PHYSICAL REVIEW D VOLUME 21, NUMBER 10 15 MAY 1980

Monopoles and dyons in the SU(5) model

Constantine P. Dokos and Theodore N. Tomaras
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachussetts 02138
(Received 18 December 1979)

The spherically symmetric monopoles and dyons of the SU(5) model of grand unification (without quarks and leptons) are discussed. It is shown that such monopoles and dyons can exist only in the sectors corresponding to magnetic charges $m = \pm 1/2e$, $\pm 1/e$, $\pm 3/2e$, and $\pm 2/e$, where e is the charge of the positron. We investigate in detail the properties of the dyons with the smallest possible magnetic charge ($|m| = 1/2e$). By semiclassical reasoning we show that apart from the magnetic charge the properties of the dyons are described by two quantum numbers n and k. The dyons come in families, denoted by $n = 0,1,2,...$, with electric charge $q_n = n(-4e/3)$, baryon minus lepton number $= n(-2/3)$, and the kth member of the nth family ($k = 0,1,2,...$) transforms according to the $(n+k,k)$ for $n \geq 0$ or the $(k,|n|+k)$ for $n<0$ representation of $SU(3)_C$. We argue that all the members of a given family are degenerate at the level we are working. This degeneracy is expected to be lifted in the full quantum theory, in which case each family collapses to one stable dyon, characterized by one integer n and whose quantum numbers are as follows: It has electric charge $= n(-4e/3)$ and baryon number minus lepton number $= n(-2/3)$, and it transforms under $SU(3)_C$ like the symmetric combination of n $\underline{3}$'s, for $n \geq 0$, or $|n|$ $\underline{\bar{3}}$'s, for $n<0$. Interesting processes involving monopoles and dyons are discussed; we show, for example, that the presence of a dyon strongly enhances baryon-number-violating processes. Finally, a less detailed discussion of poles with the other possible magnetic charges is included.

I. INTRODUCTION

Since the theoretical discovery[1] of the existence of smooth, finite-energy, particlelike, monopole solutions in the SO(3) Georgi-Glashow model, many authors[2] have worked on problems related to magnetic monopoles. The stability of the 't Hooft-Polyakov monopole has been discussed[3]; exact solutions have been found for SO(3) in an extreme case[4]; similar objects and analogous discussions have been made in the context of various other models,[5] such as $SU(N) \to U(1)$, $SU(N) \to SU(N/2) \times SU(N/2) \times U(1)$, $SU(N) \to SU(N-1) \times U(1)$, etc. Julia and Zee discovered[6] the existence of dyons in the SO(3) model. The quantization of these classical field configurations has been discussed.[7] The effects caused by the introduction of fermions[8] into the theory, as well as other interesting phenomena,[9] have also been considered.

The purpose of this paper is to discuss the properties of monopoles in the phenomenologically realistic[10] $SU(5) \to SU(3)_C \times U(1)_{em}$ grand unification model. This theory, proposed a few years ago, seems to be quite successful, as far as low-energy phenomenology is concerned and also seems to work in a nice way on problems related to the origin of the universe.[11] So, it is interesting to know the particle spectrum of this theory, including the spectrum of particles with masses as large as the monopole masses, about 100 times larger than the superheavy elementary gauge bosons and scalar particles of the model. The discussion of the properties and the cosmological effects[12] of those objects might give a clue to a deeper understanding of questions related to the principle of grand unification, to cosmology, and so on.

In Sec. II we expose our notation and discuss the features of the model in the unitary gauge. In Sec. III we give the topological argument for the existence of magnetic-monopole solutions and also determine the quantum of magnetic charge. Next, we discuss the classical properties of the monopole of lowest magnetic charge and its corresponding dyons. In Sec. IV we consider the problem of quantizing the classical solutions of the previous section and determine the quantum numbers of the dyons. A few amusing but not very important processes involving dyons and monopoles are mentioned. In Sec. V, we briefly present *Ansätze* for spherically symmetric monopoles and dyons with magnetic charges equal to $1/e$, $3/2e$, and $2/e$. We conclude with a short discussion in Sec. VI.

II. NOTATION-UNITARY GAUGE

The theory we consider is an SU(5) gauge theory,[10] spontaneously broken to $SU(3)_C \times U(1)_{em}$ by the vacuum expectation values of two Higgs multiplets, one (Φ) in the adjoint ($\underline{24}$) representation and the other (H) in the $\underline{5}$ of $\underline{SU(5)}$. The Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu}) + \mathrm{Tr}(\mathfrak{D}_\mu \Phi)^2 + |D_\mu H|^2 - V(\Phi, H), \tag{2.1}$$

$$W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu], \qquad W_\mu \equiv W_\mu^a F^a, \quad a = 1, \ldots, 24 \tag{2.2}$$

$$D_\mu H \equiv \partial_\mu H + igW_\mu H, \quad \mathfrak{D}_\mu \Phi \equiv \partial_\mu \Phi + ig[W_\mu, \Phi]. \tag{2.3}$$

The group generators F^a, with $a = 1, 2, \ldots, 24$, are chosen to satisfy $\operatorname{Tr} F^a F^b = \frac{1}{2}\delta^{ab}$, $F^{a\dagger} = F^a$, and the potential $V(\Phi, H)$ to break $SU(5) \to SU(3) \times U(1)$. The equations of motion are

$$\mathfrak{D}_\nu W^{\mu\nu} = -J^\mu \equiv -ig[\mathfrak{D}^\mu \Phi, \Phi] - igF^a[H^\dagger F^a D^\mu H - (D^\mu H)^\dagger F^a H],$$

$$\mathfrak{D}_\lambda \mathfrak{D}^\lambda \Phi = -\frac{\partial V}{\partial \Phi'}, \quad D_\mu D^\mu H = -\frac{\partial V}{\partial H^\dagger}, \tag{2.4}$$

while the energy density is given by

$$\theta^{00} = \operatorname{Tr} \vec{\underline{B}}^2 + \operatorname{Tr} \vec{\underline{E}}^2 + \operatorname{Tr}(\vec{\mathfrak{D}}\Phi)^2 + \operatorname{Tr}(\mathfrak{D}_0 \Phi)^2 + |\vec{D}H|^2 + |D_0 H|^2 + V(\Phi, H), \tag{2.5}$$

where we have defined

$$\underline{B}_i \equiv \tfrac{1}{2}\epsilon_{ijk} W_{jk} \text{ and } \underline{E}^i = -W^{0i}. \tag{2.6}$$

Since in what follows we will restrict ourselves to the discussion of the order-g^{-1} behavior of the theory, the fermions do not play any role and we do not include the quark and lepton terms in (2.1). We will refer to the fermions only in the definition of the baryon minus lepton number $(B-L)$, which we discuss extensively at the end of this section, and in the definition of the electric-charge generator.

In the so-called unitary gauge, the model has the following features. The SU(5) gauge symmetry is broken spontaneously to $SU(3)_C \times U(1)_{em}$ by the vacuum expectation values of Φ and H,

$$\Phi_0 = \nu \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2}+\epsilon & \\ & & & & -\frac{3}{2}-\epsilon \end{pmatrix}, \quad H_0 = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.7}$$

Here, we have defined the $SU(3)_C$ to act on the upper three components of the five-dimensional space, while the $U(1)_{em}$ is generated by

$$\frac{1}{e}Q \equiv t_3 - (\tfrac{5}{3})^{1/2} Y = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} \tag{2.8}$$

with

$$Y \equiv \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix} \text{ and } t_i \equiv \tfrac{1}{2} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \tau_i \end{pmatrix}, \quad i = 1, 2, 3 \tag{2.9}$$

and e is the positron charge $(e > 0)$. The definition of Q is consistent with the fact that the $\overline{5}$ of fermions contains the particles $(\bar{d}_R \bar{d}_B \bar{d}_G e\bar{\nu})_L$, R, B, G being the three colors, whose charges are the opposite of the elements of Q. Notice that Y and $\vec{t}$ are the four generators of an $SU(2)_L \times U(1)_Y$ subgroup of SU(5), which is an invariance of the vacuum in the limit $v = 0$ and $\epsilon = 0$. Let $\vec{V}_\mu(x)$ and $B_\mu(x)$ be the $SU(2)_L \times U(1)_Y$ gauge fields. Since this last symmetry is broken to $U(1)_{em}$, all those fields except for one, called the photon, become massive. Proceeding in exactly the same way as in the standard $SU(2) \times U(1)$ model, we can identify the weak and electromagnetic gauge fields as

$$Z_\mu \equiv (\tfrac{5}{8})^{1/2}[V_\mu^3 + (\tfrac{3}{5})^{1/2} B_\mu] \quad \text{with } M_Z = \frac{2}{\sqrt{5}} gv \tag{2.10}$$

$$A_\mu \equiv (\tfrac{5}{8})^{1/2}[(\tfrac{3}{5})^{1/2} V_\mu^3 - B_\mu], \text{ orthogonal to } Z_\mu, \quad \text{with } M_\gamma = 0 \tag{2.11}$$

and

$$V_\mu^\pm \equiv \frac{1}{\sqrt{2}}(V_\mu^1 \pm V_\mu^2), \text{ with } M_V = g(\tfrac{1}{2}v^2 + 4\epsilon^2\nu^2)^{1/2}. \tag{2.12}$$

The interaction of the V_μ^3 and B_μ gauge fields with the fermions can be written, symbolically, as

$$g(t_3)^\mu V_\mu^3 + gY^\mu B_\mu = (\tfrac{3}{8})^{1/2} g \left(\frac{Q}{e}\right)^\mu A_\mu + (\tfrac{5}{8})^{1/2} g [t_3 + (\tfrac{3}{5})^{1/2} Y]^\mu Z_\mu.$$

Requiring the electromagnetic coupling to have the form $e(Q/e)^\mu A_\mu$, since the electron-photon interaction is $+e\bar{\psi}\gamma_\mu\psi A^\mu$, we get

$$e=(\tfrac{3}{8})^{1/2}g\,. \tag{2.13}$$

Equation (2.13) has the form $e=g\sin\theta_W$, with θ_W defining the $V^3_\mu - B_\mu$ mixing in Z_μ and which, as we read in (2.10) satisfies $\sin\theta_W=(\tfrac{3}{8})^{1/2}$.

