0.5)$	49.4 ± 1.8
$0^+(0^-)+$	$^1S_0$	$s\bar{s}\oplus n\bar{n}\oplus c\bar{c}$	937	$\eta'(957.6\pm0.3)$	<1
$0^-(1^-)-$	$^3S_1$	$s\bar{s}\oplus n\bar{n}$	1016	$\phi(1019.7\pm0.3)$	4.1 ± 0.2
$1^+(1^+)-$ $0^-(1^+)-$	$^1P_1$	$n\bar{n}$	1106	$B(1228\pm10)$?	125 ± 10
$1^-(0^+)+$ $0^+(0^+)+$	$^3P_0$	$n\bar{n}$	1108	$\delta(976\pm10)$ $\epsilon(1100-1300)$	50 ± 20 ~600
$1^-(0^-)+$ $0^+(0^-)+$	$^1S_0$	$n\bar{n}$	1194		
$\frac{1}{2}(0^+)$	$^3P_0$	$n\bar{s}$	1252	$\kappa(1250\pm100)$	~450
$1^-(1^+)+$ $0^+(1^+)+$	$^3P_1$	$n\bar{n}$	1273	$A_1(\sim1100)$ $D(1286\pm10)$	~300 30 ± 20
$\frac{1}{2}(1^+)$	$^1P_1$	$n\bar{s}$	1289	$[Q(1200-1400)]$	
$0^+(2^+)+$ $1^-(2^+)+$	$^3P_2$	$n\bar{n}$	1328	$f(1271\pm5)$ $A_2(1310\pm5)$	180 ± 20 102 ± 5
$0^+(0^+)+$	$^3P_0$	$s\bar{s}$	1373	$[S^*(\sim993\pm5)$	$40\pm8]$
$\frac{1}{2}(1^+)$	$^3P_1$	$n\bar{s}$	1381	$[Q(1200-1400)]$	
$\frac{1}{2}(0^-)$	$^1S_0$	$n\bar{s}$	1392		
$\frac{1}{2}(2^+)$	$^3P_2$	$n\bar{s}$	1423	$K^*(1421\pm3)$	108 ± 10
$0^+(1^+)-$	$^1P_1$	$s\bar{s}$	1424		
$0^+(1^+)+$	$^3P_1$	$s\bar{s}$	1487	$[E(1416\pm10)$	$60\pm20]$
$1^+(1^-)-$	$^3S_1-^3D_1$	$n\bar{n}$	1521	$\rho'(\sim1600)$	$200-800$
$0^+(2^+)+$	$^3P_2$	$s\bar{s}$	1523	$f'(1516\pm3)$	40 ± 10
$0^-(1^-)-$	$^3S_1-^3D_1$	$n\bar{n}\oplus s\bar{s}$	1524		
$0^+(0^+)+$	$^1P_0$	$s\bar{s}$	1537		
$\frac{1}{2}(1^+)$	$^3P_1$	$n\bar{s}$	1614		
$1^-(2^-)+$ $0^-(2^-)+$	$^1D_2$	$n\bar{n}$	1675	$A_3(\sim1640)$?	~300
$1^+(2^-)-$ $0^-(2^-)-$	$^3D_2$	$n\bar{n}$	1708		
$0^-(1^-)-$	$^3S_1-^3D_1$	$s\bar{s}$	1712		

TABLE I. (*Continued*.)

$I^G(J^P)C$	$^{2S+1}L_J$	qq content	E (MeV)	Assigned meson (E (MeV))	Γ_{total} (MeV)
$0^-(3^-)-$ $1^+(3^-)-$	$^3D_3$	$n\bar{n}$	1754	$\omega(1667 \pm 10)$ $g(1690 \pm 20)$	150 ± 20 180 ± 30
$\frac{1}{2}(0^-)$	$^1S_0$	$n\bar{c}$	1875	$D^\pm(1868 \pm 1)$ [a] $D^0, \bar{D}^0(1863 \pm 1)$	
$0(0^-)$	$^1S_0$	$s\bar{c}$	1966	$F^\pm(2030 \pm 60)$ [b]	
$\frac{1}{2}(1^-)$	$^3S_1$	$n\bar{c}$	2013	$D^{*\pm}(2008.6 \pm 1.0)$ [a] $D^{*0}, \bar{D}^{*0}(2006 \pm 1.5)$	
$1^+(3^+)-$ $0^-(3^+)-$	$^1F_3$	$n\bar{n}$	2078		
$1^-(4^+)+$ $0^+(4^+)+$	$^3F_4$	$n\bar{n}$	2100	? $h(2040 \pm 20)$	193 ± 50
$0(1^-)$	$^3S_1$	$s\bar{c}$	2106	$F^{*\pm}(2140 \pm 60)$ [b]	
$1^-(4^-)+$ $0^+(4^-)+$	$^1G_4$	$n\bar{n}$	2405		
$0^+(0^-)+$	$^1S_0$	$s\bar{s} \oplus c\bar{c} \oplus n\bar{n}$	2816	$[X(2830?)]$ [c]	
$0^+(0^-)+$	$^1S_0$	$c\bar{c} \oplus n\bar{n} \oplus s\bar{s}$	2998	$\eta_c(2976 \pm 20)$ [d]	
$0^-(1^-)-$	$^3S_1$	$c\bar{c}$	3102	$\psi(3095 \pm 4)$ [e]	
$0^+(0^+)+$	$^3P_0$	$c\bar{c}$	3409	$\chi_0(3414 \pm 4)$ [f]	
$0^+(1^+)+$	$^3P_1$	$c\bar{c}$	3500	$\chi_1(3508 \pm 4)$ [f]	
$0^-(1^+)-$	$^1P_1$	$c\bar{c}$	3503	$P_c(?)$	
$0^+(2^+)+$	$^3P_2$	$c\bar{c}$	3528	$\chi_2(3552 \pm 6)$ [f]	
$0^+(0^-)+$	$^1S_0$	$c\bar{c} \oplus n\bar{n} \oplus s\bar{s}$	3630	$\eta_c'(3590?)$	
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$c\bar{c}$	3688	$\psi'(3684 \pm 5)$ [e]	
$0^-(1^-)-$	$^3D_1$-$^3S_1$	$c\bar{c}$	3761	$\psi''(3772)$ [g]	28 ± 5
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$c\bar{c}$	4049	$\psi(\sim 4040)$ [h]	
$0^-(1^-)-$	$^3D_1$-$^3S_1$	$c\bar{c}$	4085	$\psi(?)$	
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$c\bar{c}$	4330	$\psi(4410)$ [h]	33 ± 10
$\frac{1}{2}(0^-)$	$^1S_0$	$n\bar{b}$	5308	$D_b^\pm(?)$ $D_b^0, \bar{D}_b^0(?)$	
$\frac{1}{2}(1^-)$	$^3S_1$	$n\bar{b}$	5344	$D_b^{*\pm}(?)$ $D_b^{*0}, \bar{D}_b^{*0}(?)$	
$0(0^-)$	$^1S_0$	$s\bar{b}$	5388	$F_b^0, \bar{F}_b^0(?)$	
$0(1^-)$	$^3S_1$	$s\bar{b}$	5427	$F_b^{*0}, \bar{F}_b^{*0}(?)$	
$0(0^-)$	$^1S_0$	$c\bar{b}$	6293	$E^\pm(?)$	
$0(1^-)$	$^3S_1$	$c\bar{b}$	6346	$E^{*\pm}(?)$	
$0^+(0^-)+$	$^1S_0$	$b\bar{b}$	9421	$\eta_b(?)$	
$0^-(1^-)-$	$^3S_1$	$b\bar{b}$	9470	$\Upsilon(9460 \pm 10)$ [i]	
$0^+(0^+)+$	$^3P_0$	$b\bar{b}$	9727	$\chi_{0b}(?)$	
$0^+(1^+)+$	$^3P_1$	$b\bar{b}$	9839	$\chi_{1b}(?)$	

