Comments and Addenda

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Comment on Cawley's counterexample to a conjecture of Dirac

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(Received 16 July 1979)

Cawley's example still leaves open the possibility that "Dirac's test" always provides all the gauge generators. Below an example is given in which both Dirac's conjecture and Dirac's test fail.

In a recent issue of Phys. Rev. Lett.[1] Cawley considered a system for which the Lagrangian is ($\dot{x}=dx/dt$)

$$L=\dot{x}\dot{z}+\tfrac{1}{2}yz^2. \tag{1}$$

For this system the time evolution of x and z is determined by the Euler equations of motion; that of y remains indeterminate. On the other hand, following Dirac[2] one finds that in the Hamiltonian formalism corresponding to (1) there are three first-class constraints: one primary constraint $p_y\approx 0$, and two secondary constraints $z\approx 0$ and $p_x\approx 0$. (If not stated explicitly otherwise, the notations and the terminology in this note are the same as in Ref. 1, except that the unimportant index n of the variables is suppressed.)

According to a conjecture of Dirac, all the first-class constraints should be generators of gauge transformations.[2] Cawley's example shows that the conjecture fails, since in his example only p_y is a gauge generator, corresponding to y being the only gauge-dependent variable.

Cawley also pointed out that in his example "Dirac's test" provides only the constraint p_y. On the other hand, in those well-known cases (in electrodynamics,[2] in Yang-Mills theory,[3] and in the theory of the Dirac monopole[4]) in which Dirac's conjecture seemed to be supported, Dirac's test provided all the first-class constraints. This might suggest that in general all the gauge generators are provided by Dirac's test. The following example shows that this is not the case.

Consider a system with the Lagrangian

$$L=\dot{x}\dot{z}^2+\tfrac{1}{2}yz^2. \tag{2}$$

(This Lagrangian is not bilinear in the velocities, but Dirac's procedure[2] is valid for such systems, too.) The Euler equations read

$$\frac{d\dot{z}^2}{dt}=0,\ \ z=0,\ \ zy-2\frac{d(\dot{x}\dot{z})}{dt}=0\,. \tag{3}$$

Since $z=0$, the time evolution of x and y remains indeterminate; they are gauge-dependent coordinates. Therefore, we must have two gauge generators p_x and p_y in the Hamiltonian formalism. Let us first see how many first-class constraints we do have.

The canonical momenta are

$$p_x=\dot{z}^2,\ \ p_y\approx 0,\ \ p_z=2\dot{x}\dot{z}\,, \tag{4}$$

so that the total Hamiltonian is

$$H_T=\dot{x}p_x+\dot{z}p_z-L+vp_y=H'+vp_y\,, \tag{5}$$

where v is an arbitrary function of the coordinates and of the momenta and

$$H'=p_x{}^{1/2}p_z-\tfrac{1}{2}yz^2\,. \tag{6}$$

[For (1), $H'=p_xp_z-\frac{1}{2}yz^2$.] The consistency condition to the primary constraint

$$p_y\approx 0 \tag{7}$$

reads

$$\dot{p}_y\approx\{p_y,H_T\}\approx\tfrac{1}{2}z^2\approx 0\,. \tag{8}$$

We have here a secondary constraint which emerged in quadratic form. Let us follow Dirac[5] and Cawley[1] in rewriting it in the linearized form

$$z\approx 0\,. \tag{9}$$

The consistency condition for this new constraint gives

$$\dot{z} \approx \{z, H_T\} \approx p_x^{1/2} \approx 0, \qquad (10)$$

or, in linearized form,

$$p_x \approx 0. \qquad (11)$$

There are no more constraints, since

$$\dot{p}_x \approx \{p_x, H_T\} \approx 0. \qquad (12)$$

So, we have three first-class constraints, p_y, z, and p_x, but only two of them are gauge generators. Dirac's conjecture fails for this example, too.

Let us now turn to Dirac's test. As noticed in Ref. 1, $z \approx 0$ implies $z^2 \simeq 0$, and the test

$$\{p_y, H'\} \simeq 0, \;\; \{p_y, p_y\} \simeq 0 \qquad (13)$$

provides us only with the primary constraint p_y. It fails to give the gauge generator p_x. Our example shows that neither the extended Hamiltonian H_E, nor the generalized Hamiltonian H_G is in general suitable for taking into account the full gauge freedom of the system.

The value of Dirac's conjecture would have been that it would give all the gauge generators without dealing explicitly with the often complicated equations of motion. No simple substitute seems to exist for this conjecture.

(In an earlier version[6] of the present note some remarks concerning the form of the constraints are also included. Otherwise, the earlier version coincides with the comment published here.)

Note added in proof. This pessimistic statement proved to be unfounded, since the algorithm proposed by Cawley in the next paper does provide a relatively simple substitute for Dirac's conjecture. It remains to be seen whether the algorithm, tested so far only for a certain class of Lagrangians, is correct in general. Cawley proposes in his new example to substitute an undetermined function of time $u(t)$ for $p_x^{-1/2}p_z$ in H. In this case the Poisson brackets (Pb's) of $u(t)$ with the dynamical variables are ill-defined, and this might lead to difficulties in calculating time derivatives. I think this substitution should not be made in expressions which, like H, come under Pb's. It seems sufficient to require that $p_x^{-1/2}p_z = u(t)$ be a smooth function of time on the constraint surface *after* all the Pb's have been opened. Of course, the "subsecondary constraints" of Cawley must be introduced, so that the algorithm would be only slightly modified.

The author is indebted to R. Casalbuoni, L. Lusanna, and P. Hraskó for valuable discussions, and to P. Hasenfratz for a critical reading of the manuscript.

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