Phenomenologically interesting[10] is the case in which we have $\nu\sim O(10^{16}\text{ GeV})$, $\epsilon\sim O(v/\nu)$ $\sim O(10^{-14})$. But, as far as our discussion is concerned, since we will be interested only in the massless gauge fields and the long-range properties of monopoles and dyons, we do not have to choose any particular set of parameters. Everything we will say below is independent of the magnitude of ϵ, as long as it is nonzero.

Now, as we promised, we turn to the discussion of the $B-L$ symmetry of the theory. First of all, remember that the full Lagrangian of the model we are interested in contains a $\underline{5}$ (Ψ_α) and a $\underline{10}$ ($\Psi^{\alpha\beta}=-\Psi^{\beta\alpha}$) of fermions for each family of quarks and leptons.[10] Those are coupled to the H field through the Yukawa couplings $\bar{\Psi}_{\alpha\beta}\Psi^\alpha H^\beta$ and $\epsilon_{\alpha\beta\gamma\delta\lambda}\Psi^{\alpha\beta}\Psi^{\gamma\delta}H^\lambda$. Notice that $\mathcal{L}$ is then invariant under

$$H^\lambda\to e^{-i\omega}H^\lambda,\quad \Phi\to\Phi,\quad W\to W,$$
$$\Psi^{\alpha\beta}\to e^{i\omega/2}\Psi^{\alpha\beta}\text{ and }\Psi^\alpha\to e^{3i\omega/2}\Psi^\alpha, \tag{2.14}$$

a global U(1) transformation. This symmetry is broken by H_0, but, as we will now explain, it is unbroken if combined with a certain gauge symmetry. Clearly, any such combination will be a symmetry of the Lagrangian. So, let us combine a U(1) rotation (2.14) with angle $\tfrac{2}{5}\omega$ with another, generated by the SU(5) generator Y and with angle $-4\omega/\sqrt{15}$. The net result of this transformation on H, Ψ^α, Φ, and W is

$$H'^\lambda=e^{i\omega B^H}H^\lambda,$$
$$B^H=-\frac{2}{5}-\frac{4}{\sqrt{15}}Y=\text{diag}(-\tfrac{2}{3},-\tfrac{2}{3},-\tfrac{2}{3},0,0), \tag{2.15}$$

$$\Psi'^\alpha=e^{i\omega B^5}\Psi^\alpha,\quad B^5=\frac{3}{5}-\frac{4}{\sqrt{15}}Y=\text{diag}(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},1,1)\,, \tag{2.16}$$

$$\Phi'=e^{i\omega B}\Phi e^{-i\omega B},\quad W'_\mu=e^{i\omega B}W_\mu e^{-i\omega B},\quad B=-\frac{4}{\sqrt{15}}Y, \tag{2.17}$$

respectively. This is a U(1) symmetry of the theory with the following properties: (a) The form of B^H and B in (2.15) and (2.17), respectively, is such that this symmetry remains unbroken by the vacuum expectation values Φ_0 and H_0. (b) The values of the diagonal elements of B^5, which are the baryon minus lepton numbers of the fermions in the 5-plet $(d_R\, d_B\, d_G e^+\bar{\nu})_R$, provide an explanation of why this symmetry is called $B-L$. (c) The $B-L$ quantum numbers of the 5-plet of Higgs we read from B^H, while those of Φ and W_μ are

$$\begin{bmatrix} & 0 & & -\tfrac{2}{3} & -\tfrac{2}{3}\\ & & & -\tfrac{2}{3} & -\tfrac{2}{3}\\ & & & -\tfrac{2}{3} & -\tfrac{2}{3}\\ \tfrac{2}{3} & \tfrac{2}{3} & \tfrac{2}{3} & & \\ \tfrac{2}{3} & \tfrac{2}{3} & \tfrac{2}{3} & & 0\end{bmatrix}.$$

(d) Incidentally, this discussion answers a question one might have, namely, what happened to the Goldstone boson associated with the breaking of the U(1) in (2.14). We know that (2.14) is a linear combination of $B-L$, which remains unbroken, and Y, which is spontaneously broken, and the corresponding Goldstone boson, via the Higgs mechanism, gives mass to the Z_μ gauge field.

III. CLASSICAL MONOPOLES–DYONS

According to the general topological arguments,[2] the very presence of the $U(1)_{em}$ factor in the unbroken gauge group guarantees the existence of smooth, finite-energy, topologically stable, particlelike solutions of the equations of motion with quantized magnetic charge. To determine the unit m_0 of magnetic charge, it is important to realize that the unbroken group $\mathfrak{h}$ of the theory is only locally isomorphic to SU(3)×U(1). The group $\mathfrak{h}$, the subgroup of SU(5) which remains unbroken by Φ_0 and H_0 in (2.7), is the set of matrices of the form

$$\begin{bmatrix} ue^{-i\alpha} & & \\ & e^{3i\alpha} & \\ & & 1\end{bmatrix}$$

with $u\in$SU(3). The mapping from SU(3)×U(1) to $\mathfrak{h}$, defined by $(u,e^{i\alpha})\to\text{diag}(ue^{-i\alpha},e^{3i\alpha},1)$ of $\mathfrak{h}$, is $\kappa=3$ to 1, since the three elements $(u,e^{i\alpha})$, $(ue^{2\pi i/3},e^{i(\alpha+2\pi/3)})$, and $(ue^{4\pi i/3},e^{i(\alpha+4\pi/3)})$ of SU(3)×U(1) are mapped to the same element of $\mathfrak{h}$. As a consequence of this, the minimum magnetic charge m_0 is given by[2,13]

$$m_0=\frac{1}{2Q_{\min}\kappa} \tag{3.1}$$

with $\kappa=3$ and $Q_{\min}$ the smallest positive charge that can exist in the theory. In our case we have $Q_{\min}=\tfrac{1}{3}e$. So, the quantum of magnetic charge is $m_0=1/2e$ and the possible magnetic charges are integral multiples of m_0.

There exists another way to prove that the smallest possible value of the magnetic charge is $m_0 = 1/2e$. Corrigan and Olive[14] have shown that the only possible values of magnetic charge m, are those which satisfy the condition $e^{4\pi i m Q} = k$, with k an element of the center of the $SU(3)_C$ subgroup of SU(5). This is an operator equation. Acting on the $|e^+\rangle$ state, the right-hand side gives $|e^+\rangle$, since the positron is a color singlet, and the above condition reduces to $\exp(4\pi i m e) = 1$, which implies $m = n/2e$ for any integer n.

Let us now proceed to the explicit construction and discussion of properties of monopoles and dyons which exist in this theory. Since it is easier to work with symmetric objects and, on the other hand, we believe that the larger the symmetry of a solution the smaller its energy will be, we ask for time-independent solutions of the equations of motion with the highest possible symmetry, which, in addition, are topologically nontrivial. If $\vec{L} \equiv -i\vec{r}\times\vec{\nabla}$ and T^a, $a = 1, 2, 3$, the three generators of an SU(2) embedding in SU(5), we ask for the most general *Ansatz* satisfying

$$[L_i + T_i, W_j] = i\epsilon_{ijk}W_k, \quad [L_i + T_i, W_0] = 0,$$
$$[L_i + T_i, \Phi] = 0, \quad (L_i + T_i)H = 0 . \tag{3.2}$$

A configuration with the transformation properties (3.2) is what we call spherically symmetric. To maximize the symmetry of the solutions we are after, we also require them to be invariant under the largest possible subgroup Γ of U(5) [the U(1) factor is related to the $B - L$ symmetry] which is compatible with the spherical symmetry $\vec{L}+\vec{T}$, i.e.,

$$[\Gamma_i, \Phi(\vec{r})] = 0, \quad \Gamma_i H(\vec{r}) = 0, \quad [\Gamma_i, W_\mu(\vec{r})] = 0$$
$$[\Gamma_i, L_j + T_j] = 0, \quad j = 1, 2, 3, \quad i = 1, 2, \ldots , \tag{3.3}$$

where Γ_i are the generators of Γ.

In this section, we will construct a particular monopole with its corresponding dyons. In fact, we will always be considering the pure magnetic poles as special cases of dyons with $W_0 = 0$. We will determine the classical properties of those objects in this section. We leave for the next section the discussion of the quantum properties of this family of dyons. Other possibilities and some of their properties are discussed in Sec. V.

