Quaternionic chromodynamics as a theory of composite quarks and leptons

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I make a preliminary study of quaternionic chromodynamics [i.e., U(2) algebraic chromodynamics] as a theory of composite quarks and leptons. In the unphysical symmetric static limit in which the gluon field is massless and the fundamental two-internal-component spinor is infinitely massive, I compute the internal-symmetry structure of the residual interactions of three-spinor composites, using the heuristic quark and lepton identifications proposed by Harari and Shupe. Three types of interactions appear: (i) a color-singlet, flavor-diagonal photon, coupling to the electron, quarks, and neutrino with the correct charge assignments, and a second photon coupling to the neutrino and quarks, but not to the electron; (ii) color-changing, flavor-diagonal gluons, coupling to the quarks in a pattern resembling, but not identical to, SU(3) quantum chromodynamics; (iii) color-changing, flavor-changing gluons, three exchanges of which can produce a weak flavor-changing transition between color-singlet states, without requiring the existence of conventional intermediate bosons. While certain aspects of the symmetric static limit are clearly at variance with standard phenomenology, the results make it plausible that a more realistic calculation, taking symmetry breaking into account, may reproduce the observed features of the usual $SU(3)_{\text{color}} \times [SU(2) \times U(1)]_{\text{weak-electromagnetic}}$ model. I briefly discuss some ideas about symmetry breaking, and describe a mechanism leading to topologically inequivalent quark-lepton generations.

I. INTRODUCTION

It is now widely accepted that the observed properties of matter can be explained as arising from the interactions, through gauge-field intermediaries, of spin $-\frac{1}{2}$ leptons and quarks. These fermions appear[1] to occur in three families, or generations, containing four members each, together with their corresponding antiparticles, as indicated in Table I. Within each generation there is a charge-multiplicity regularity which has been graphically depicted by Glashow[2] as a cube on its end (Fig. 1). The appearance of such regularities, and their associated weight diagrams, has in the past always been indicative of composite structure, and so it is natural to seek an explanation for the regularities of the quark-lepton generations by postulating the existence of further substructure. Recently, Harari[3] and Shupe[4] have proposed a set of simple heuristic rules for building composite quarks and leptons. They postulate the existence of two types of fundamental spin $-\frac{1}{2}$ objects, which in this paper I will call simply U and D spinors, carrying, respectively, electric charges 1 and 0 in units of $e/3$. They then construct the quarks and leptons in the lowest generation as three-spinor composites as indicated in Table II, and postulate the two higher generations to be internal, dynamical excitations of the lowest generation. These rules clearly give a simple accounting (based essentially on the fact that $8=2^3$) for the cubic regularity depicted in Fig. 1. What is needed to carry the Harari-Shupe idea further is a plausible underlying dynamics for the constituent U and D spinors.

My aim in this paper is to give a preliminary analysis of what I believe to be a promising candidate for the dynamics of quark-lepton constituents. In a recent paper,[5] I proposed a generalization of the usual SU(n) quantum chromodynamics, which I called "algebraic chromodynamics", in which the equations of motion are covariant under local *operator-valued* gauge transformations. I further suggested that the $n=2$ version of algebraic chromodynamics might provide a suitable dynamics for quark-lepton constituents, and this is the idea which I wish to pursue further here. Let me begin by recapitulating the central arguments of Ref. 5, as specialized to the $n=2$ case. Let τ^a, $a=0,1,2,3$ be the usual Hermitian bases for U(2),

$$\tau^0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \tau^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$
$$\tau^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \tau^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{1}$$

TABLE I. The three fermion generations, according to standard phenomenology.

Charge (units of $e/3$)	-3	-2	-1	0	0	1	2	3
Generation 3	τ^-	$\bar{t}$	b	$\bar{\nu}_\tau$	ν_τ	$\bar{b}$	t	τ^+
Generation 2	μ^-	$\bar{c}$	s	$\bar{\nu}_\mu$	ν_μ	$\bar{s}$	c	μ^+
Generation 1	e^-	$\bar{u}$	d	$\bar{\nu}_e$	ν_e	$\bar{d}$	u	e^+

