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Augmented algorithm for the Hamiltonian

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An example is given, additional to Frenkel's example in the preceding paper, in which Dirac's test gives *too much* gauge arbitrariness, signaling a breakdown of the constraint algorithm. A common pathology of the two examples is that H_T is not differentiable along the constraint hypersurface, and this causes the canonical equations to be inherently singular. An augmented algorithm for the Hamiltonian is proposed which coincides with Dirac's in the differentiable case, which works correctly in a class of examples having nondifferentiable H_T, and in which the usual algorithm fails. The proposed algorithm appears to be generally applicable.

In Frenkel's example[1] H_G contains only one of the two required gauge generators with arbitrary coefficients, so Dirac's test has failed to produce the full set. I will give a further example that is only slightly different, but in which too many gauge generators result, a circumstance similar to that encountered in the (counter) example I gave earlier.[2,3] But the present situation raises a serious problem of consistency for the constraint algorithm since the conditions embodied in Dirac's test are almost certainly minimal. The two examples (Frenkel's and the one I will present here), however, share a common and general kind of pathology, and, as I will show, the difficulties they present possess a common resolution. The remedy I propose for the general case is a modification of Dirac's procedure, an "augmented algorithm," to cover cases where the canonical equations are inherently singular on the constraint submanifold identified by Dirac's algorithm. All the important features of the usual theory of constraints, including Dirac's test, are preserved for the two examples, and for a general class of examples given at the end. The level of rigor employed is in the classical spirit of Dirac's original methods of analysis.[4,5]

My example is specified by the Lagrangian

$$L = \dot{x}\dot{z}^2 + yz\ , \tag{1}$$

the equations of motion being

$$\frac{d}{dt}\dot{z}^2 = 0, \quad z = 0, \quad \frac{d}{dt}(2\dot{x}\dot{z}) - y = 0, \tag{2}$$

so that $x(t)$ is undetermined while $y(t) = z(t) = 0$. The total Hamiltonian is

$$H_T \simeq H + vp_y, \tag{3}$$

p_y being the only primary constraint, and H given by

$$H = -yz + p_x^{1/2} p_z\,. \tag{4}$$

The consistency condition for the primary constraint $\varphi = p_y$ is

$$\dot{\varphi} = \dot{p}_y \approx \{p_y, H_T\} \approx z \approx 0, \tag{5}$$

that for the first secondary constraint $\chi_1 \equiv z$ is

$$\dot{\chi}_1 = \dot{z} \approx \{\chi_1, H_T\} \approx p_x^{1/2} \approx 0, \tag{6}$$

viz.,[2]

$$\chi_2 = p_x \approx 0, \tag{7}$$

and the algorithm terminates here with[6] $H' \to H$. So there are three first-class constraints to consider in the implementation of Dirac's test. The present example differs from that of Frenkel and my previous one in the yz term present in H'. Accordingly, the Poisson bracket relation of the primary constraint p_y with H' sets in motion a chain enveloping both z and p_x as gauge generators, and one finds that H_G becomes the extended Hamiltonian

$$H_T \to H_G \simeq up_x + vp_y + wz\,. \tag{8}$$

The canonical equations now tell us, incorrectly, that the $y(t)$ and $p_z(t)$ are arbitrary as well as $x(t)$.

The trouble, plainly put, can be traced to the ("proper") fractional power of the secondary constraint $p_x^{1/2}$ entering (by way of H) into H_T in both Frenkel's example and the present one; for the decision to "doctor" H_T by augmenting it with additional, secondary first-class constraints as determined from Dirac's test is predicated upon the assumption that H_T provides a serviceable starting point. But H_T is not differentiable, for the canonical equations from H_T in both examples include

$$\dot{x} \approx \{x, H_T\} \approx \tfrac{1}{2} p_x^{-1/2} p_z\,, \tag{9}$$

the right-hand side of which is infinite on the constraint submanifold, where p_x vanishes, except for that portion of it along which, additionally,

$$\psi_1 \equiv p_z \approx 0\,, \tag{10}$$

where the right-hand side of Eq. (9) will assume an indeterminate form 0/0. If we do not accept the indeterminate form, then we must accept $\dot{x}=\infty$, which is (inspiringly) uninteresting.