TABLE I. (*Continued.*)

$I^G(J^P)C$	$^{2S+1}L_J$	qq content	E (MeV)	Assigned meson (E (MeV))	Γ_{total} (MeV)
$0^+(2^+)+$	$^3P_2$	$b\bar{b}$	9878	$\chi_{2b}(?)$	
$0^-(1^+)-$	$^1P_1$	$b\bar{b}$	9899	$P_b(?)$	
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$b\bar{b}$	10021	$\Upsilon'(10\,016 \pm 20)$[i]	
$0^+(0^-)+$	$^1S_0$	$b\bar{b}$	10027	$\eta_b'(?)$	
$0^-(1^-)-$	$^3D_1$-$^3S_1$	$b\bar{b}$	10071	$\Upsilon''(?)$	
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$b\bar{b}$	10330	$\Upsilon(10\,380 \pm 40)$[j]	
$0^-(1^-)-$	$^3D_1$-$^3S_1$	$b\bar{b}$	10350	$\Upsilon(?)$	
$0^-(1^-)-$	$^3S_1$-$^3D_1$	$b\bar{b}$	10549	$\Upsilon(?)$	

[a]Reference 37.
[b]Reference 38.
[c]Reference 39.
[d]Reference 40.
[e]Reference 3.
[f]References 39 and 41.
[g]Reference 42.
[h]Reference 43.
[i]Reference 44.
[j]Reference 45.

rameters were adjusted until the ρ, ω, and ϕ mesons were in reasonable agreement with experiment. All other $S=1$ mesons were then calculated with no further parameter adjustment.

For $S=0$ mesons only the gluon-annihilation parameter K_2 and the coupling-constant scaling parameters b_0 and A_0 differ from the parameters found for $S=1$ mesons. Using the parameters found for $S=1$ mesons except allowing α_{eff} to vary, the masses of the $\pi, K,$ and D mesons can be fitted. Unlike the $S=1$ case the gluon-annihilation term was found to weakly influence the $c\bar{c}$ system. Thus K_2 for $S=0$ mesons was determined by including $n\bar{n}$, $s\bar{s}$, and $c\bar{c}$ systems and by attempting to fit the η and η' masses. All other $S=0$ mesons were then calculated using the chosen

TABLE II. Pseudoscalar mesons $[I^G(J^P)C=0^+(0^-)+]$ mixed through two-gluon annihilation.

$E_{unmixed}$ (MeV)	E_{mixed} (MeV)	Dominant admixtures of interest		
132	565 (η)	$0.71n\bar{n}(1)$	$-0.69s\bar{s}(1)$	$-0.01c\bar{c}(1)$
699	937 (η')	$0.63n\bar{n}(1)$	$+0.69s\bar{s}(1)$	$-0.02c\bar{c}(1)$
1194	1250	$0.94n\bar{n}(2)$	$+0.18s\bar{s}(1)$	
			$-0.11s\bar{s}(2)$	
1537	1554	$-0.11n\bar{n}(1)$	$+0.99s\bar{s}(2)$	
1840	1849	$\sim1.00n\bar{n}(3)$		
2052	2056		$\sim1.00s\bar{s}(3)$	
2336	2339	$\sim1.00n\bar{n}(4)$		
2455	2457		$\sim1.00s\bar{s}(4)$	
2763	2765	$\sim1.00n\bar{n}(5)$		
2815	2816 (X)	$0.02n\bar{n}(5)$	$+0.99s\bar{s}(5)$	$-0.02c\bar{c}(1)$
2986	2998 (η_c)	$+0.03n\bar{n}(1)$	$+0.02s\bar{s}(5)$	$+0.99c\bar{c}(1)$
			$-0.02s\bar{s}(6)$	
3156	3157	$\sim1.00n\bar{n}(6)$		
3166	3167		$\sim1.00s\bar{s}(6)$	
3537	3538		$\sim1.00s\bar{s}(7)$	
3556	3556	$\sim1.00n\bar{n}(7)$		
3627	3630 (η_c')	$+0.01n\bar{n}(1)$	$+0.01s\bar{s}(7)$	$+0.99c\bar{c}(2)$
		$+0.01n\bar{n}(7)$		
3910	3911		$\sim1.00s\bar{s}(8)$	
3941	3942	$\sim1.00n\bar{n}(8)$		
3977	3978			$\sim1.00c\bar{c}(3)$
4244	4245			$\sim1.00c\bar{c}(4)$

TABLE III. Parameters.

Quark masses	
$m_u = m_d \equiv m_n = 240$ MeV	
$m_s = 460$ MeV	
$m_c = 1700$ MeV	
$m_b = 5050$ MeV	
Potential terms	
$A_0(480\text{ MeV}) = 1.840$	$b_0 = 0.1111$
$A_1(480\text{ MeV}) = 1.726$	$b_1 = 0.0741$
$r_0 = 3.753\text{ GeV}^{-1}$	$\beta = 1.659\text{ GeV}^{2/3}$
$C_{SS} = 0.814\text{ GeV}^{-4/3}$ $C_T = 0.588\text{ GeV}^{-4/3}$	$C_{LS} = 0.188\text{ GeV}^{-4/3}$
$K_2 = 0.160$	$K_3 = 0.018$

parameters.

Eigenvalues of the $q\bar{q}$ systems considered are summarized in Table I. When the state has appreciable mixing of two orbital angular momenta, both are given as the spectroscopic description with the largest-amplitude term occurring first. Observed mesons which we feel correspond to these states are shown with their mass and total width. Assignments enclosed in square brackets are felt to be somewhat questionable. Mesons whose energies are listed as a question mark are not observed experimentally. However, these states would be expected if the $q\bar{q}$ model of mesons is valid and their widths are small enough. For several $n\bar{n}$ systems, two mesons, one with $I=0$ and one with $I=1$, should be observed but only one is known at the present time. The unobserved $n\bar{n}$ meson is denoted with a question mark. Total decay widths are given because for those mesons with large widths, which arise mostly from hadronic decays, there may be energy shifts from the unperturbed mass due to the decay. Mesons whose widths are indicated by a blank space have measured widths which are so small that no significant energy shift is expected or is unknown.

During the fitting procedure, if the state had a small total width, an attempt was made to reproduce the observed mass of the meson to within ~20 MeV. As can be seen in Table I, the agreement of the calculated mass with the measured mass is generally better than 20 MeV. Of the light, narrow mesons only the η and η' are off by as much as 20 MeV. However, these mesons are strongly mixed by the two-gluon annihilation term $\langle V_A \rangle$ which is only iterated once (see Sec. II A above). Essentially all broad, light mesons are within their total width of agreeing with experiment.