Consider the $\underline{5} \to \underline{2} + \underline{1} + \underline{1} + \underline{1}$ embedding of SU(2) into SU(5), given by

$$\vec{T} = \frac{1}{2}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \vec{\tau} & \\ & & & 0 \end{pmatrix}, \tag{3.4}$$

where τ^a, $a = 1, 2, 3$, are the Pauli matrices. The U(5) subgroup Γ which commutes with $\vec{T}$ is quite large, but in order to be able to get a nontrivial *Ansatz* for $H(\vec{r})$, we restrict Γ to be an SU(2)×U(1)×U(1) generated by

$$\vec{\Gamma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \Gamma_4 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_5 = \frac{1}{2}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} .$$

The general form of a field configuration, spherically symmetric, Γ-invariant, and also invariant under simultaneous inversion of $\vec{r}$ and $\vec{T}$ is

$$H(\vec{r}) = \frac{1}{g}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h(r) \end{pmatrix}, \quad \Phi(\vec{r}) = \frac{1}{g}\begin{pmatrix} \phi_1(r) & & & \\ & \phi_1(r) & & \\ & & \phi_2(r) + \phi_3(r)\hat{r}\cdot\vec{\tau} & \\ & & & -2(\phi_1 + \phi_2) \end{pmatrix},$$
$$W_i(\vec{r}) = (\vec{T}\times\hat{r})_i \, \frac{K(r) - 1}{gr} \quad \text{and} \quad W_0(\vec{r}) = \frac{1}{g}\begin{pmatrix} J_1(r) & & & \\ & J_1(r) & & \\ & & J_2(r) + \frac{J(r)}{2}\,\hat{r}\cdot\vec{\tau} & \\ & & & -2(J_1 + J_2) \end{pmatrix} . \tag{3.5}$$

The functions $K(r)$, $\phi_j(r)$, and $J_j(r)$ are real, as required by the Hermiticity of W_i, Φ, and W_0, respectively. We also consider $h(r)$ to be real. The reason for this is the following. Consider instead of $h(r)$, the complex function $h(r)e^{i\alpha(r)}$. The exponent $\alpha(r)$ appears in the energy functional only through the term $|\vec{D}H|^2 = h'^2(r) + \alpha'^2(r)h^2(r)$. Thus, the field equations are consistent with α = constant and, furthermore, the energy becomes smaller in this case. Without loss of generality we can take $\alpha = 0$.

The inversion symmetry we required from W_i forbids the appearance of terms with the structure $\hat{r}_i$ or T_i. Also, terms of the form $(\hat{r}\cdot\vec{T})(\hat{r}\times\vec{T})_i$ are not independent since the matrices (3.4) satisfy $T_aT_b + T_bT_a = \frac{1}{2}\delta_{ab}$ $(a, b = 1, 2, 3)$. Finally, notice that (3.5) automatically satisfies the gauge-fixing condition $\partial_i W^i = 0$.

As usual, we insert the *Ansatz* (3.5) into the energy functional and minimize with respect to the radial functions to get the field equations for symmetric solutions.[2] The boundary conditions at infinity are determined from the fact that finiteness of the energy requires the fields to approach the Higgs vacuum away from the origin, i.e., to satisfy $V(\Phi, H) = 0$, $\mathfrak{D}_\mu\Phi = 0$, and $D_\mu H = 0$. Along the $\hat{z}$ direction, Φ and H must approach at infinity Φ_0 and H_0, respectively, from which we conclude that

$$\phi_1(r) \underset{r\to\infty}{\longrightarrow} \nu g, \quad \phi_2 \to \tfrac{1}{2}\nu g(-\tfrac{1}{2}+\epsilon), \quad \phi_3(r) \to \tfrac{1}{2}\nu g(\tfrac{5}{2}-\epsilon) \tag{3.6}$$

and $h(r) \to vg$. As for the boundary conditions for the functions $K(r)$, $J(r)$, and $J_i(r)$, $i = 1, 2$, the obvious similarity of our dyon to the one of Julia and Zee,[6] makes it reasonable to assume that there exists a solution of the field equations of the form (3.5) with[15]

$$K(r) \underset{r\to\infty}{\sim} e^{-ar}, \; J(r) \sim M + \frac{b}{r} + O(r^{-2}), \;\; J_i(r) \sim O(r^{-2}) \,. \tag{3.7}$$

The parameter a depends, as in the Julia-Zee case, upon ν, v, M, and g. Since we are interested in having $a > 0$, the possible values of M are restricted. We expect that M has to satisfy $|M| \lesssim g\nu$. On the other hand, there is no restriction on the continuous parameter b. For the discussion that follows we do not need to know what the short-range behavior of the solution is. Of course, the radial functions are expected to behave smoothly for the energy of the solution to be finite. But, apart from this, the asymptotic behavior of the fields is all we need to know in order to investigate the magnetic and electric properties of the dyons, as well as their B-L charge. This is exactly what we will do next.

Let us start with the magnetic charge. This we calculate by integrating the ordinary magnetic field over a sphere at infinity. The electromagnetic field strengths are defined to be

$$F_{\mu\nu} \equiv \frac{2}{g}\,\mathrm{Tr}(W_{\mu\nu}(\vec{r})Q(\vec{r})) \tag{3.8}$$

with $Q(\vec{r})$ the spherically symmetric version of (2.8), which in this case has the behavior

$$\frac{1}{e}Q(\vec{r}) \underset{r\to\infty}{\sim} \begin{pmatrix} -\frac{1}{3} & & & \\ & -\frac{1}{3} & & \\ & & \frac{1}{3}-\frac{2}{3}\hat{r}\cdot\vec{\tau} & \\ & & & 0 \end{pmatrix} . \tag{3.9}$$

Notice, as a consistency check of our definition, that in the unitary gauge the photon field A_μ can be written in the form $A_\mu = (2/g)\mathrm{Tr}(W_\mu(x)Q)$, where Q is now the matrix (2.8).

Using (3.5) and (3.7), we get

$$W_{ij}(\vec{r}) \underset{r\to\infty}{\sim} -\frac{2}{gr^2}T_a(\epsilon_{jab}\hat{r}_b\hat{r}_i - \epsilon_{iab}\hat{r}_b\hat{r}_j - \epsilon_{ija}) - \frac{1}{gr^2}\epsilon_{ijb}\hat{r}_b\hat{r}\cdot\vec{T} \,. \tag{3.10}$$

The magnetic fields $\underline{B}^i \equiv -\frac{1}{2}\epsilon_{ijk}W_{jk}$ are at infinity

$$\vec{\underline{B}} \underset{r\to\infty}{\sim} -\frac{1}{gr^2}\hat{r}\hat{r}\cdot\vec{T} \,. \tag{3.11}$$

This magnetic field has, obviously, both ordinary magnetic and color-magnetic components. The electromagnetic part of the field $\vec{\underline{B}}$ is, according to (3.8) given by

$$\vec{B} \equiv \frac{2}{g}\,\mathrm{Tr}(\vec{\underline{B}}(\vec{r})Q(\vec{r})) \underset{r\to\infty}{\sim} \frac{1}{2e}\frac{\hat{r}}{r^2}. \tag{3.12}$$

Thus, the magnetic charge of our field configuration is

$$m = \frac{1}{2e} \tag{3.13}$$

equal to the smallest possible, according to the topological argument. The long-range color-magnetic fields, carried by the dyon configuration (3.5)–(3.7), is analogously given by

$$\vec{B}^a \equiv 2\,\mathrm{Tr}(\vec{\underline{B}}(\vec{r})\lambda^a(\vec{r})) \underset{r\to\infty}{\sim} \frac{1}{\sqrt{3}}\delta^{a8}\frac{\hat{r}}{gr^2}. \tag{3.14}$$

We now proceed to the investigation of the electric properties, both ordinary and color, of our dyons. Here we need to know the asymptotic behavior of $J(r)$ and $J_k(r)$, $k=1,2$. Using (3.7), we conclude that

$$W_0(\vec{r}) \underset{r\to\infty}{\sim} \frac{1}{g}(M+b/r)\hat{r}\cdot\vec{T}+O(r^{-2})\,. \qquad (3.15)$$

From (3.15) we can immediately calculate

$$W_{0i}(\vec{r}) \underset{r\to\infty}{\sim} \frac{b}{gr^2}\hat{r}_i\hat{r}\cdot\vec{T} \qquad (3.16)$$

and again, we can split it into ordinary electric and color-electric fields. The first is

$$E^i(\vec{r}) \equiv F_{0i}(\vec{r}) = \frac{2}{g}\mathrm{Tr}(W_{0i}(\vec{r})Q(\vec{r})) \underset{r\to\infty}{\sim} -\frac{4\pi b}{2e}\frac{\hat{r}_i}{4\pi r^2}\,,$$

so the dyon electric charge is

$$q=-4\pi b/2e\,. \qquad (3.17)$$

It is simpler to calculate the asymptotic behavior of the color-electric fields by looking at what happens along the $+z$ axis at infinity. From (3.16) we conclude

$$W_{0i}(r\hat{z}) \underset{r\to\infty}{\sim} \frac{b}{gr^2}\hat{z}_i\left(-\frac{1}{\sqrt{3}}\lambda^8-\frac{1}{2e}Q\right)\,.$$