Now $p_x^{-1/2}p_z$ should not be regarded as a "limit" to be fixed by some application of l'Hopital's rule in Eq. (9), and none of the other canonical equations in fact fix such a limit, i. e., none exists. On the other hand, the functions $\chi_2=p_x$ and $\psi_1=p_z$ are functionally independent, so any motion of the quantity $\chi_2^{-1/2}\psi_1$ may be assumed arbitrarily. If we do this and implement the decision by writing for H_T in Eq. (3)

$$H_T\equiv H_S\simeq -yz+up_x+vp_y\,, \tag{11}$$

where I have replaced $p_x^{-1/2}p_z$ by an undetermined function of t, $u=u(t)$, then we have a new, "sanitized" total Hamiltonian functional form H_S, from which to start again.

Evidently, this is the general situation. The function H is computed from L via an elimination of $\dot{q}$'s in favor of p's, and it cannot be guaranteed to be free of "messy" functional dependence upon the secondary constraints, which appear later. Hence H_T cannot be guaranteed differentiable, and it might have Poisson brackets with some of the canonical variables that are infinite on the constraint submanifold identified by Dirac's algorithm. On the other hand, analyticity is not necessary to produce finite canonical equations; differentiable expressions like $\chi^{3/2}$ present in H_T offer only vanishing contributions. It can happen that "sanitization" does not produce any new constraints (ψ's), but only new gauge arbitrariness. Thus, if a term like $(\chi_1\chi_2)^{1/2}$ is present, it may produce either or both of the $\chi_1^{-1/2}$ and $\chi_2^{-1/2}$ terms in some of the canonical equations, so H_T would have to be reexpressed. If χ_2 is first class, this could be done by writing

$$(\chi_1\chi_2)^{-1/2}=(\chi_1^{1/2}\chi_2^{-1/2})\chi_2=u(t)\chi_2 \tag{12}$$

and no new constraint is needed. But also the other choice, having χ_1 and χ_2 interchanged, might be made instead of Eq. (12), and χ_1 would become the gauge generator. Here two distinct serviceable Hamiltonians H_{S_1} and H_{S_2} result and each must be regarded on separate and equal bases. In this instance, the augmented algorithm presented below would branch, generally not an uncommon kind of occurrence for singular systems.

I will call any ψ's which arise from additional requirements such as Eq. (10) *subsecondary* constraints. Just as the starting form of H_T involves a specified function H plus a linear sum of primary constraints with undetermined multipliers, but no secondary constraints, H_S starts with a new specified function, which is H' with the "messy" parts subtracted off, plus a linear combination of primary *and* secondary constraints with undetermined multipliers, but with no subsecondary constraints.

Now for the augmented algorithm. It is necessary, since H_S is a different function from H_T, to impose new consistency conditions on all the original primary and secondary constraints, i.e.,

$$\dot{\varphi}_i\approx\{\varphi_i,H_S\}\approx 0\,,\quad \dot{\chi}_j\approx\{\chi_j,H_S\}\approx 0\,, \tag{13}$$

and, in addition, to require

$$\dot{\psi}_k\approx\{\psi_k,H_S\}\approx 0 \tag{14}$$

for each of any subsecondary constraints that may have arisen from the reexpression of H_T. The φ's, χ's, and ψ's all are to be treated on an equal footing, Dirac's algorithm is to be implemented in full for H_S, and then it is to be augmented with the check for further nondifferentiabilities and further ψ's. If any are found, H_S again must be reexpressed and the steps just described repeated for the new version and for all the constraints so far produced. The process is to be continued until a differentiable Hamiltonian is found and the procedure terminates. In this way one arrives at a final H_S, which I write as

$$H_S\simeq H''+\sum_b u_b\phi_b\,, \tag{15}$$

where the u_b are the multipliers whose values have remained arbitrary.

Equation (15) may be compared with Dirac's equation,[5]

$$H_T\simeq H'+\sum_a v_a\varphi_a\,. \tag{16}$$

Dirac's theorem that H' and the φ_a are first class followed from analysis of the form of the consistency conditions, and it applies to the subset embodied in the primary and secondary constraints of the first, "unsanitized," total Hamiltonian. Since now all the constraints satisfy the same consistency conditions but are based on the final H_S, the same theorem again follows for H'' and the ϕ_b over the *full* constraint set in Eq. (15). But H_S is guaranteed now not to give canonical equations that imply infinite velocities for the canonical variables.

The natural procedure from here is to implement Dirac's test by replacing H' by H'' in the test and the primary subset of the φ_a by the collection of ϕ_b:

$$H_S\to H_G\simeq H''+\sum_g u_g\phi_g\,, \tag{17}$$

as in Ref. 2.