The light, strange mesons are composed of a quark and antiquark of different mass. Because the averaging coefficient β_i is mass dependent and because of the unequal masses, these mesons are a good test of whether the averaging and the coupling constant α_{eff} are being scaled consistently. All of the known, light, strange mesons are well reproduced by our potential. The assignment of the Q region to both ${}^1P_{1^+}$ and ${}^3P_{1^+}$ is made because this region appears to contain two resonances of $J^P = 1^+$. The charmed mesons are also in good agreement with experiment. The one possible discrepancy is the $F^{\pm}$ which is reported at 2030 ± 60 MeV and calculated to be at 1966 MeV. Since the width is unknown and the error on the energy is so large this does not appear to be a serious disagreement.

Because of the success with charmed and strange mesons we are encouraged that the same potential will do well in predicting the masses of mesons which contain a b quark with one of the other antiquarks or vice versa. We find pseudoscalar (vector) states of $n\bar{b}$ at 5308 (5344) MeV, of $s\bar{b}$ at 5388 (5427) MeV, and of $c\bar{b}$ at 6293 (6346) MeV. It is interesting to note that a recent experiment[24] at CERN has suggested that "bare b" has been observed near 5300 MeV.

The experimental situation concerning pseudoscalar $c\bar{c}$ states is unfortunately not as well determined as for the vector $c\bar{c}$ states. However, because vector states have apparently scaled correctly between light mesons and heavy mesons, and because the pseudoscalar mesons up through the charmed mesons are well reproduced, we feel the predictions of the present potential are meaningful for $c\bar{c}$ and $b\bar{b}$ systems. In the region below the $\psi(3095)$ we have two states with $I^G(J^P)C = 0^+(0^-)+$. The first is at 2816 MeV and is composed mostly of $s\bar{s}$ with ~2% $c\bar{c}$. The second state is at 2998 MeV and is predominantly $c\bar{c}$ in composition. A state which has been observed at DESY in the process $\psi \to X + \gamma$ at 2830 MeV would be consistent with a pseudoscalar with some small component of $c\bar{c}$. However, a 2% $c\bar{c}$ component in the wave function does not seem to be large enough to explain the observed transition. There is recent evidence[25] of a narrow state at 2976 ± 20 MeV in the photon decay of ψ'. Our calculation of a predominantly $c\bar{c}$ pseudoscalar at 2998 MeV suggests that this is the η_c.

Shifts in masses of the $I^G(J^P)C = 0^+(0^-)+$ mesons due to the two-gluon-annihilation term $\langle V_A \rangle$ are summarized in Table II. Energies before and after mixing are given. Where an assignment has been made in Table I, that meson is given in parentheses next to the energy of the state. Dominant amplitudes of the $n\bar{n}$, $s\bar{s}$, and $c\bar{c}$ components are given also. The number in parentheses after the $q\bar{q}$ component is the radial quantum number of the dominant amplitude of that $q\bar{q}$ component. Notice that the η and η' mesons are predicted to be of almost equal amounts of $n\bar{n}$ and $s\bar{s}$ with 1 or 2% $c\bar{c}$. 1% $c\bar{c}$ component in the η would be helpful to an understanding of the observed decay $\psi' \to \psi + \eta$ with 4% branch. If the η were all $c\bar{c}$, this would correspond to a width of ~100 MeV which is comparable to observed allowed hadronic decays.

Three-gluon annihilation mixes the $I^G(J^P)C = 0^-(1^-)-$ mesons. The strength of this term was adjusted by considering the masses and decays of the ρ, ω, and ϕ mesons. In principle, $c\bar{c}$ and $b\bar{b}$ vector mesons will mix with $n\bar{n}$ and $s\bar{s}$ mesons through this term. However, mixing of $c\bar{c}$ and $b\bar{b}$ with the lighter flavor systems was found to be negligible. Diagonal energy shifts of $c\bar{c}$ and $b\bar{b}$ due to this term were less than 1 MeV. Masses of ω and ϕ mesons including mixing of $n\bar{n}$ and $s\bar{s}$ only are given in Table I. The wave functions of these states are $0.99n\bar{n}(1) - 0.07s\bar{s}(1)$ for ω and $-0.07n\bar{n}(1) - 0.99s\bar{s}(1)$ for ϕ. Energy shifts are $755 \to 787$ MeV for ω and $1009 \to 1016$ MeV for ϕ.

Octet-singlet mixing angles can be calculated from the ratios of admixture amplitudes for the $n\bar{n}$ and $s\bar{s}$ components. From the eigenstates given in Table II and the preceding paragraph we find the octet-singlet mixing angle to be $+39°$ for isoscalar vector mesons and $-10° \pm 2°$ for isoscalar pseudoscalar mesons. The uncertainty in the pseudoscalar result is due to the different amount of $c\bar{c}$ in the η and η' mesons as shown in Table II. These are in excellent agreement with the values obtained for the mixing angle (θ_{quad}) from a quadratic mass formula[36] as would be suggested by a relativistic mass dependence. Extracted values are[36] $\theta_{\text{quad}}(\text{vector}) = +40° \pm 1°$ and $\theta_{\text{quad}}(\text{pseudoscalar}) = -11° \pm 1°$.

B. Radiative transitions

1. Magnetic dipole

The radiative widths of vector mesons (V) decaying to lower-lying pseudoscalar mesons (P) have been calculated here by a simple generalization of the usual charmonium formula for $M1$ transitions:

$$\Gamma(V \to P + \gamma) = \tfrac{16}{3}\mu^2\omega^3 I_{\text{LW}}^2 . \tag{21}$$

For charmonium viewed as a pure $c\bar{c}$ configuration the various terms in Eq. (21) are given by (in units with $\hbar = c = 1$)

$$\omega = \Delta E(1 - \Delta E/2M_V),$$

$$\mu = \left(\frac{2}{3}\frac{\sqrt{\alpha}}{2m_c}\right)\mu_c,$$

$$I_{\text{LW}} = \int_0^\infty R_V(r)R_p(r)r^2 dr,$$

wherein ΔE is the mass difference $(M_V - M_P)$ between the vector and pseudoscalar mesons, m_c is the c-quark mass (1.7 GeV), α is the fine-structure constant (e^2), and μ_c is the multiplier describing the effective moment in units of the Dirac moment ($Q_c\sqrt{\alpha}/2m_c$). The overlap integral I_{LW} is the radial overlap for the $M1$ operator in the long-wavelength approximation. The expression for ω used here allows for relativistic kinematics in the two-body final states.

Two modifications are necessary in order to extend the above formulas to meson decays involving multiflavor configurations. The first modification improves on the long-wavelength approximation by replacing I_{LW} by

$$I_f = \int_0^\infty R_{Vf}(r)j_0(\omega r/2)R_{Pf}(r)r^2 dr, \tag{22}$$

in which the radial solutions may now depend upon the flavor of the $q\bar{q}$ pair involved in a given component.