[The quantity in parentheses, with $\lambda^8 \equiv (1/2\sqrt{3})$ diag $(1,1,-2,0,0)$ and Q given by (2.8), is equal to T_3.] The color-electric field is then, using the spherical symmetry,

$$\vec{E}^a(\vec{r}) \underset{r\to\infty}{\sim} \frac{4\pi b}{g^2}\left(-\frac{g}{\sqrt{3}}\delta^{a8}\right)\frac{\hat{r}}{4\pi r^2},\quad a=1,2,\ldots,8\,. \qquad (3.18)$$

Notice that (3.18) is exactly the solution of the classical Poisson equation for the color-electric fields created by a source at the origin with density

$$J_0^a(\vec{x}) = -\frac{4\pi b}{\sqrt{3}\,g^2}\delta^{a8}\delta(\vec{x})\,. \qquad (3.19)$$

What is left is the computation of the $B-L$ (baryon minus lepton number) of the configuration (3.5)–(3.7). As we will show, the $B-L$ of the dyon is $1/2e$ times its electric charge. Notice, first of all, that we can equally well calculate the electric and $B-L$ charges of the dyon simply by integrating over space the zeroth component of the Noether current associated with the transformations generated by $Q(\vec{r})$ and $B(\vec{r})$, respectively. $Q(\vec{r})$ and $B(\vec{r})$ represent the spherically symmetric forms of the generators of $\mathrm{U}(1)_{\mathrm{em}}$ and $B-L$. Thus,

$$q=\int d^3x\, J^0_{\mathrm{em}}(x),$$

with

$$-J^0_{\mathrm{em}} = \frac{\partial\mathcal{L}}{\partial\dot{W}^a_\mu}\delta_Q W^a_\mu + \frac{\partial\mathcal{L}}{\partial\dot{\Phi}^a}\delta_Q\Phi^a + \frac{\partial\mathcal{L}}{\partial\dot{H}}\delta_Q H + \frac{\partial\mathcal{L}}{\partial\dot{H}^\dagger}\delta H^\dagger\,.$$

We now observe that $\partial\mathcal{L}/\partial\dot{\Phi} = \mathfrak{D}_0\Phi = 0$ for our dyon and, similarly, $\delta_Q H = 0 = \delta_Q H^\dagger$, since $\delta_Q H = iQ(\vec{r})H(\vec{r}) = 0$. Since we also have $\partial\mathcal{L}/\partial\dot{W}_i = -W^{0i}$, we end up with

$$q=2\int d^3x\,\mathrm{Tr}(W^{0i}\delta_Q W_i)$$

and by the same token we have

$$B-L=2\int d^3x\,\mathrm{Tr}(W^{0i}\delta_B W_i)\,.$$

The transformations generated by $Q(\vec{r})$ and $B(\vec{r})$ are local gauge transformations. This means that $\delta_Q W_i$ and $\delta_B W_i$ are given by $\delta_Q W_i = i[Q(\vec{r}),W_i(\vec{r})] - (1/g)\partial_i Q(\vec{r})$ and $\delta_B W_i = i[B(\vec{r}),W_i(\vec{r})] - (1/g)\partial_i B(\vec{r})$. If we make use of the field equations $(\mathfrak{D}_i W_{0i}=0)$, we arrive at the expressions

$$q=-\frac{2}{g}\int d^2S_i\,\mathrm{Tr}(W^{0i}(\vec{r})Q(\vec{r})) = \int d\vec{S}\cdot\vec{E}(\vec{r})\,, \qquad (3.20)$$

the same with the formula used previously and

$$B-L=-\frac{2}{g}\int d^2S_i\,\mathrm{Tr}(W^{0i}(\vec{r})B(\vec{r}))\,. \qquad (3.21)$$

In formulas (3.20) and (3.21) only the $\vec{\tau}$ parts of the generators $Q(\vec{r})$ and $B(\vec{r})$ give nonzero contributions to q and $B-L$, respectively. From (2.8) and (2.17) one can easily check that the τ_3 part of B is $1/2e$ times that of Q. By spherical symmetry we conclude that everywhere at infinity the $\hat{r}\cdot\vec{\tau}$ part of $B(\vec{r})$ is $1/2e$ times that of $Q(\vec{r})$, which leads to

$$B-L=\frac{1}{2e}q=-\frac{\pi b}{e^2}\,. \qquad (3.22)$$

This completes our discussion of the classical properties of the dyon. A few comments are in order. (a) In the special case $W_0=0$ we can still find a solution of the field equations with W_i, Φ, and H having the form (3.5). This represents a pure magnetic pole without electric fields and with $B-L=0$. (b) The electric properties, as well as the $B-L$ of the dyons, depend upon the continuous parameter b. Thus, the dyon electric properties and $B-L$ are not quantized at the classical level, unlike the magnetic charge, which is related to topology. (c) Had we started with an ansatz differing from (3.5) only in the sign of W_i, we would arrive at another monopole and its corresponding tower of dyons with $m=-1/2e$.

IV. DYON QUANTUM NUMBERS

To determine the properties of our dyons at the quantum level, we will use the existing[16,9] semiclassical reasoning. We will apply the Bohr-

Sommerfeld quantization condition, according to which, if we have a one-parameter family of periodic solutions, labeled by the period T, then an energy eigenstate occurs whenever

$$S(T)+E(T)T=2\pi n,\quad n=\text{integer} \tag{4.1}$$

with S and E the action and energy of the solution, respectively.

Where is the one-parameter family of periodic solutions in our case?[17] Let us perform the local gauge transformation

$$V(\vec{r},t)=e^{iMt\hat{r}\cdot\vec{T}} \tag{4.2}$$

on the static dyon field configurations (3.5)–(3.7). The fields Φ and H remain the same, W_0 tends now to zero at infinity like $1/r$, and the W_i become periodic. Although their period is $2\pi/M$, as we argue in Ref. 17, what actually matters is the period of the gauge transformation itself, and this is

$$T=4\pi/M\,. \tag{4.3}$$

Thus, the family of the static solutions parametrized by M can be thought of as a family of periodic solutions parametrized by T given in (4.3).

On the other hand, the quantity action plus period times energy, on the left-hand side of Eq. (4.1), is gauge invariant. So, we will calculate it using the time independent form (3.5) of the *Ansatz*. It is easy to show that[6]

$$\begin{aligned}S(T)+E(T)T&\equiv\int_0^T dt\int d^3x(\mathcal{L}+\theta^{00})\\&=\int_0^T dt\int d^3x\left[\sum_a \pi^a\partial_0\phi^a\right]-\Sigma(T)\,,\end{aligned} \tag{4.4}$$

where the first term on the right represents the integral of the sum of terms of the form (time derivative of a field ϕ^a)×(its conjugate momentum π^a), while the second is

$$\Sigma(T)=\int_0^T dt\int d^2S_i\,\mathrm{Tr}(W_{0i}W_0+f\partial_0 W_i) \tag{4.5}$$

(f is related to the longitudinal part of W_{0i} by $-\partial_i f=W_{0i}^L$). For the static solution only the first term of $\Sigma(T)$ survives, since all the time-derivative terms vanish, and is easily calculable. The computation gives $\Sigma(T)=2\pi bMT/g^2$ and by using (4.3) the quantization condition reads

$$4\pi b=ng^2,\quad n \text{ is any integer}\,. \tag{4.6}$$

The quantization of b leads, through (3.17)–(3.19) and (3.22), to the quantization of the electric charge B-L and color properties of the dyons. If we combine the above results, we conclude that the dyons have (a) electric charge $q_n=n(-4e/3)$, (b) $B\text{-}L=n(-\frac{2}{3})$, (c) magnetic charge, of course, $m=1/2e$. (d) To determine their color properties, we argue as follows. Combining (4.6) with (3.19) we get $J_0^a(\vec{x})=(n/\sqrt{3})\delta^{a8}\delta(\vec{x})$. This is the color density of a "color-isospin" singlet [due to the fact that the Γ symmetry of the *Ansatz* contains the color-isospin SU(2) factor], with $\lambda^8=-n/\sqrt{3}$. Thus, our dyons transform according to the $(n+k,k)$ for $n\geq 0$, or $(k,|n|+k)$ for $n<0$ $(k=0,1,2,\ldots)$ representation of $SU(3)_C$, since all those have their SU(2) singlet at the position $(0,-n/\sqrt{3})$ on the weight diagram in the (λ^3,λ^8) coordinate system.

The conclusion is that our dyons, all with magnetic charge $m=1/2e$, come in families, denoted by $n=0,1,2,\ldots$, with electric charge $q_n=n(-\frac{4}{3}e)$, $B\text{-}L=n(-\frac{2}{3})$, and the kth member of the nth family $(k=0,1,2,\ldots)$ transforms according to the $(n+k,k)$ for $n\geq 0$, or the $(k,|n|+k)$ for $n<0$ representation of $SU(3)_C$.