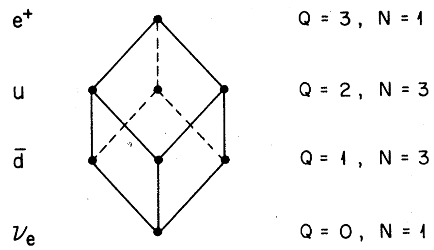

FIG. 1. Charge-multiplicity pattern within a generation (ignoring the missing helicity components of the neutrino).

normalized so that

$$\mathrm{tr}(\tau^a\tau^b) = 2\delta^{ab}. \tag{2}$$

The τ's satisfy the algebra

$$\begin{aligned} &\tfrac{1}{2}\tau^a\tfrac{1}{2}\tau^b = q^{abc}\tfrac{1}{2}\tau^c, \\ &q^{abc} = q^{bca} = q^{cab}, \\ &q^{0bc} = \tfrac{1}{2}\delta^{bc}, \\ &q^{abc} = \frac{i}{2}\epsilon^{abc}, \quad 1 \leq a,b,c \leq 3 \end{aligned} \tag{3}$$

and obey the completeness relation

$$\begin{aligned} &\sum_a \tau^a_{AB}\tau^a_{CD} = 2\,\delta_{AD}\delta_{BC}, \\ &\sum_a \tau^a_{AB}\tau^{*a}_{CD} = 2\delta_{AC}\delta_{BD}. \end{aligned} \tag{4}$$

Since the algebra of Eq. (3) is just the algebra of quaternions, the theory which I am about to describe can be viewed as a quaternionic generalization of spinor quantum electrodynamics—whence the name "quaternionic chromodynamics"—and in fact is similar to the quaternionic quantum mechanics proposed in 1962 by Finkelstein *et al.*[6] However, I will adhere to the Pauli matrix realization for quaternions, since the use of Hamilton's abstract notation obscures the central role which spinors play in the theory. To proceed, I introduce a quartet gauge potential or connection operator b^a_ν, and the corresponding field strength $f^a_{\mu\nu}$, related by

$$\begin{aligned} &F_{\mu\nu} = f^a_{\mu\nu}\tfrac{1}{2}\tau^a = \frac{\partial B_\mu}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu} - ig[B_\mu, B_\nu], \\ &B_\nu = b^a_\nu\tfrac{1}{2}\tau^a, \end{aligned} \tag{5a}$$

which implies that

$$f^a_{\mu\nu} = \frac{\partial b^a_\mu}{\partial x^\nu} - \frac{\partial b^a_\nu}{\partial x^\mu} - igq^{abc}(b^b_\mu b^c_\nu - b^b_\nu b^c_\mu). \tag{5b}$$

When the potential B^μ is varied, the change in the field strength is given by

$$\delta F_{\mu\nu} = D_\nu\delta B_\mu - D_\mu\delta B_\nu, \tag{6}$$

with D_ν the covariant derivative defined by

$$D_\nu W = \frac{\partial}{\partial x^\nu}W + ig[B_\nu, W] \tag{7}$$

when acting on an arbitrary quartet operator

$$W = w^a\tfrac{1}{2}\tau^a. \tag{8}$$

From the Jacobi identity for the commutator, it is readily verified that $F_{\mu\nu}$ satisfies

$$\begin{aligned} &D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} + D_\mu F_{\nu\lambda} = 0, \\ &[D_\mu, D_\nu]W = -ig[F_{\mu\nu}, W]. \end{aligned} \tag{9}$$

For the dynamical equation I take[7]

$$D_\mu F^{\nu\mu} = -gJ^\nu, \quad J^\nu = j^{a\nu}\tfrac{1}{2}\tau^a, \tag{10}$$

with the source current $j^{a\nu}$ constructed from a Dirac spinor field χ_A (with two internal components labeled by $A = 1, 2$) according to

$$j^{a\nu} = \bar\chi_A(-\tfrac{1}{2}\tau^*)^a_{AB}\gamma^\nu\chi_B = \bar\chi(-\tfrac{1}{2}\tau^*)^a\gamma^\nu\chi. \tag{11}$$

Using the completeness relation of Eq. (4), Eq. (10) can be written in the equivalent form

$$(J^\nu)_{AB} = j^{a\nu}\tfrac{1}{2}(\tau^a)_{AB} = -\tfrac{1}{2}\bar\chi_A\gamma^\nu\chi_B. \tag{12}$$