The second modification is to allow for the various combinations of $q\bar{q}$ favors which occur due to the two- and three-gluon annihilation interactions discussed in Sec. II A above. We replace $(\mu I_{\text{LW}})^2$ by

$$(\mu I_{\text{LW}})^2 = \left[\sum_f a_{Vf}a_{Pf}\left(\frac{Q_f\sqrt{\alpha}}{2m_f}\right)\mu_f I_f\right]^2 \tag{23}$$

and use the admixture coefficients a_{Vf} and a_{Pf} obtained in the eigenfunctions (Sec. III A) for each meson. For example, in the $\rho \to \pi + \gamma$ situation only the $u\bar{u}$ and $d\bar{d}$ configurations occur and we find

$$a_{Pu}=+a_{\pi u}=\frac{1}{\sqrt{2}},\quad Q_u=\tfrac{2}{3},$$

$$a_{Pd}=+a_{\pi d}=-\frac{1}{\sqrt{2}},\quad Q_d=-\tfrac{1}{3},$$

wherein the admixture coefficients are the appropriate isospin vector coupling coefficients for isovector mesons. If we assume $\mu_u/m_u \simeq \mu_d/m_d$ for simplicity in the present example then we obtain, with $I_u=I_d=I$,

$$\Gamma(\rho\to\pi+\gamma)=\tfrac{16}{3}\,\mu_{\rm eff}{}^2\omega^3 I^2\,, \qquad (24)$$

in which

$$\mu_{\rm eff}{}^2=\left[\left(\frac{1}{6}\frac{\sqrt{\alpha}}{2m_u}\right)\mu_u\right]^2$$

agrees with the results used by other workers.[10,12]

In the calculations carried out here we have used quark masses from Sec. II A, i.e., $m_u=m_d=0.24$ GeV, $m_s=0.46$ GeV, $m_c=1.70$ GeV, and $m_b=5.05$ GeV. To avoid additional parameters the values of μ_f for $f=u,d,s$ have been taken from a fit to the static baryon magnetic moments for the proton, neutron, and Λ. Such a fit was made using only the sum over quark spin operators[26] and yields

$$\mu_u=0.711,\quad \mu_d=0.736,\quad \mu_s=0.897\,.$$

Consequently the first seven transitions shown in Table IV are calculated with no new parameters since the various amplitudes needed are determined by diagonalization of the energy matrix and the quark magnetic moments are fixed from other data. For the five measured $M1$ widths we see the calculations are in excellent agreement with the data.

In the case of the $\omega\to\pi+\gamma$ transition the improved agreement with experiment compared to previous calculations[27,28] is due largely to the use of perturbed radial solutions for R_V and R_P which reduced I^2 by almost 30%. The $\phi\to\pi+\gamma$ transition is nonzero because of the small (~6% amplitude) ω-like admixture which occurs via three-gluon-annihilation mixing. The $K^{+*}\to K^+ +\gamma$ was calculated approximately by use of an overlap integral like Eq. (22) above which neglects the mass difference between the u and s quarks. We expect a more correct treatment will not change the results by more than 20%. The unequal-mass $q\bar{q}$ pair (e.g., $u\bar{s}$) also involves a small change in the calculation since the individual magnetic moments in the $q\bar{q}$ pair are no longer equal in magnitude.

The remaining transitions shown in Table IV involve the unknown factor μ_c for charmed quarks. Naively we would expect $\mu_c \lesssim 1$ (as occurs for μ_u, μ_d, and μ_s) so that the assignment of the η_c to the recently observed state at 2976 ± 20 GeV does have the right order of magnitude for the $\psi'\to\eta_c+\gamma$ decay width.[25] The small predicted magnitudes for the $M1$ widths for the last two transitions shown in Table IV suggest that these will prove to be difficult to observe experimentally. Until μ_c is determined it appears to be uninteresting to estimate decays such as $D^*\to D+\gamma$.

TABLE IV. Magnetic dipole transitions $V\to P+\gamma$.

Transition	$\Gamma_{\rm expt}$ (keV)	$\Gamma_{\rm theory}$ (keV)
$\omega\to\pi+\gamma$	880 ± 80 [a]	868
$\rho\to\pi+\gamma$	37 ± 11 [a]	77
$\phi\to\pi+\gamma$	5.7 ± 2.2 [a]	7.5
$\omega\to\eta+\gamma$	<50 [a]	11
$\rho\to\eta+\gamma$	?	70
$\phi\to\eta+\gamma$	82 ± 20 [a]	97
$K^{+*}\to K^+ +\gamma$	74 ± 39 [a]	94
$\psi'\to\eta_c+\gamma$	~1 [b]	$9.2\,\mu_c^2$
$\psi'\to\eta_c'+\gamma$	(<5) [c]	$1.2\,\mu_c^2$
$\psi\to\eta_c+\gamma$	?	$1.5\,\mu_c^2$
$\eta_c'\to\psi+\gamma$	?	$6.0\,\mu_c^2$
$\Upsilon'\to\eta_b+\gamma$	?	$0.3\,\mu_b^2$

[a]Reference 36.
[b]References 25 and 40.
[c]See Refs. 10 and 12.

2. Electric dipole

In this work we restrict our attention to $E1$ transitions in charmonium. At present there are no measured results for $E1$ for any other $q\bar{q}$ systems. The results given in Table V are calculated

TABLE V. Electric dipole transitions in charmonium.

Transition	$\Gamma_{\rm expt}$ (keV) [a]	$\Gamma_{\rm theory}$ (keV) Present work	Ref. 10
$\psi'\to\chi_{2^+}$ (3.55)	16 ± 8	40 (23)	36 (36)
$\psi'\to\chi_{1^+}$ (3.51)	18 ± 8	53 (35)	50 (40)
$\psi'\to\chi_{0^+}$ (3.41)	16 ± 8	7.4 (5)	58 (41)
χ_{2^+} (3.55) $\to\psi$	?	304	460
χ_{1^+} (3.51) $\to\psi$	?	236	350
χ_{0^+} (3.41) $\to\psi$	?	145	170
$\eta_c'\to P_c(3.5)$	?	44	
$P_c(3.5)\to\eta_c$	?	351	

[a]Reference 41.

from the expressions for a single flavor pair ($f\bar{f}$) transition:

$$\Gamma(J_i \to J_f; E1) = \tfrac{4}{3}\omega^3 Q_f^2 (2J_f+1) \left| \sum_{L_i L_f} G_{L_i L_f} \right|^2 ,$$
$$G_{L_i L_f} = \delta_{S_i S_f} (-)^{L_I} (2L_i+1)^{1/2} \langle L_i 100 | L_f 0 \rangle \times W(J_i J_f L_i L_f; 1S_i) I_{L_i L_f} , \quad (25)$$

in which the overlap integral is given by

$$I_{L_i L_f} = \int R_{L_f}(f) \frac{3}{\omega}\left[\frac{2}{\omega r} j_0\left(\frac{\omega r}{2}\right) - j_1\left(\frac{\omega r}{2}\right)\right] R_{L_i}(r) r^2 dr$$

and includes an improvement[29] over the long-wavelength approximation. The multiple values of L_i and L_f are allowed for triplet spin states ($S_i = 1$ and $S_f = 1$) only when the tensor interaction can mix $L = J \pm 1$ states. The quantities $\langle L_i 100 | L_f 0\rangle$ and $W(J_i J_f L_i L_f; 1S_i)$ are the usual vector coupling coefficient and Racah coefficient, respectively.

Also shown in Table V are the results obtained by the simpler model of Krammer and Krasemann[10] which ignores spin-dependent distortions and relativistic corrections. In most cases the present calculations are within 35% of the previous $E1$ calculations although this is not true for the $\psi' \to \chi_{0^+} + \gamma$ decay. The large difference in this transition strength is due to the coherent addition of spin-orbit and tensor terms for the χ_{0^+} state which significantly contracts the $L = 1$ radial function. This happens to be a large effect for the $\psi' \to \chi_{0^+} + \gamma$ decay because of the node in the ψ' radial solution with $L_i = 0$ which allows strong destructive interference effects to arise. This particular result suggests that the use of "unperturbed" radial solutions can lead to large errors in γ-decay calculations. Unfortunately, the experimental data are not yet accurate enough to distinguish such differences in a clear-cut way.