We would like at this point to sketch another line of reasoning[2] which leads to the same conclusion. Let us go to the $W_0=0$ gauge with the Higgs fields pointing everywhere in the same direction of internal space and think of the dyons as being rotational excitations in internal space of the pure monopole. The global vacuum symmetry in our theory is $SU(3)_C\times U(1)_{em}\times U(1)_{B\text{-}L}$ and the classical monopole $(W_0=0)$ is invariant under $\Gamma=SU(2)\times U(1)_{\Gamma_4}\times U(1)_{\Gamma_5}\subset SU(3)_C\times U(1)_{em}\times U(1)_{B\text{-}L}$. Any quantum state of the theory will be characterized by its $SU(3)_C\times U(1)_{em}\times U(1)_{B\text{-}L}$ quantum numbers, i.e., it will have the form $|(p,l;\lambda^3,\lambda^8),q,B\text{-}L\rangle$, where (p,l) with $p,l\geq 0$ is the $SU(3)_C$ representation, (λ^3,λ^8) is the particular state in that representation, and q, B-L are the electric charge and the B-L quantum numbers, respectively. Since the classical dyons are Γ invariant, the corresponding quantum states will be Γ singlets. Every SU(3) representation (p,l) contains one and only one SU(2) singlet; its position in the weight diagram is $(0,-(p-l)/\sqrt{3})$ in the (λ^3,λ^8) coordinate system. On the other hand, since $\Gamma_4=(1/2\sqrt{2})\,\mathrm{diag}(1,1,-1,-1,0)=(\frac{2}{3})^{1/2}\lambda^8-(1/2\sqrt{2})Q/e$, the requirement $\Gamma_4=0$ leads to the further restriction on the dyon states $|\psi\rangle$, $[\Lambda^8-(\sqrt{3}/4)Q/e]|\psi\rangle=0$, with Λ^8 and Q being the "color-hypercharge" and the electric charge operators, respectively. This implies $q=(p-l)(-\frac{4}{3}e)$.

Finally, $\Gamma_5=0$ leads to the condition $B\text{-}L=(1/2e)q$. Thus, in agreement with our previous results the dyon states are

$$\left|\left(n+k,k;0,-\frac{n}{\sqrt{3}}\right),\ q=n\left(-\frac{4e}{3}\right),\ B\text{-}L=n(-\tfrac{2}{3})\right\rangle$$

for $n \geq 0$

and

$$\left|\left(k,|n|+k;0,-\frac{n}{\sqrt{3}}\right),\ q=n\left(-\frac{4e}{3}\right),\ B\text{-}L=n(-\tfrac{2}{3})\right\rangle$$

for $n<0$,

where we have defined $n \equiv p-l$ and $k=0,1,2,\ldots$.

We will now argue that all the dyons with the same n are degenerate at the level at which we are working. The reason is that their mass splitting is[18] $\Delta M \sim 1/I_0$, where I_0 is the classical moment of inertia of the dyon around any of the 4th through 7th directions in color space. But,

$$I_0 \propto \int d^3\vec{r}[\mathrm{Tr}(\delta W_i)^2 + \mathrm{Tr}(\delta\Phi)^2 + |\delta H|^2],$$

where by $\delta\phi$ we mean the change in ϕ due to SU(3) rotations generated by λ^4, λ^5, λ^6, or λ^7. It is easy to see that under such rotations our monopole (with $W_0=0$) gives $\delta H=0$, $\mathrm{Tr}(\delta\Phi)^2 \sim e^{-\beta r}$, because all the functions of the Φ-*Ansatz* are expected to approach the vacuum expectation value (VEV) exponentially fast, but $\mathrm{Tr}(\delta W_i)^2 \sim 1/r^2$. This last makes $I_0=\infty$ and thus $\Delta M=0$. The dyons with different n's are not degenerate because they carry different electric and color charge and, at the same time, the moments of inertia around the electromagnetic and the 8th SU(3) directions are both finite.

The analysis of the quantum problem, at the level we discussed it in this section, is certainly not adequate to answer the question of stability of the dyons. It is expected that in the full quantum theory of the dyons the infinite k degeneracy, mentioned above, will be lifted and the energy of the dyons will increase with k. But then the $(n+k,k)$ dyon will decay through the emission of k gluons to the $(n,0)$ one, without any change in q, B-L, and m.

Recall that this model contains a fundamental superheavy gauge particle (X) with quantum numbers[10] $q=-4e/3$, $B\text{-}L=-\frac{2}{3}$ and color triplet. Thus, the above results can be equivalently stated: The nth stable dyon has the quantum numbers of the symmetric combination of n X's for $n \geq 0$ or of $|n|$ $\bar{X}$'s for $n<0$, and magnetic charge $m=1/2e$.

We can now get some partial information concerning the stability issue of those dyons by the following heuristic discussion, based on simple energetic considerations.[6] Notice that the mass M_0 of the pure monopole is of order $O(\alpha^{-1}M_X) \sim O(10^2 M_X)$, since in our model and at the grand unification mass scale we have $\alpha=g^2/4\pi \simeq \frac{1}{50}$ (M_X is the mass of the superheavy gauge boson X). On the other hand, the mass splitting between two successive dyons is expected[16] to be of the order $O(\alpha M_X)$, i.e., $O(10^{-2}M_X)$. This rough estimate makes almost certain the stability of the nth dyon (D_n) against decays to D_{n-1} and X. However, it is not conclusive about the possibility of the decay $D_n \to D_{n-1}+H_3+V^-$, where H_3 is a color triplet of Higgs fields with $q=-\frac{1}{3}$ and $B-L=-\frac{2}{3}$ (Ref. 10) which also exists in our model, and V^- the weak gauge field defined in (2.12). The reason is that, presumably,[11] the mass of H_3 is smaller than M_X by one or two powers of ten, i.e., $M_{H_3} \sim O(10^{-2} - 10^{-1}M_X)$.

We can also discuss at this level certain processes involving quarks and leptons. Consider, for example,[19] a d quark plunging through the center of the first positively charged dyon, which we treat for the moment as a fixed static potential. Since this potential, according to (3.5), mixes the third and fourth components of the 5-plet, the d quark will be replaced by a positron in the final state. This process must be accompanied by a transfer of charge and color to the dyon. So, we actually have the process $d+D_{-1} \to e^+ + D_0$, or more generally we can have $d+D_n \to e^+ + D_{n+1}$. For $n<0$ this process is exothermic, while for $n \geq 0$ the center-of-mass energy of the system has to be high enough to compensate for the mass difference of the D_{n+1} and D_n. Analogously, we can have the reaction $u+D_n \to \bar{u}+D_{n-1}$, since the u and $\bar{u}$ are both in the $\underline{10}$ representation of SU(5) (Ref. 10):

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \bar{u}_G & -\bar{u}_B & -u_R & -d_R \\ -\bar{u}_G & 0 & \bar{u}_R & -u_B & -d_B \\ \bar{u}_B & -\bar{u}_R & 0 & -u_G & -d_G \\ u_R & u_B & u_G & 0 & -e^+ \\ d_R & d_B & d_G & e^+ & 0 \end{pmatrix}_L$$

and the dyons can mix them. Since those processes are all possible at the classical level, their cross sections are of the order of the geometrical area of the dyons, i.e., $O(1/M_X^2)$. This must be compared to baryon-number-violating processes in the absence of a dyon, which are $O(1/M_X^4)$. Therefore, the dyon acts as a catalyst—the presence of a dyon strongly enhances baryon-number-violating processes.

Incidentally, the above discussion is an independent consistency check of the quantum numbers we have found for the dyons. Since $d \to e^+$ in a dyon field there must exist neighboring dyons which differ in quantum numbers by an amount corresponding to the difference between d and e^+. But this agrees with what we found using the semiclassical argument.

Notice, though, that if we take into account the fermion effects in their full glory, interesting and amusing phenomena will occur.[20] For example, since the dyons have the same quantum numbers as combinations of X's they decay to monopoles through the emission of quarks and leptons. Their width is estimated to be of O(1 GeV), which is tiny compared to the grand unification mass scale. We can have, in analogy to charmonium, unstable dyon-antidyon bound systems, as well as dyon-quark and dyon-lepton meson bound states. A detailed analysis of the above processes, as well as other phenomenological properties of those dyons, are included in Ref. 20.