If the augmented algorithm proposed above is applied to Frenkel's example, Eq. (11) is replaced by

$$H_S \simeq up_x + vp_y\,, \tag{18}$$

and the new consistency conditions on p_y, z, p_x, and the subsecondary constraint p_z, all are already satisfied. So H_S is the (final) "serviceable" total Hamiltonian, Dirac's test gives $H_G \simeq H_S$, and the canonical equations correctly reproduce the content of the Euler equations, as may be verified.

The application to my example goes as follows: The consistency conditions to the primary and secondary constraints are computed from Eq. (11) as

$$\begin{aligned} \dot{p}_y &\approx \{p_y, H_S\} \approx z \approx 0\,, \\ \dot{z} &\approx \{z, H_S\} \approx 0\,, \\ \dot{p}_x &\approx \{p_x, H_S\} \approx 0 \end{aligned} \tag{19}$$

and the additional consistency condition is

$$\dot{\psi}_1 = \dot{p}_z \approx \{p_z, H_S\} \approx y \approx 0\,, \tag{20}$$

which gives a new constraint. Continuing further,

$$\dot{\psi}_2 = \dot{y} \approx \{y, H_S\} \approx v \approx 0 \tag{21}$$

and the Dirac procedure stops. Since now $yz \simeq 0$, Eq. (11) becomes

$$H_S \simeq up_x, \tag{22}$$

so that, with $H'' \simeq 0$ there can be no more subsecondary constraints, Eq. (22) gives a serviceable Hamiltonian and the augmented algorithm also stops. Dirac's test trivially gives $H_G \simeq H_S$ and this produces the correct canonical equations, as is again easily checked. It is worth noting that the constraints, p_y, z, and p_x in Eqs. (19), were they to have formed the complete set, would have been first class; while at the end of the algorithm, with the addition of p_z and y to the collection, only p_x remains as first class.

The Lagrangian of Ref. 2, Frenkel's Lagrangian, and Eq. (1) can be generalized to a class[7]

$$L = \dot{x}\dot{z}^l + yz^m, \quad l \geq 0, \quad m > 0, \tag{23}$$

where the conditions on l and m are needed for consistency of the Euler-Lagrange equations ($-\infty < x, y, z, < +\infty$). Corresponding to $l \gtreqless 1$, $m \gtreqless 1$, and $l = 0$, Eq. (23) specifies a large number of distinct examples with varying properties. H_T is not differentiable if either $l > 1$ or $m < 1$ holds, and is differentiable otherwise. Dirac's algorithm works correctly in the differentiable cases and fails in all the nondifferentiable cases, while the proposed augmented algorithm with its corresponding Dirac test works correctly in every case.

It seems a reasonable guess that the sanitization of H_T to H_S and the augmented algorithm presented here which, in conjunction with the corresponding Dirac test, seems to be the natural implementation of this approach to the determination of a Hamiltonian, is a complete and generally justifiable scheme. A key ingredient has been the introduction of "gauge functions" to deal with nondifferentiable or messy functional combinations of the constraints when they occur in H, the possibility for which the examples show to be nonvacuous.

Note added in proof. The nondifferentiable case $l = 0$, $m < 0$ in Eq. (23) has a notable feature. H_T must be sanitized owing to a term $-yz^m$, and the secondary constraint $z \approx 0$. But $p_z \approx 0$ also, so z is second class; hence the coefficient $-yz^{m-1}$ is not a gauge function and must be set equal to zero. As $y \approx 0$ is already a (second class) secondary constraint here, no subsecondaries arise from this.

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[1]A. Frenkel, preceding paper, Phys. Rev. D 21, 2986 (1980).

[2]Robert Cawley, Phys. Rev. Lett. 42, 413 (1979).

[3]Since the publication of my counterexample to Dirac's conjecture concerning the extended Hamiltonian, I learned of the work of G. R. Allcock, Philos. Trans. R. Soc. London A279, 33 (1975), in which the failure of Dirac's conjecture also was recognized.

[4]P. A. M. Dirac, Can. J. Math. 2, 147 (1950); Proc. R. Soc. London A246, 326 (1958).

[5]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964). My notation will be that of Dirac's book, or of Refs. 1 and 2, as appropriate.

[6]The notation H' is that of Dirac's book (Ref. 5); viz. $H' \simeq H + U_m \phi_m$, where the U_m are the Lagrange multipliers that are determined by the algorithm.

[7]The case $l = 4$, $m = 2$ was considered in the earlier version of Frenkel's note.