In comparing theory with experiment it should be pointed out that we have used model masses to determine the radiative transition strengths. If we use experimental masses instead of model masses to determine the ω^3 factor we would obtain somewhat closer agreement with experiment, i.e., 7 keV (0^+), 44 keV (1^+), and 23 keV (2^+). Only the 1^+ case appears to be in disagreement with experiment and even in this case it is only 3 standard deviations. Applying the corrections of Novikov *et al.*[12] arising from $M2$ amplitudes and the interaction of the quark magnetic moments with the electric vector of the photon field yields the results shown in parentheses in Table V.

C. Annihilation widths

1. Annihilation into e^+e^-

For point quarks of a definite flavor the e^+e^- decay of vector mesons with $J^{PC} = 1^{--}$ are assumed to proceed via a virtual photon according to the decay-width formula[12]:

$$\Gamma(1^{--} \to e^+e^-) = \frac{4\alpha^2 Q_f^2}{M_V^2} \left| R_S(0) + \frac{\sqrt{50}}{M_V^2} R_D''(0) \right|^2 . \quad (26)$$

This simple expression depends upon the charge (Q_f) of the quark with definite flavor f, the mass of the vector meson (M_V), and the two amplitudes at the origin: the S-wave radial function and the second derivative of the D-wave radial function.

The above formula in its basic form was introduced by Van Royen and Weisskopf[28] and is valid in the nonrelativistic domain. It has been pointed out by Barbieri *et al.*[30] that there are important relativistic corrections to the Van Royen–Weisskopf expression which are significant in charmonium—at least for absolute values of the various e^+e^- decay widths.

In the calculations carried out here we have attempted to improve on the Van Royen–Weisskopf formula by replacing $R_S(0)$ by its average value:

$$\langle R_S(0)\rangle = \int R_S(\vec{r}) \int U^*(\vec{r} - \vec{r}')\delta(\vec{r}' - \vec{r}'') \times U(\vec{r}'' - \vec{r}) d\vec{r}' d\vec{r}'' d\vec{r} . \quad (27)$$

This replacement is consistent with the averaging procedure used to obtain the interaction operators in Sec. II A. Since the averaging procedure is a parametric method of including relativistic and finite-size effects we expect the above formula to be a reasonably accurate estimate of absolute values of e^+e^- decay widths.

For mesons with several flavor components (e.g., ρ, ω, ϕ) the only additional modification is to replace

$$Q_f^2 \left| R_S(0) + \frac{\sqrt{50}}{M_V^2} R_D''(0) \right|^2$$

in the Van Royen–Weisskopf formula by

$$\left| \sum_f Q_f a_{Vf} \left(\langle R_{Sf}(0)\rangle + \frac{\sqrt{50}}{M^2} R_{Df}''(0) \right) \right|^2 ,$$

in which a_{Vf} is the admixture coefficient for a particular $q\bar{q}$ flavor. For simplicity we have not averaged the small correction term arising from the D-state annihilation.

The results of our calculations for the most interesting cases are compared with experimental values in Table VI. The calculated widths are entirely determined by the results of the matrix diagonalization discussed in Sec. III A. These results are in remarkable agreement with experiment not only for charmonium and upsilonium but also for the light vector mesons. Except as noted above for the ψ particle these calculations are parameter free. Since relativistic effects are

TABLE VI. Lepton annihilation $V \to e^+e^-$.

Transition	Γ_{expt} (keV)	Γ_{theory} (keV)
$\rho(773) \to e^+e^-$	6.54 ± 0.90 [a]	6.00
$\omega(783) \to e^+e^-$	0.76 ± 0.17 [a]	0.79
$\phi(1\,020) \to e^+e^-$	1.31 ± 0.15 [a]	1.62
$\psi(3\,095) \to e^+e^-$	4.8 ± 0.6 [b]	4.71
$\psi(3\,684) \to e^+e^-$	2.1 ± 0.3 [b]	1.31
$\psi(3\,772) \to e^+e^-$	(0.37) [b]	0.40
$\psi(4\,040) \to e^+e^-$	0.75 ± 0.10 [c]	0.67
$\psi(4\,159) \to e^+e^-$	0.77 ± 0.20 [c]	0.42
$\psi(4\,414) \to e^+e^-$	0.44 ± 0.14 [b]	0.40
$\Upsilon(9\,460) \to e^+e^-$	1.2 ± 0.2 [d]	0.79
$\Upsilon(10\,020) \to e^+e^-$	0.33 ± 0.10 [d]	0.20
$\Upsilon(10\,330) \to e^+e^-$	?	0.12

[a]Reference 36.
[b]Reference 46.
[c]Reference 43.
[d]Reference 44.

strongest in the light-quark mesons the present approach must be including these effects reasonably accurately.

The occurence of S-D mixing is important for the $\psi(3772)$ decay since it is due to the tensor interaction that this predominantly D state gains enough S-wave strength to be observable. The similar state in the $b\bar{b}$ system has a calculated energy of 10.071 GeV and a small theoretical e^+e^- width of 48 eV—which presumably explains why it has not been observed as yet. The $b\bar{b}$ analogs of the $c\bar{c}$ states at 4.085 and 4.330 GeV are predicted to be at 10.350 and 10.549 GeV with e^+e^- widths of 16 and 84 eV, respectively. The state at 10.549 GeV may eventually be observable although this would depend upon whether it lies above the actual D_b-$\bar{D}_b$ threshold or not. According to the current model calculations (see Table I), the D_b-$\bar{D}_b$ threshold is near 10.6 GeV.

2. *Annihilation into two photons*

As in Sec. III C 1 the situation for two-photon decay is well developed for nonrelativistic situations with point quarks. For pseudoscalar mesons with $J^{PC} = 0^{-+}$ the annihilation into two photons occurs[28] via the $M1$ emission of one photon leading to intermediate $q\bar{q}$ states with $J^{PC} = 1^{--}$ followed by the annihilation of this virtual vector meson into a photon. Following Van Royen and Weisskopf we assume the $M1$ transition only occurs to vector mesons of the same multiplet. For point quarks of a definite flavor we have the decay width into two photons given by

$$\Gamma(P \to 2\gamma) = \frac{3Q_f^4\alpha^2}{m_f^2}\left(\frac{M_P}{M_V}\right)^3 |R_{Sf}(0)|^2, \quad (28)$$

which reduces to the result given by Novikov *et al.*[12] for heavy quarks because the approximation $M_P \simeq M_V \simeq 2m_f$ is reasonably appropriate.

In a similar manner to the e^+e^- annihilation we attempt to improve the above result by replacing $R_{Sf}(0)$ by the average value $\langle R_{Sf}(0)\rangle_V$ for the vector meson of flavor f. For the multiflavor pseudoscalar mesons we replace the single-flavor factor

$$\left[\frac{Q_f^2}{m_f}\left(\frac{1}{M_{Vf}}\right)^{3/2} R_{Sf}(0)\right]^2$$

by the expression

$$\left|\sum_V \left[\sum_f a_{Pf}a_{Vf}\frac{Q_f}{m_f}\mu_f I_f\right] M_V^{-3/2} \times\left[\sum_{f'} a_{Vf'}Q_{f'}\langle R_{Sf'}(0)\rangle_V\right]\right|^2,$$

in which the first term in brackets describes the $M1$ transition $P \to V + \gamma$ in terms of the same quantities used for $M1$ transitions in Sec. III B 1 above. The second term in brackets describes the one-photon annihilation of the intermediate vector meson V.