V. OTHER MONOPOLE ANSÄTZE

In this section we will discuss briefly other *Ansätze* corresponding to monopoles and dyons with spherical symmetry and magnetic charge larger than the one considered before. By definition, the spherical symmetry is meant under $\vec{L}+\vec{T}$, a combination of ordinary spatial rotations and an SU(2) embedding in SU(5) generated by $\vec{T}$. For every such $\vec{T}$ embedding we can write the most general spherically symmetric *Ansatz* for the fields W_μ, Φ, and H. We want, of course, the *Ansatz* to have nonzero magnetic charge in order to be topologically stable and also to have the maximum possible symmetry Γ, since we believe that the higher the symmetry of a solution, the less its mass will be. According to this last belief, whenever we have two possible spherically symmetric *Ansätze* in the same topological sector, we will be considering the one with higher Γ symmetry as the interesting one, while the other will be considered unstable. The asymptotic behavior of the radial functions, which appear in the Higgs fields, are completely determined by the requirement that at spatial infinity along the z axis they approach the values Φ_0 and H_0 given in (2.7). As for the gauge fields, although we cannot determine the asymptotic behavior of W^0 without actually solving the field equations, there exists only a small number of possible behaviors of the W^i fields for every $\vec{T}$ embedding, and those we can specify in the following way. The method makes use of a rather obvious generalization of a theorem due to Wilkinson and Goldhaber,[21] to the case of interest, in which we also have the Higgs field H in the $\underline{5}$ representation of SU(5). Let us consider first the case of monopoles ($W^0=0$). We start with the observation that for any constant matrix $\tilde{Q}$, which satisfies

$$[\tilde{Q},\Phi_0]=0 \text{ and } \tilde{Q}H_0=0\,, \tag{5.1}$$

the configuration

$$\Phi(\vec{r})=\Phi_0,\quad H(\vec{r})=H_0,\quad \text{and } \vec{W}(\vec{r})=\frac{1}{g}\tilde{Q}\vec{A}_D \tag{5.2}$$

[with Φ_0, H_0 given by (2.7), and $\vec{A}_D=\hat{\phi}(1-\cos\theta)/r\sin\theta$, the singular Dirac vector potential of a unit charge monopole] is a solution of the field equations. The requirement that the string be unobservable implies the further condition on $\tilde{Q}$

$$e^{4\pi i\tilde{Q}}=1\,. \tag{5.3}$$

Since Φ_0 and H_0 break the symmetry down to $SU(3)_C\times U(1)_{em}$ and $\tilde{Q}$ is, according to (5.1), a symmetry of both Φ_0 and H_0, $\tilde{Q}$ must be a linear combination of the generator Q/e of $U(1)_{em}$, given by (2.8), and an element C of the $SU(3)_C$ algebra, i.e.,

$$\tilde{Q}=me\frac{Q}{e}+C\,. \tag{5.4}$$

Making use of the definition (3.8) of the electromagnetic fields and of the fact that $\vec{A}_D$ represents the vector potential of a unit of magnetic charge, it is easy to see that the factor m in (5.4) is the magnetic charge of the configuration (5.2). We now state the previously mentioned theorem of Wilkinson and Goldhaber, as applied in our case.

Let $\vec{T}$ be the generators of any SU(2) subgroup of SU(5) (not necessarily irreducible). There exists a gauge transformation which transforms the solution (5.2) to a spherically symmetric one under $\vec{L}+\vec{T}$, if and only if there exists another SU(2) embedding in SU(5) generated by $\vec{I}$ (also not necessarily irreducible), satisfying

$$\tilde{Q}=I_3-T_3,\quad [\vec{I},\tilde{Q}]=0,\quad [\vec{I},\Phi_0]=0,\quad \text{and } \vec{I}H_0=0\,. \tag{5.5}$$

The form of that gauge transformation is

$$\begin{aligned}&\Lambda(\hat{r})=\Omega(\hat{r})\omega^{-1}(\hat{r})\,,\\ &\Omega(\hat{r})=e^{-i\phi T_3}e^{-i\theta T_2}e^{i\phi T_3}\,,\\ &\omega(\hat{r})=e^{-i\phi I_3}e^{-i\theta I_2}e^{i\phi I_3}\,,\end{aligned} \tag{5.6}$$

and the spherically symmetric equivalent form of the solution (5.2) is

$$\begin{aligned}&\vec{W}=\frac{1}{gr}(\vec{I}(\hat{r})-\vec{T})\times\hat{r}\,,\quad \Phi(\hat{r})=\Omega(\hat{r})\Phi_0\Omega^{-1}(\hat{r})\,,\\ &H(\vec{r})=\Omega(\hat{r})H_0,\quad \text{with } \vec{I}(\hat{r})=\Lambda(\hat{r})\vec{I}\Lambda^{-1}(\hat{r})\,.\end{aligned} \tag{5.7}$$

An immediate consequence of the above theorem is the following. Consider H_0, Φ_0, and Q as given by (2.7) and (2.8). Since, according to the theorem $[\vec{I},\Phi_0]=0$, $\vec{I}H_0=0$, $\vec{I}$ is restricted to act only on the upper three components of the 5-plet of SU(5). The form of Φ_0 and H_0, on the other hand, specifies that $SU(3)_C$ also acts in the same three-dimensional subspace. Thus, the 4-4 element of $\tilde{Q}$

as given by (5.4) is (me), and as given by $\tilde{Q}=I_3-T_3$ is $-(T_3)_{44}$. This implies that the possible values of (me), for a given $\vec{T}$ embedding, are the eigenvalues of T_3.

In what follows, we will write down the smooth, finite-energy monopole and dyon *Ansätze* satisfying the conditions and the symmetries mentioned in the beginning of this section, with asymptotic behavior determined by (5.7). The time-independent dyon *Ansätze* have, of course, $W_0\neq 0$. The form of W_0 is dictated by the already mentioned symmetries, but its asymptotic behavior can actually be determined only by the field equations, except for some partial information we extract from the condition $D_0H\sim_{r\to\infty}O(1/r^2)$, which is required for the energy to be finite.

As will become clear from the following discussion of all the $\vec{T}$ embeddings, there can possibly exist spherically symmetric monopoles and dyons only in the sectors corresponding to magnetic charges[22] $m=\pm 1/2e$, $\pm 1/e$, $\pm 3/2e$, and $\pm 2/e$. Although in each of those sectors there exists more than one spherically symmetric dyon, by using the criteria discussed in the beginning of this section, we will uniquely determine and write the *Ansätze* of the stable ones.

(A) In the $\underline{5}\to\underline{2}+\underline{1}+\underline{1}+\underline{1}$ embedding the only possible nonzero magnetic charges are $\pm 1/2e$. This is the monopole (and its antimonopole) we discussed in Secs. III and IV, where we saw that the Γ symmetry was SU(2)×U(1). No other possibility for a stable monopole exists in this embedding.

(B) The possible magnetic charges in the $\underline{5}\to\underline{2}+\underline{2}+\underline{1}$ case are $\pm 1/2e$. Since here we cannot have a Γ larger than the U(1)×U(1) generated by $(1/2\sqrt{2})\,\mathrm{diag}(1,1,-1,-1,0)$ and $\frac{1}{2}\mathrm{diag}(1,1,0,0,0)$ the monopoles and dyons for this are expected to be unstable and decay to the previous one.

(C) Let us now consider the $\underline{5}\to\underline{3}+\underline{1}+\underline{1}$ embedding $\vec{T}=\mathrm{diag}(0,0,\vec{T}_{(1)})$, with $\vec{T}_{(1)}$ the standard spin-1 representation of the generators of SU(2). Here, the highest possible residual Γ symmetry is an SU(2)×U(1) acting on the upper two components of the 5-plet, and is achieved only for $\vec{I}=0$, $H_0=\mathrm{col}(0,0,0,v,0)$, and $(1/v)\Phi_0=\mathrm{diag}(1,1,1,-\frac{3}{2}-\epsilon,-\frac{3}{2}+\epsilon)$. Notice that what we have actually done in this case in order to simplify, by having standard $\vec{T}$'s, the construction of the *Ansatz*, is to consider $SU(3)_C$ again acting on the upper three components of the 5-plet, but $(1/e)Q=\mathrm{diag}(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,1)$. Since $\tilde{Q}=-T_3=\mathrm{diag}(0,0,-1,0,1)$, the magnetic charge of this monopole is $m=1/e$, and the color-magnetic fields do not vanish at infinity. The spherically symmetric *Ansatz* in this case is[23]

$$\frac{g}{v}\Phi(\vec{r})=\begin{pmatrix}-\frac{3}{2}\phi_1-\phi_3 &&&&\\ & -\frac{3}{2}\phi_1-\phi_3 &&&\\ && \phi_1(r) &&\\ &&& \phi_1(r) &\\ &&&& \phi_1(r)\end{pmatrix}+\phi_2(r)\hat{r}\cdot\vec{T}+\phi_3(r)(\hat{r}\cdot\vec{T})^2\,,$$

$$H(\vec{r})=\frac{1}{g}h(r)\begin{pmatrix}0\\0\\-Y_{1-1}(\theta,\phi)\\Y_{10}(\theta,\phi)\\-Y_{11}(\theta,\phi)\end{pmatrix}\,,\quad Y_{lm}\text{ are the spherical harmonics} \tag{5.8}$$

$$\vec{W}(\vec{r})=-\frac{\vec{T}\times\hat{r}}{gr}+\frac{1}{gr}\{K_0(r)+K_1(r)\hat{r}\cdot\vec{T}+K_2(r)(\hat{r}\cdot\vec{T})^2,\vec{T}\times\hat{r}\}+\frac{1}{gr}\{\vec{T}-\hat{r}(\hat{r}\cdot\vec{T}),L_0(r)+L_1(r)\hat{r}\cdot\vec{T}+L_2(r)(\hat{r}\cdot\vec{T})^2\}\,.$$

$W_0(\vec{r})$ = of the same form as $(1/v)\Phi(\vec{r})$, with the replacement

$$\phi_i\to J_i(r),\quad i=1,2,3$$

where $\{A,B\}\equiv AB+BA$. A few comments are in order:

(a) $\Phi(\vec{r})$, $W^i(\vec{r})$, and $W^0(\vec{r})$ are traceless, as they should be.