For the fermion equation of motion I take (with m_0 a bare mass)

$$\begin{aligned} &\bar\chi_B(\overleftarrow{\partial\!\!\!/} + im_0) = -igB_{\nu BC}\bar\chi_C\gamma^\nu = -igB_\nu\bar\chi\gamma^\nu, \\ &(\partial\!\!\!/ - im_0)\chi_A = ig\gamma^\nu\chi_C B_{\nu CA} = ig\gamma^\nu\chi B_\nu, \\ &\partial\!\!\!/ = \gamma^\nu\partial/\partial x^\nu. \end{aligned} \tag{13}$$

Taking the covariant derivative of Eq. (10) and using Eq. (9), one finds that the source current is covariantly conserved,

$$D_\nu J^\nu = 0, \tag{14}$$

and this may be verified directly from Eqs. (7), (12), and (13), checking the consistency of the field equations.

It is now straightforward to show that the above system of field equations is covariant under general operator-valued gauge transformations. That

TABLE II. The Harari (Ref. 3)-Shupe (Ref. 4) assignments for composite leptons and quarks. The fundamental spinors U and D have charges $Q=1$, $Q=0$, respectively, in units of $e/3$.

e^+	$U_1U_2U_3$
u_1, u_2, u_3	$D_1U_2U_3$, $U_1D_2U_3$, $U_1U_2D_3$
$\bar d_1, \bar d_2, \bar d_3$	$U_1D_2D_3$, $D_1U_2D_3$, $D_1D_2U_3$
ν_e	$D_1D_2D_3$

is, defining the gauge variations δ_g of χ, $\overline{\chi}$, and B_ν by

$$\delta_g\chi=-i\chi U,\quad \delta_g\overline{\chi}=iU\overline{\chi},\quad \delta_g B_\nu=-g^{-1}D_\nu U, \tag{15}$$

one can show that the field equations are gauge covariant, *without ever reordering operator terms*. The details of this calculation, and the construction of an appropriate action principle for algebraic chromodynamics, are given in Ref. 5. The significance of operator gauge covariance is that the gauge invariance of the theory does not depend on the assumption that operator factors with spacelike-separated arguments commute. In fact, in the calculations of Sec. II I will explicitly violate the conventional microscopic causality postulate by letting the gauge potential components b^a_ν be noncommuting matrix-valued variables. It is this freedom for the gauge potential components to take on matrix values that permits the generation of effective gauge interactions for composites, such as an $\mathrm{SU(3)}_{\mathrm{color}}$ interaction, which involve gauge groups not explicitly appearing in the formulation of the underlying field equations.

Because quaternionic chromodynamics is a nonlocal field theory, one suspects that the *TCP* theorem will be violated. To see explicitly that this does happen, let us calculate the action of *P*, *T*, and *C* conjugations on the equations given above. Since the space-spin structure of the theory is that of standard Yang-Mills quantum chromodynamics, calculations identical to those for the Yang-Mills case show that quaternionic chromodynamics is *P* and *T* invariant. To check *C* invariance, let us make the charge-conjugation substitutions (with superscript *T* indicating the Dirac-spinor-index transpose)

$$\chi_B=C\overline{\chi}^T_{cB},\quad \overline{\chi}_A=-\chi^T_{cA}C^{-1},\quad C\gamma^\nu C^{-1}=-\gamma^{\nu T}. \tag{16}$$

Working in an operator gauge in which the fermion fields obey canonical anticommutation relations,[8] the source current becomes

$$\begin{aligned} j^{a\nu}&=\overline{\chi}_A(-\tfrac{1}{2}\tau^*)^a_{AB}\gamma^\nu\overline{\chi}_B\\ &=\chi^T_{cA}C^{-1}(\tfrac{1}{2}\tau^*)^a_{AB}\gamma^\nu C\overline{\chi}^T_{cB}\\ &=\overline{\chi}_{cB}(\tfrac{1}{2}\tau^a)_{BA}\gamma^\nu\chi_{cA}=\overline{\chi}_c\tfrac{1}{2}\tau^a\gamma^\nu\chi_c\,, \end{aligned} \tag{17}$$