As an example consider the case of $\pi^0 \to 2\gamma$ where we have only two flavors in P,

$$a_{\pi u} = \frac{1}{\sqrt{2}}, \quad a_{\pi d} = -\frac{1}{\sqrt{2}}.$$

However, due to the three-gluon-annihilation mixing interaction we have to consider ρ, ω, and ϕ mesons in the sum over V. Neglecting the ω-ϕ mixing leads (with the approximation $\mu_u \simeq \mu_d$) to the result

$$\Gamma(\pi^0 \to 2\gamma) \simeq \frac{3e^2}{16}|\langle R_S(0)\rangle_V|^2 \times\left(\frac{\sqrt{2}}{3}M_\rho^{-3/2} + \frac{\sqrt{2}}{3}M_\omega^{-3/2}\right)^2\left(\frac{e\mu_u}{m_u}\right)^2 M_\pi^3, \quad (29)$$

which except for averaging $R_S(0)$ is just 3 times larger than the result given by Van Royen and Weisskopf due to the inclusion of color. The same factor of 3 occurs between positronium and quarkonium as indicated by Novikov *et al.* for colored quarks.

Using the general formula our calculations include ω-ϕ mixing as a two-state problem and η-η'-η_c mixing as a three-state problem. The results for four pseudoscalar-meson annihilations are shown in Table VII. In view of the fact that the π meson is described entirely as $q\bar{q}$ here, the $\pi^0 \to 2\gamma$ decay is in fair agreement with experiment. The possibility that the π meson has significant admixtures of $qq\bar{q}\bar{q}$ configurations is not unlikely,[31] and we suspect such higher-order configurations would serve to reduce our value for the $\pi^0 \to 2\gamma$ decay. Although the value we obtain

TABLE VII. Anihilation into two photons.

Transition	Γ_{expt} [a]	Γ_{theory}
$\pi^0 \to 2\gamma$	7.95 ± 0.55 eV	11.6 eV
$\eta \to 2\gamma$	0.33 ± 0.06 keV	0.96 keV
$\eta' \to 2\gamma$	<20 keV	3.9 keV
$\eta_c \to 2\gamma$	?	$4.5\,\mu_c^2$ keV

[a]Reference 36.

for the $\eta \to 2\gamma$ decay is too large, it is important to realize that the results are quite sensitive to the pseudoscalar mixing. For example, if we use pure SU(3) admixture coefficients for η we obtain a factor of 2 reduction in the calculated width. Such a sensitivity to the admixture coefficients was also found by Van Royen and Weisskopf. The result for $\eta_c \to 2\gamma$ is similar to the value suggested by the simpler calculations of Novikov *et al.*

3. *Annihilation into gluons*

For heavy mesons where α_{eff} is small an estimate of the hadronic annihilation width via two and three gluons has been given in the literature.[32] We restrict our attention to pseudoscalar and vector mesons in charmonium which have decay widths given by[33]

$$\Gamma(P \to 2g) \simeq \frac{9}{8}\frac{\alpha_{\text{eff}}^2}{\alpha^2}\Gamma(P \to 2\gamma), \tag{30}$$

$$\Gamma(V \to 3g) \simeq \frac{5}{18\pi}(\pi^2 - 9)\frac{\alpha_{\text{eff}}^3}{\alpha^2}\Gamma(V \to e^+e^-). \tag{31}$$

In the literature the last formula has been used to calculate α_{eff} at *annihilation distances*. One finds $\alpha_{\text{eff}} \sim 0.19$ from the J/ψ decays.[12] Recent calculations have noted that this value of α_{eff} is not the same as the running coupling constant (which we call A in Sec. II) appropriate to the potential energy. According to the discussion in Sec. II we have the relation

$$A_1 = \lambda_1 \alpha_{\text{eff}}(V),$$

which for vector mesons determines $\lambda_1 = 6.1$. Such a result implies a strong gluon-cloud effect in charmonium and also allows us to predict the number of flavors N_f from the relation

$$b_1 = 0.074 = (11 - \tfrac{2}{3}N_f)/4\pi\lambda_1, \tag{32}$$

in which b_1 is given its numerical value from the results in Sec. II A. We find $N_f = 8$ almost exactly.

It is also likely that $\alpha_{\text{eff}}(P) \to 2g$ is not the same as $\alpha_{\text{eff}}(V)$ for $V \to 3g$ although we would expect them to be quite similar. From the relations

$$\frac{b_1}{b_0} = \frac{\lambda_0}{\lambda_1} \text{ and } A_0 = \lambda_0 \alpha_{\text{eff}}(P),$$

we find with $b_0 = 0.1111 = 1.50\, b_1$ that $\lambda_0 = \frac{2}{3}\lambda_1$. For charmonium this means $\alpha_{\text{eff}}(P) = A_0(c\bar{c})/(4.06) = (1.02)/(4.06) = 0.25$ which is not dissimilar to $\alpha_{\text{eff}}(V)$ in charmonium. Using this value of $\alpha_{\text{eff}}(P)$ we predict $\Gamma(\eta_c \to 2g)$ from Eq. (30) above and the value of $\Gamma(\eta_c \to 2\gamma)$ from Sec. III B 2 to be 6.0 μ_c^2 (MeV). For $\mu_c^2 \sim 1$ this estimate is in good agreement with previous estimates[12] of the $\eta_c \to 2g$ decay.

D. Pion form factor

The pion form factor in the spacelike region has been extracted from pion-electron scattering and from model-dependent analyses of pion electroproduction. From these analyses the rms radius of the pion was found to be in the range $0.52 \leq \langle r_\pi^2 \rangle^{1/2} \leq 0.79$ F. The lower values are from the most recent pion-electron scattering experiment.[34] Electroproduction experiments tend to find[35] $\langle r_\pi^2 \rangle^{1/2} \approx 0.71$ F. Our model gives the pion rms radius to be $\langle r_\pi^2 \rangle^{1/2} = 0.425$ F. Also, our calculated form factor overestimates the "measured" form factor from pion electroproduction by approximately a factor of 2 at a momentum transfer $q^2 = 2.0$ GeV^2.

The pion form factor from electroproduction is reasonably reproduced by the vector-dominance model in which the pion couples to an $I = 1$ vector meson which then couples to the photon. In principle the linear confining potential in our model parameterically includes the effects of coupling at large distances to configurations other than simple $q\bar{q}$. However, it would seem to be overextending the applicability of the model to expect an average term such as our linear potential to account for a specific $qq\bar{q}\bar{q}$ configuration. Since we appear to overestimate the pion form factor, the results we get are somewhat encouraging. Any additional degrees of freedom will tend to reduce the form factor. We must therefore leave any serious comparison with experiment until more complex configurations than $q\bar{q}$ can be explicitly included in the calculations. Suggestions of how to include $q\bar{q}$ creation processes have been given by Eichten *et al.*,[7] but it remains to be seen whether the atomic quark models can eventually be made consistent with the collective pictures[31] based on PCAC (partially conserved axial-vector currents) and chiral symmetry.