(b) Notice that we have required the *Ansatz* to satisfy the gauge-fixing condition $\hat{r}\cdot\vec{W}(\vec{r})=0$. This is compatible with spherical symmetry.[19]

(c) All the radial functions are real, for the same reasons as in the $\underline{5}\to\underline{2}+\underline{1}+\underline{1}+\underline{1}$ *Ansatz*.[24]

(d) Also, notice that in this case we cannot require invariance of the *Ansatz* under simultaneous inversion of $\hat{r}$ and $\vec{T}$, since $H(\vec{r})$ does not have this invariance for $h(r)\neq 0$.

(e) Observe that the chosen form of H_0 forces the last three components of $H(\vec{r})$ to have $|\vec{T}|=1$. In order for $H(\vec{r})$ to be a singlet under $\vec{L}+\vec{T}$, we had to require $|\vec{L}|=1$ also. This is why we used the Y_{lm}'s in the *Ansatz* for H.

(f) Along the $+\hat{z}$ axis at infinity $\Phi(r\hat{z})\to\Phi_0$, and $H(r\hat{z})\to H_0$. This leads to

$$\phi_1(r)\to ag,\quad \phi_2(r)\to g\left[2+\frac{a}{2}\right],\quad \phi_3(r)\to g\left[-\frac{3a}{2}-1\right],$$
$$a\equiv-\tfrac{3}{2}-\epsilon\,,$$
and $$h(r)\to(4\pi/3)^{1/2}vg\,. \tag{5.9}$$

The asymptotic form of $\vec{W}$ has to be the one given in (5.7) with $\vec{I}(\hat{r})=0$. This leads to

$$K_i(r)\underset{r\to\infty}{\longrightarrow}0,\quad L_i(r)\underset{r\to\infty}{\longrightarrow}0,\quad i=0,1,2\,. \tag{5.10}$$

Finally, $D_0H\underset{r\to\infty}{\sim}O(1/r^2)$ for the energy to be finite and as a consequence

$$J_1(r)\underset{r\to\infty}{\sim}O(1/r^2)\,. \tag{5.11}$$

(D) The embedding $\underline{5}\to\underline{3}+\underline{2}$ does not lead to anything interesting to us, the reason being that the possible magnetic charges in this case are $\pm1/e$, $\pm1/2e$, and the Γ symmetry cannot be larger than the U(1) generated by $\frac{1}{2}\operatorname{diag}(1,1,0,0,0)$. As a consequence, we expect those monopoles to be unstable and decay to the ones we have already discussed.

(E) We now come to the case $\underline{5}\to\underline{4}+\underline{1}$, with $\vec{T}=\operatorname{diag}(\vec{T}_{(3/2)},0)$ and $\vec{T}_{(3/2)}$ the $\mathrm{SU}(2)$ generators in the spin-$\frac{3}{2}$ representation. The Γ symmetry in this case can at most be the U(1) generated by $(1/2\sqrt{2})\operatorname{diag}(1,1,1,1,0)$, and if we require invariance under simultaneous inversion of $\hat{r}$ and $\vec{T}$, we have the most general *Ansatz*:

$$\frac{g}{v}\Phi(r)=\begin{bmatrix}\phi_0(r)&&&&\\&\phi_0(r)&&&\\&&\phi_0(r)&&\\&&&\phi_0(r)&\\&&&&-4\phi_0(r)-5\phi_2(r)\end{bmatrix}+\phi_1(r)\hat{r}\cdot\vec{T}+\phi_2(r)(\hat{r}\cdot\vec{T})^2+\phi_3(r)(\hat{r}\cdot\vec{T})^3\,,$$

$$H(\vec{r})=\frac{1}{g}\begin{bmatrix}0\\0\\0\\0\\h(r)\end{bmatrix},\quad W_0(\vec{r})=\text{the same form as }\frac{1}{v}\Phi(\vec{r}),\quad\text{with }\phi_k\to J_k(r),\quad k=0,1,2,3 \tag{5.12}$$

and

$$\vec{W}(\vec{r})=-\frac{\vec{T}\times\hat{r}}{gr}+\frac{1}{gr}\{K(r)+K_1(r)\hat{r}\cdot\vec{T}+K_2(r)(\hat{r}\cdot\vec{T})^2+K_3(r)(\hat{r}\cdot\vec{T})^3,\vec{T}\times\hat{r}\}$$

with all the radial functions real. The possible magnetic charges in this case are $\pm3/2e$ and $\pm1/2e$. In the sectors with $m=\pm1/2e$ we have already found more symmetric monopoles. There exists more than one possibility of spherically symmetric monopoles with $m=\pm3/2e$. All those have the same Γ symmetry, and we cannot use this criterion to decide which the stable one is. Notice, though, that in one of the possible *Ansätze*, namely the one with $\vec{I}=\operatorname{diag}(\vec{I}_{(1)},0,0)$, Φ_0, and H_0 as given by (2.7), we have $\tilde{Q}=I_3-T_3=\operatorname{diag}(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{3}{2},0)=\frac{3}{2}Q/e$, which is purely electromagnetic. At infinity the corresponding monopole has no color-magnetic field. Considering this as an indication of smaller energy, we give the asymptotic behavior of the radial functions appropriate for this monopole. Following the same procedure as before, we are led to

$$\phi_0(r)\to\frac{g}{16}(a+20),\quad \phi_1(r)\to-g\frac{a+4}{24},\quad \phi_2(r)\to-g\frac{a+4}{4},$$
$$\phi_3(r)\to g\frac{a+4}{6},\quad h(r)\to gv,\quad 4J_0+5J_2\to O(1/r^2)\,, \tag{5.13}$$
$$K(r)\to\frac{\sqrt{6}}{48}(5\sqrt{3}-1),\quad K_1(r)\to\frac{1}{2\sqrt{3}},\quad K_2(r)\to\frac{\sqrt{6}}{12}(1-\sqrt{3}),\quad K_3(r)\to0\,.$$

(F) Finally, we consider the embedding $\underline{5}\to\underline{5}$. The $\vec{T}$'s now have the standard form of the generators of

SU(2) in the spin-2 representation. The possible magnetic charges now are $m=\pm 2/e, \pm 1/e$. Again, since there is no residual Γ symmetry a nontrivial solution can have in this embedding, the monopole we found in (C) with $m=\pm 1/e$ and $\Gamma=\mathrm{SU}(2)\times\mathrm{U}(1)$ is expected to be the stable one in this sector. Thus, we are left with the new option $m=\pm 2/e$. Again, there are more than one possible spherically symmetric *Ansätze* with these last magnetic charges. Our guess is that we will have the stable monopole if we take $H_0=\mathrm{col}(0,0,v,0,0)$, $(1/v)\Phi_0=\mathrm{diag}(1,1,-\frac{3}{2}-\epsilon,1,-\frac{3}{2}+\epsilon)$, $(1/e)Q=\mathrm{diag}(-\frac{1}{3},-\frac{1}{3},0,-\frac{1}{3},1)$, $\mathrm{SU}(3)_C$ acting on the first, second, and fourth components of the 5-plet, and $\vec{\mathrm{I}}=\frac{1}{2}\mathrm{diag}(\vec{\sigma},0,0,0)$ with $\vec{\sigma}$ the Pauli matrices. The corresponding $\tilde{Q}=\mathrm{diag}(-\frac{3}{2},-\frac{3}{2},0,1,2)=2Q/e-(5/\sqrt{3})\lambda^8$, where we used $\lambda^8=(1/2\sqrt{3})\,\mathrm{diag}(1,1,0,-2,0)$. Consequently, these monopoles and the associated dyons have magnetic charge $m=2/e$, and their form is

$$\frac{g}{v}\Phi(\vec{r})=-2\phi_2(r)-\tfrac{34}{5}\phi_4(r)+\phi_1(r)\hat{r}\cdot\vec{T}+\phi_2(r)(\hat{r}\cdot\vec{T})^2+\phi_3(r)(\hat{r}\cdot\vec{T})^3+\phi_4(r)(\hat{r}\cdot\vec{T})^4,$$

$$H(\vec{r})=\frac{1}{g}h(r)\begin{pmatrix}Y_{2-2}(\theta,\phi)\\-Y_{2-1}(\theta,\phi)\\Y_{20}(\theta,\phi)\\-Y_{21}(\theta,\phi)\\Y_{22}(\theta,\phi)\end{pmatrix},\quad W_0(\vec{r})=\text{the same form as }\frac{1}{v}\Phi(\vec{r}),\quad\text{with }\phi_k\to J_k(r),\quad k=1,2,3,4$$

and

$$\vec{W}(\vec{r})=-\frac{\vec{T}\times\hat{r}}{gr}+\frac{1}{gr}\{K_0(r)+K_1(r)\hat{r}\cdot\vec{T}+K_2(r)(\hat{r}\cdot\vec{T})^2+K_3(r)(\hat{r}\cdot\vec{T})^3+K_4(r)(\hat{r}\cdot\vec{T})^4,\vec{T}\times\hat{r}\}$$
$$+\frac{1}{gr}\{\vec{T}-\hat{r}(\hat{r}\cdot\vec{T}),L_0(r)+L_1(r)(\hat{r}\cdot\vec{T})+L_2(r)(\hat{r}\cdot\vec{T})^2+L_3(r)(\hat{r}\cdot\vec{T})^3+L_4(r)(\hat{r}\cdot\vec{T})^4\}$$

with all the radial functions being real and $\vec{W}(\vec{r})$ satisfying the gauge-fixing condition $\hat{r}\cdot\vec{W}(\vec{r})=0$. It is easy to check that $(L_\pm+T_\pm)H=0$ and $(L_3+T_3)H=0$. Finally, the fields $\Phi(\vec{r})$, $W_0(\vec{r})$, and $W^i(\vec{r})$ are all traceless.