where I have dropped an infinite internal-symmetry singlet c-number piece[8]. Since

$$\begin{aligned} &\tfrac{1}{2}\tau^a=\epsilon_a(-\tfrac{1}{2}\tau^{*a})\quad (a \text{ not summed}),\\ &\epsilon_{0,1,3}=-1,\quad \epsilon_2=+1, \end{aligned} \tag{18}$$

the gluon-fermion interaction term is left invariant in form if we make the substitutions

$$\left.\begin{aligned} b^a_\nu&=\epsilon_a b^a_{c\nu}\\ f^a_{\mu\nu}&=\epsilon_a f^a_{c\mu\nu}\end{aligned}\right\}\ (a\text{ not summed}). \tag{19}$$

However, Eq. (19) is not an invariance of the field-strength–potential relation of Eq. (5), which is changed to read

$$\begin{aligned} &f^a_{c\mu\nu}=\frac{\partial b^a_{c\mu}}{\partial x^\nu}-\frac{\partial b^a_{c\nu}}{\partial x^\mu}-ig\,\overline{q}^{\,aef}(b^e_{c\mu}b^f_{c\nu}-b^e_{c\nu}b^f_{c\mu}),\\ &\overline{q}^{\,abc}=\overline{q}^{\,bca}=\overline{q}^{\,cab},\\ &\overline{q}^{\,0bc}=-\tfrac{1}{2}\delta^{bc},\\ &\overline{q}^{\,abc}=\tfrac{1}{2}i\epsilon^{abc},\quad 1\le a,b,c\le 3 \end{aligned} \tag{20}$$

and involves the structure constant $\overline{q}^{\,abc}$ appearing in the multiplication law for conjugated quanternions. Hence quaternionic chromodynamics is *C* violating (and in a sense, maximally *C* violating), and since it is *P* and *T* conserving, it is *CPT* violating as well. To see that this is consistent with the usual *CPT* theorem for local field theories, we note that the *C* violation and *CPT* violation come entirely from the term

$$q^{0bc}(b^b_\mu b^c_\nu-b^b_\nu b^c_\mu)=\tfrac{1}{2}[b^b_\mu,b^b_\nu] \tag{21}$$

in Eq. (5b), which would vanish in a local field theory, and which can be nonvanishing in quaternionic chromodynamics only because the microscopic causality postulate has been dropped.

To make a connection between the two-internal-component spinor of quaternionic chromodynamics and the *U*, *D* states of the Harari-Shupe scheme, I take

$$\begin{aligned}\chi_{c1}&=U,\\ \chi_{c2}&=D\,.\end{aligned} \tag{22}$$

Referring to Eq. (17), we see that the charge matrix acting on the spinor

$$\begin{pmatrix}U\\ D\end{pmatrix} \tag{23a}$$

is

$$Q^a=\tfrac{1}{2}\tau^a, \tag{23b}$$

while the corresponding charge matrix acting on the antispinor

$$\begin{bmatrix}\overline{U}\\ \overline{D}\end{bmatrix} \tag{24a}$$

is

$$\overline{Q}^a=-\tfrac{1}{2}\tau^{*a}. \tag{24b}$$

I will do all the calculations for the case of composites formed from three spinors, and then infer results for the antispinor case by using the fact that

$$\begin{aligned}&\overline{Q}^{\,0}=Q^0-1=\tau^2Q^0\tau^2-1\,,\\ &\overline{Q}^{\,a}=\tau^2Q^a\tau^2,\quad a=1,2,3\,.\end{aligned} \tag{25}$$

That is, in quaternionic chromodynamics, antispinor charges[9] are related to spinor charges by a rotation of the vector formed from the $a=1,2,3$ components by the angle π about the internal $a=2$ axis, followed by a shift in the $a=0$ component of the charge by -1; these operations are easily implemented in the color-charge algebra calculations of Sec. II. Since the theory is not C invariant, one of course cannot use C conjugation of the gluon field to relate the spinor and antispinor calculations.