IV. CONCLUSIONS

The Hamiltonian for $q\bar{q}$ systems presented here involves several important revisions from those used previously by other authors

(i) Quarks are given effective finite extent to account for self-interactions and Chraplyvy transformations which eliminate small components.

(ii) All interactions are treated exactly using a nonperturbative matrix diagonalization with a harmonic-oscillator basis set.

(iii) Kinetic energy is solved relativistically.

(iv) Two- and three-gluon annihilation graphs are included as effective potentials and orthogonality of eigenstates maintained.

(v) Mass-dependent parameters are scaled according to linear-confinement expectations rather than according to free-particle recipes.

(vi) The Coulomb-type strong-interaction potential is allowed more strength than has been customary and is weakly spin dependent.

(vii) The transition point (between the short-range r^{-1} interaction and the linear confinement term) at $r=r_0$ is taken to be flavor and mass independent. This in turn means that the overall potential including the linear potential scales with the strength of the running coupling constant rather than just the short-range component.

(viii) No constant potential terms are used to adjust each $q\bar{q}$ multiplet spectrum.

The above list appears to us to be a minimum set of revisions which need to be adopted so that a potential model approach can successfully describe both light and heavy mesons yet remain reasonably within the guidelines of QCD. Item (vii) corresponds to the result suggested by potentials of the flavor-independent logarithmic type $[\ln(r/r_0)]$ which are also zero at $r=r_0$ in a mass-independent manner. Since QCD is not a complete theory it is not clear whether (vii) and the spin dependence noted in item (vi) are outside of the QCD guidelines or not. It is clear from our calculations, however, that within the framework of our potential model a satisfactory description of light and heavy mesons cannot be obtained with a constant linear potential nor with a spin-independent running coupling constant. It is possible and perhaps even quite likely that these parameter dependences arise because we have not considered all degrees of freedom in the meson system. In particular we have ignored the explicit coupling[7] between $q\bar{q}$ and more complicated multiquark configurations such as $q\bar{q}q\bar{q}$. Such couplings are indeed spin dependent (as are the gluon annihilation terms already considered) and in general not restricted to short distances. Item (viii) above not only reduces the number of parameters in the model, but also removes an unattractive component of the potential occasionally used by previous workers.

The results obtained can be summarized in terms of the data fitted which includes 45 meson masses and over 20 decay widths. Overall we introduced eleven potential parameters and four quark masses so we have over 4 data points per parameter. Previous potential models have at best fitted about two data points per parameter. In view of this we regard the results as much more than an exercise in curve fitting. In fact, our results would appear to represent a strong endorsement of the underlying QCD approach to all strong interactions.

We should hasten to add at this point that it is very unlikely that the present model is unique in its ability to represent the data considered. We did not perform enough adjustments of parameters or of the potential form to rule out the existence of equivalent potentials. On the other hand, it is not clear at present whether such extensive searches are really worthwhile. Until QCD or some alternate basic theory can provide more definitive guidance it appears unlikely that a unique $q\bar{q}$ potential model should be expected. Finding even one potential model based on QCD which can successfully describe the entire range of meson masses is already a minor triumph.

It is certainly valid to question how complete our model description of mesons actually is. Clearly there is room for further refinements.[47] In order to describe the full range of hadron dynamics involved in meson spectroscopy it will be necessary[7] to include the explicit coupling of $q\bar{q}$ configurations to the more complex $q\bar{q}q\bar{q}$ configurations. Such multiquark configurations require an understanding of qq interactions as well as $q\bar{q}$ interactions and, moreover, will involve pairs of particles in symmetric and octet color states as well as totally antisymmetric color states.

An appropriate first step in the study of q-q interactions will be to study the basic three-quark system associated with baryons where the only additional complication expected is the possibility of a genuine three-quark interaction. Because of the need for a finite size for "dressed" quarks it is no longer clear that the simple $\frac{1}{2}$ factor expected in the ratio of q-q to q-$\bar{q}$ short-range potentials will prove to be useful. In spite of these uncertainties, it is apparent that the present type of model is convenient and numerically feasible for the three-quark problem. The initial work by us to calculate the baryon spectrum nonperturbatively with relativistic kinematics is already underway. Only when this step is achieved can the exciting possibilities of coupling in more complicated configurations and examining hadron-hadron systems such as π-π, π-N, and N-N be carried through with some hope of success.

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APPENDIX

Introduction of a finite-size distribution rather than assuming that the quarks behave as point particles requires the evaluation of the integral

$$\langle V_{SR}\rangle = \int d\vec{r}'\, d\vec{r}''\, U_1(\vec{r}-\vec{r}')V(\vec{r}'-\vec{r}'', \vec{\sigma}_1, \vec{\sigma}_2)U_2(\vec{r}''-\vec{r}).$$

If we use the Yukawa form factor $U_i(s)$ given in Sec. II A and use Fourier transforms, the integral can be evaluated analytically in momentum space. The averaged short-range interaction then takes the form

$$\langle V_{SR}\rangle = -\frac{4A_S}{3r}(m_1+m_2)\Bigg(f_c(r)+\left[\frac{A(m_1+m_2)}{m_1 m_2}\right]^{1/3} \times\Bigg\{\frac{C_{SS}}{6}\vec{\sigma}_1\cdot\vec{\sigma}_2 f_{SS}(r)-\frac{3}{4r^2}\Bigg[C_T\left(\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2}-\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{3}\right)f_T(r) + C_{LS}\vec{L}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)f_{LS}(r)\Bigg]\Bigg\}\Bigg), \tag{A1}$$

where

$$f_c(r) = 1-\frac{(\beta_1^2 e^{-\beta_2 r}-\beta_2^2 e^{-\beta_1 r}}{(\beta_1^2-\beta_2^2)}, \tag{A2}$$

$$f_{SS}(r) = \frac{\beta_1^2\beta_2^2}{(\beta_1^2-\beta_2^2)}(e^{-\beta_1 r}2^r), \tag{A3}$$

$$f_{LS}(r) = 1-\frac{\beta_1^2}{(\beta_1^2-\beta_2^2)}(1+\beta_2 r)e^{-\beta_2 r} + \frac{\beta_2^2}{(\beta_1^2-\beta_2^2)}(1+\beta_1 r)e^{-\beta_1 r}, \tag{A4}$$

and

$$f_T(r) = 1-\frac{\beta_1^2}{(\beta_1^2-\beta_2^2)}(1+\beta_2 r+\tfrac{1}{3}\beta_2^2 r^2)e^{-\beta_2 r} + \frac{\beta_2^2}{(\beta_1^2-\beta_2^2)}(1+\beta_1 r+\tfrac{1}{3}\beta_1^2 r^2)e^{-\beta_1 r}. \tag{A5}$$

In the limit $\beta_2\to\beta_1$ corresponding to equal-mass $q\bar{q}$ systems the above reduce to the simpler forms

$$f_c(r) = 1-e^{-\beta_1 r}\tfrac{1}{2}\beta_1 r e^{-\beta_1 r}, \tag{A6}$$

$$f_{SS}(r) = -\frac{\beta_1^3 r}{2}e^{-\beta_1 r}, \tag{A7}$$

$$f_{LS}(r) = 1-e^{-\beta_1 r}-\beta_1 r e^{-\beta_1 r}-\tfrac{1}{2}\beta_1^2 r^2 e^{-\beta_1 r}, \tag{A8}$$

and

$$f_T(r) = 1-e^{-\beta_1 r}-\beta_1 r e^{-\beta_1 r} -\tfrac{1}{2}\beta_1^2 r^2 e^{-\beta_1 r} -\tfrac{1}{6}\beta_1^3 r^3 e^{-\beta_1 r}. \tag{A9}$$

[1]M. Gell-Mann, Phys. Lett. 8, 214 (1964); G. Zweig, CERN Reports Nos. TH401 and 412, 1964 (unpublished).