The asymptotic behavior of the radial functions in the *Ansatz* is

$$\phi_1(r)\to-\frac{4+a}{12}g,\quad\phi_2(r)\to\frac{34-29a}{24}g,\quad\phi_3(r)\to\frac{4+a}{12}g,\quad\phi_4(r)\to\frac{5a-10}{24}g,$$

$$h(r)\to(4\pi/5)^{1/2}vg,\quad K_0(r)\to-\tfrac{1}{32},\quad K_1(r)\to-\tfrac{1}{24},\quad K_2(r)\to\tfrac{1}{16},\quad K_3(r)\to\tfrac{1}{24},\quad K_4(r)\to0,\quad L_i(r)\to0,\quad i=0,1,2,3,4.$$

How fast the asymptotic values are approached is, as always, determined by the field equations.

VI. SUMMARY AND DISCUSSION

We showed that spherically symmetric monopole and dyon *Ansätze* exist only in the sectors with magnetic charge $\pm 1/2e$, $\pm 1/e$, $\pm 3/2e$, and $\pm 2/e$. The existence of a spherically symmetric *Ansatz* does not guarantee the existence of a solution of the field equations. Using the intuitive argument that the larger the symmetry of a configuration, the smaller its energy is, we were able to uniquely specify and write down the *Ansatz* for the monopole and dyons that have a greater chance of being stable in each sector. We concentrated on the dyons with the smallest possible magnetic charge $(1/2e)$. We discussed their properties in detail. We used the Bohr-Sommerfeld quantization condition with a plausible definition of periodicity in gauge theories to determine the stable dyon quantum numbers, which turned out to be normal. The first stable dyon has, apart from the magnetic charge, the same quantum numbers as the superheavy X gauge boson. Thus, it can be involved in processes analogous to the ones which employ the X's. The fact, on the other hand, that their density in the universe seems to be extremely small, makes them phenomenologically uninteresting.

We determined the magnetic properties, both ordinary and color, of the stable dyons in every sector. Those properties depend only upon the asymptotic behavior of W_i, and this is uniquely given by $\tilde{Q}$. But, for the electric properties as well as the $B-L$ of the dyons we need the asymptotic behavior of W_0, and this we do not know *a priori* without actually solving the field equations.

ACKNOWLEDGMENTS

We wish to thank E. Witten and S. Coleman for many valuable discussions. This research was supported in part by the National Science Foundation under Grant No. PHY77-22864.

[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[2]It is impossible to make a complete list of references related to monopoles. For general reviews, see S. Coleman, in *New Phenomena in Subnuclear Physics*, proceedings of the 14th Course of the International School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977); R. Rajaraman, Phys. Rep. 21C, 227 (1975); R. Jackiw, Rev. Mod. Phys. 49, 681 (1977); P. Goddard and D. Olive, Rep. Prog. Phys. (to be published).

[3]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)]; S. Coleman, S. Parke, A. Neveu, and C. M. Sommerfield, Phys. Rev. D 15, 544 (1977); R. Brandt and F. Neri, Nucl. Phys. B161, 253 (1979).

[4]M. Prasad and C. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[5]Some relevant papers are G. 't Hooft, Nucl. Phys. B105, 538 (1976); A. Goldhaber and D. Wilkinson, *ibid.* B114, 317 (1976); D. Wilkinson and F. Bais, Phys. Rev. D 19, 2410 (1979); D. Wilkinson, Nucl. Phys. B125, 423 (1977); F. Bais and J. Primack, *ibid.* B123, 253 (1977); Z. Horváth and L. Palla, Phys. Lett. 69B, 197 (1977); P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. B125, 1 (1977); E. Corrigan, D. Olive, D. B. Fairlie, and J. Nuyts, *ibid.* B106, 475 (1976).

[6]B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[7]Again, a few relevant references are E. Tomboulis and G. Woo, Nucl. Phys. B107, 221 (1976); E. Tomboulis, Phys. Rev. D 12, 1678 (1975); J. Gervais and B. Sakita, *ibid.* 11, 2943 (1975); N. Christ, A. Guth, and E. Weinberg, Nucl. Phys. B114, 61 (1976); J. Gervais, A. Jevicki, and B. Sakita, Phys. Rev. D 12, 1038 (1975); J. Goldstone and R. Jackiw, *ibid.* 11, 1486 (1975); R. Rajaraman and E. Weinberg, *ibid.* 11, 2950 (1975).

[8]R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976); Phys. Rev. Lett. 36, 1116 (1976); P. Hasenfratz and G. 't Hooft, *ibid.* 36, 1119 (1976); A. Goldhaber, *ibid.* 36, 1122 (1976).

[9]E. Witten, Phys. Lett. 86B, 283 (1979).

[10]H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974); H. Georgi, H. R. Quinn and S. Weinberg, *ibid.* 33, 451 (1974); A. Buras, J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135, 66 (1978).

[11]S. Weinberg, Phys. Rev. Lett. 42, 850 (1979) and references therein. D. V. Nanopoulos and S. Weinberg, Phys. Rev. D 20, 2484 (1979); A. Yildiz and P. Cox, *ibid.* 21, 906 (1980).

[12]J. Preskill, Phys. Rev. Lett. 43, 1365 (1979).

[13]Z. Horváth and L. Palla, Phys. Lett. 69B, 197 (1977).

[14]E. Corrigan and D. Olive, Nucl. Phys. B110, 237 (1976).

[15]In fact, it is irrelevant in the discussion that follows how fast $K(r)$ goes to zero at infinity. In order for the energy to be finite, $D_0 H \sim 1/r^2$ at infinity. This leads to $J_1(r)+J_2(r) \to O(1/r^2)$. We expect that $J_1(r)$ and $J_2(r)$ will have this behavior separately.

[16]E. Tomboulis and G. Woo, MIT report, 1976 (unpublished).

[17]To apply the Bohr-Sommerfeld quantization condition in a gauge theory, we need a reasonable definition of what we mean by a periodic field configuration. If we are free to perform time-dependent gauge transformations, we can go from a static field (any time interval is a period of this) to another with definite period or to a third with no period at all. To fix this gauge freedom, let us go to the gauge $W_0=0$, which is the one used in the canonical quantization of gauge theories. To transform to this gauge, our dyon solution (3.5)–(3.7), we must perform the gauge transformation

$$U(\vec{r},t)=e^{igW_0(\vec{r})t} \tag{1}$$

with $W_0(\vec{r})$ given in (3.5)–(3.7). The new field configuration is, of course, not periodic in the ordinary sense of classical mechanics. The fields never return everywhere to their initial values at $t=0$. But given the asymptotic forms of the radial functions (3.7) and the fact that $\hat{r}\cdot\vec{T}$ has eigenvalues $\pm 1/2$, we observe that after a time $T=4\pi/M$ the fields are gauge equivalent to the $t=0$ ones, with a gauge transformation $U(\vec{r},t=4\pi/M)$, which tends to the identity at spatial infinity. We consider the field configurations which are gauge equivalent via a gauge transformation that tends to the identity at spatial infinity as being the same. Thus, the periods of the dyon configurations in the $W_0=0$ gauge are

$$T=\frac{4\pi}{M} \tag{2}$$

and the family of the dyon solutions given in (3.5)–(3.7) can be thought of as a family of periodic solutions with period given in (2).

[18]R. Jackiw in Ref. 2.

[19]For an analogous discussion, see A. Goldhaber and D. Wilkinson, Nucl. Phys. B114, 317 (1976).

[20]C. Dokos and T. Tomaras (unpublished).

[21]D. Wilkinson and A. Goldhaber, Phys. Rev. D 16, 1221 (1977).

[22]For analogous results in the SO(3) case, see E. Weinberg and A. Guth, Phys. Rev. D 14, 1660 (1976).

[23]For similar examples see, e.g., A. Goldhaber and D. Wilkinson, Nucl. Phys. B114, 317 (1976); D. Wilkinson, *ibid.* B125, 423 (1977); F. Bais and J. Primack, *ibid.* B123, 253 (1977).

[24]The reality of ϕ_k, J_k, $k=1,2,3$, and K_i, L_i, $i=0,1,2$ is required by the Hermiticity of $\Phi(\vec{r})$, $W_0(\vec{r})$, and $\vec{W}(\vec{r})$, respectively. As for the function $h(r)$, we use the following argument. Let us allow for a complex phase in $h(r)$ and consider $H'(\vec{r})=e^{i\alpha(r)}H(\vec{r})$. The function $\alpha(r)$ appears in the energy density only through the term $|D_i H'|^2$. Using the condition $\hat{r}\cdot\vec{W}(\vec{r})=0$ one gets $|D_i H'|^2=|D_i H|^2+\alpha'^2(r)H^\dagger H$. The field equation for $\alpha(r)$ $[(d/dr)(\alpha'(r)H^\dagger H)=0]$ is consistent with $\alpha(r)=$const and at the same time the energy is lower. Without loss of generality we take $\alpha(r)=0$.