Before turning to the detailed calculations which follow, let me conclude this section with a simple, heuristic argument which shows that quaternionic chromodynamics gives the U, D states the electric charge assignments needed for the Harari-Shupe scheme. Consider the lowest-order Feynman diagram for the interaction of two spinor lines (Fig. 2), which gives the matrix element

$$\mathfrak{M} \propto \bar{u}_{(2')}\gamma^{\nu}\tau^{a}_{(2)}u_{(2)}\frac{1}{q^2}\bar{u}_{(1')}\gamma_{\nu}\tau^{a}_{(1)}u_{(1)}\,. \tag{26}$$

The sum over the internal index a can be rewritten in the form

$$\tfrac{1}{2}\tau^{a}_{(2)}\tau^{a}_{(1)} = \tfrac{1}{2}\tau^{0+3}_{(2)}\tfrac{1}{2}\tau^{0+3}_{(1)} + \tfrac{1}{2}\tau^{0-3}_{(2)}\tfrac{1}{2}\tau^{0-3}_{(1)} + \tfrac{1}{2}\tau^{1+i2}_{(2)}\tfrac{1}{2}\tau^{1-i2}_{(1)} + \tfrac{1}{2}\tau^{1-i2}_{(2)}\tfrac{1}{2}\tau^{1+i2}_{(1)}\,, \tag{27}$$

which can be regarded as consisting of four different types of vector particle exchange. The $1\pm i2$ gluons are flavor changing, and in the more detailed analysis of Sec. II contribute to the weak force. The $0+3$ gluon is a photon, coupling to the U with charge 1 and to the D with charge 0, as postulated in the Harari-Shupe rules. The $0-3$ gluon is a second photon, coupling to the U with charge 0 and to the D with charge 1.[10] The existence of two photons with these coupling patterns will emerge again from the analysis of Sec. II, where the residual interactions acting on three-spinor components are calculated in a careful way so that coherence effects, such as color, can be seen. The existence of a second photon is of

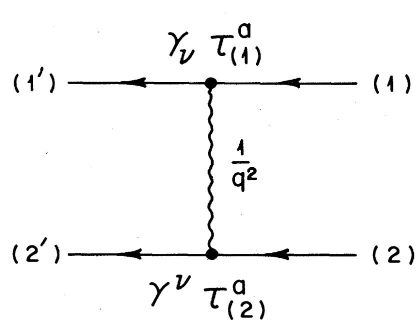

FIG. 2. Lowest-order Feynman diagram for the interaction of two spinor lines.

course an undesirable feature of the symmetric limit of the theory; a discussion of symmetry-breaking mechanisms is given in Sec. III.

II. STRUCTURE OF THE LEADING RESIDUAL INTERACTIONS ACTING ON THREE-SPINOR COMPOSITES

In this section I analyze the internal-symmetry structure of the leading residual interactions acting on three-spinor composites in quaternionic chromodynamics. In order to obtain a problem which can be studied using existing computational machinery,[5,11] I consider only the (unphysical) symmetric static limit of the theory, in which the gluons are massless while the two components U, D of the fundamental spinor have equal, infinite masses. In the static limit with three spinor sources present, the source current may be represented in the form[12]

$$j^{ai}=0,\qquad j^{a0}=\sum_{i=1}^{3}\tfrac{1}{2}\tau^{a}_{i}\delta^3(x-x_i), \tag{28}$$

where the τ^a_i are three independent sets of $U(2)$ matrices,

$$[\tau^a_i,\tau^b_j]=0,\quad i\neq j,\qquad \tfrac{1}{2}\tau^a_i\tfrac{1}{2}\tau^b_i = q^{abc}\tfrac{1}{2}\tau^c_i\quad (i\text{ not summed}). \tag{29}$$

The current of Eq. (28) is evidently an operator acting on the 8-dimensional Hilbert space spanned by the 8 states

$$\begin{aligned}
e^+ &= \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_3,\\
u_1 &= \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_3,\quad u_2 = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_3,\\
u_3 &= \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_3,\\
\bar{d}_1 &= \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_3,\quad \bar{d}_2 = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_3,\\
\bar{d}_3 &= \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_3,\\
\nu_e &= \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_3,
\end{aligned} \tag{30}$$

where I have labeled the states to correspond with Table II. A key feature of algebraic chromodynamics[5] is that the field equations with sources can be solved in terms of operators acting on the finite-dimensional Hilbert space of Eq. (30), by expanding the potential b^a_ν and the field strength $f^a_{\mu\nu}$ in the form

$$b_\nu^a = \sum_i b_\nu^i w_i^a ,$$
$$f_{\mu\nu}^a = \sum_i f_{\mu\nu}^i w_i^a . \tag{31}$$