[2]J. D. Bjorken and S. L. Glashow, Phys. Lett. 11, 255 (1964); S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970); M. K. Gaillard, B. W. Lee, and J. L. Rosner, Rev. Mod. Phys. 47, 277 (1975).

[3]J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1404 (1974); J. E. Augustin *et al.*, *ibid.* 33, 1406 (1974); G. S. Abrams *et al.*, *ibid.* 33, 1453 (1974); G. J. Feldman and M. L. Perl, Phys. Rep. 33C, 285 (1977).

[4]S. W. Herb *et al.*, Phys. Rev. Lett. 39, 252 (1977); W. R. Innes *et al.*, *ibid.* 39, 1240 (1977); Ch. Berger *et al.*, Phys. Lett. 76B, 243 (1978); C. W. Darden *et al.*, *ibid.* 76B, 246 (1978); J. K. Bienlein *et al.*, *ibid.* 78B, 360 (1978); C. W. Darden *et al.*, *ibid.* 78B, 364 (1978).

[5]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975); Phys. Rev. D 12, 1404 (1975).

[6]T. Appelquist *et al.*, Phys. Rev. Lett. 34, 365 (1975).

[7]E. Eichten *et al.*, Phys. Rev. Lett. 36, 500 (1976); Phys. Rev. D 17, 3090 (1978).

[8]A. De Rújula and S. L. Glashow, Phys. Rev. Lett. 34, 46 (1975); C. J. Callan *et al.*, *ibid.* 34, 52 (1975); B. J. Harrington, S. Y. Park and A. Yildiz, *ibid.* 34, 168, 706 (1975).

[9]J. S. Kang and H. J. Schnitzer, Phys. Rev. D 12, 841 (1975); 12, 2791 (1975).

[10]M. Krammer and H. Krasemann, DESY Report No. 78/66, 1978 (unpublished); although the name "quark-onium" appears to have been used first on p. 663 of Ref. 28.

[11]G. Goldhaber *et al.*, Phys. Rev. Lett. 37, 255 (1976).

[12]V. A. Novikov *et al.*, Phys. Rep. 41C, 1 (1978).

[13]J. K. Bienlein *et al.*, Phys. Lett. 78B, 360 (1978); C. W. Darden *et al.*, *ibid.* 78B, 364 (1978).

[14]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. B47, 365 (1973); D. J. Gross and F. Wilczek,

Phys. Rev. D 8, 3497 (1973); S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[15]C. Quigg and J. L. Rosner, Phys. Rep. 56C, 167 (1979).

[16]G. Bhanot and S. Rudaz, Phys. Lett. 78B, 119 (1978).

[17]A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. D 12, 147 (1975).

[18]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957).

[19]Z. V. Chraplyvy, Phys. Rev. 91, 388 (1953); 92, 1310 (1953); see also G. Feldman, T. Fulton, and J. Townsend, Phys. Rev. A 8, 1149 (1973).

[20]M. A. Stroscio, Phys. Rep. 22C, 215 (1975).

[21]See, for example, a standard text such as A. Messiah, *Quantum Mechanics* (Wiley, New York, 1958), Vol. II.

[22]K. G. Wilson, Phys. Rev. D 10, 2445 (1974); J. Kogut and L. Susskind, *ibid.* 11, 395 (1975).

[23]The specific convention we have chosen is that given by Amos de-Shalit and Igal Talmi, *Nuclear Shell Theory* (Academic, New York, 1963).

[24]R. Barate *et al.*, CERN Report No. EP/79-113, 1979 (unpublished).

[25]See E. Eichten and Frank L. Feinberg, Phys. Rev. Lett. 43, 1205 (1979).

[26]G. Morpurgo, Physics (N.Y.) 2, 95 (1965).

[27]C. Becchi and G. Morpurgo, Phys. Rev. 140B, 687 (1965).

[28]R. Van Royen and V. F. Weisskopf, Nuovo Cimento 50A, 617 (1967).

[29]H. Hebach, A. Wortberg, and M. Gari, Nucl. Phys. A267, 425 (1976).

[30]R. Barbieri, R. Gatto, R. Kögerler, and Z. Kunszt, Phys. Lett. 57B, 455 (1975).

[31]Y. Nambu and G. Jona-Lasino, Phys. Rev. 122, 345 (1961); 124, 246 (1961); D. G. Caldi and H. Pagels, Phys. Rev. D 14, 809 (1976); H. Pagels, *ibid.* 14, 2748 (1976).

[32]See Ref. 12 and references therein.

[33]It should be kept in mind that Eqs. (30) and (31) may not always be accurate. Gluon annihilations necessarily involve color octet states for the $q\bar{q}$ pair at some stage of the process. For Eqs. (30) and (31) to have validity it is necessary for the mass splitting between octet and singlet $q\bar{q}$ color states to be small relative to the quark masses involved. Since octet states never appear as physical mesons we have no *a priori* way of knowing whether the splitting is indeed small. Similar doubts occur concerning the simple form of V_A assumed in Sec. II A. Hopefully the parameters K_2, K_3 have absorbed the majority of the uncertainty involved in those calculations.

[34]E. B. Dally *et al.*, Phys. Rev. Lett. 39, 1176 (1977).

[35]C. J. Bebek *et al.*, Phys. Rev. D 17, 1693 (1978).

[36]Particle Data Group, Rev. Mod. Phys. 48, S1 (1976), for octet-singlet mixing angles see also S. L. Glashow and R. H. Socolow, Phys. Rev. Lett. 15, 329 (1966).

[37]I. Peruzzi, Phys. Rev. Lett. 39, 1301 (1977).

[38]E. Lohrman, in *Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies, Hamburg, 1977*, edited by F. Gutbrod (DESY, Hamburg, 1977).

[39]W. Braunschweig *et al.*, Phys. Lett. 57B, 407 (1975).

[40]E. Bloom, in Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies, Batavia, Illinois, 1979 (unpublished).

[41]J. S. Whitaker *et al.*, Phys. Rev. Lett. 37, 1596 (1976).

[42]Particle Data Group, Phys. Lett. 75B, 1 (1978).

[43]B. Wiik and G. Wolf, DESY Report No. 78/23 (unpublished).

[44]G. Flügge, in *Proceedings of the 19th International Conference on High Energy Physics, Tokyo, 1978*, edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Physical Society of Japan, Tokyo, 1979), p. 255; W. Schmidt-Purzefall, *ibid.*, p. 260.

[45]K. Ueno *et al.*, Phys. Rev. Lett. 42, 486 (1979).

[46]P. A. Rapidis *et al.*, Phys. Rev. Lett. 39, 526 (1977); 39, 974(E) (1977).

[47]The phenomenological approach to Chraplyvy transformations used here needs to be related to earlier approaches. For example, corrections to $M1$ transition rates suggested by G. Feinberg and J. Sucher [Phys. Rev. Lett. 35, 1740 (1975)] via expansions in powers of V^2/c^2 are not easily related to our empirical formulation. A more detailed study of these different formulations is in progress.