Here the b_ν^i, $f_{\mu\nu}^i$ are c-number fields, while the w_i^a form a complete basis for the (3,0) color charge algebra, defined as the minimal algebra of quartet operators containing the three spinor source charges

$$Q_1^a = \tfrac{1}{2}\tau_1^a, \quad Q_2^a = \tfrac{1}{2}\tau_2^a, \quad Q_3^a = \tfrac{1}{2}\tau_3^a \tag{32}$$

and closed under composition with the outer product

$$P^a(u,v) = 2q^{abc}(u^b v^c - v^b u^c). \tag{33}$$

For future reference, it will be useful to have the following explicit form of P, obtained by substituting Eq. (3) into Eq. (33) and regrouping into 0 ± 3 and $1\pm i2$ components:

$$\begin{aligned}
P^{0+3}(u,v) &= [u^{0+3}, v^{0+3}] + u^{1-i2}v^{1+i2} - v^{1-i2}u^{1+i2},\\
P^{0-3}(u,v) &= [u^{0-3}, v^{0-3}] + u^{1+i2}v^{1-i2} - v^{1+i2}u^{1-i2},\\
P^{1+i2}(u,v) &= u^{0-3}v^{1+i2} - v^{1+i2}u^{0+3}\\
&\quad - v^{0-3}u^{1+i2} + u^{1+i2}v^{0+3},\\
P^{1-i2}(u,v) &= u^{0+3}v^{1-i2} - v^{1-i2}u^{0-3}\\
&\quad - v^{0+3}u^{1-i2} + u^{1-i2}v^{0-3},
\end{aligned} \tag{34}$$

with [,] the commutator of its arguments

$$[u,v] = uv - vu . \tag{35}$$

Since the computation which follows makes explicit use of the structure of the (3,0) color charge algebra, I will pause briefly at this point to review the isomorphism[11] between the general $(N,0)$ algebra and the group algebra of the permutation group S_{N+1}. Letting τ_0^a denote the "carrier" matrix which appears in Eq. (5), and using the fact that

$$\tau_0^a \tau_0^b = 2q^{abc}\tau_0^c , \tag{36}$$

we can rewrite the outer product of Eq. (33) in the matrix commutator form

$$\begin{aligned}
P &= [u,v],\\
P &= \tau_0^a P^a(u,v),\\
u &= \tau_0^a u^a,\\
v &= \tau_0^a v^a .
\end{aligned} \tag{37}$$

In this form, the source charge $Q_j^a = \tfrac{1}{2}\tau_j^a$ corresponds to the matrix

$$Q_j = \tau_0^a \tfrac{1}{2}\tau_j^a \tag{38a}$$

with matrix elements

$$\begin{aligned}
(Q_j)_{A_0\cdots A_N B_0 \cdots B_N} &= (\tau_0^a)_{A_0B_0}(1_1)_{A_1B_1}\cdots(1_{j-1})_{A_{j-1}B_{j-1}}(\tfrac{1}{2}\tau_j^a)_{A_jB_j}(1_{j+1})_{A_{j+1}B_{j+1}}\cdots(1_N)_{A_NB_N}\\
&= \delta_{A_1B_1}\cdots\delta_{A_{j-1}B_{j-1}}\delta_{A_0B_j}\delta_{A_jB_0}\delta_{A_{j+1}B_{j+1}}\cdots\delta_{A_NB_N} ,
\end{aligned} \tag{38b}$$

where I have used the completeness relation of Eq. (4). Clearly, the effect of Q_j is just that of the permutations $(0j)$, in cycle notation, on the numbers $0,1,\ldots, N$ labeling the spinor indices. This permits one to set up an isomorphism between elements of the $(N,0)$ color charge algebra and the permutations on the set of $N+1$ objects, with the algebra outer product P just corresponding to a commutator of permutations. When worked out in full, the correspondence (in the $n=2$ case[13]) reads as follows:

permutation in cycle notation	*color charge algebra element*
$(0m)$	$\rightarrow Q_m^a = \tfrac{1}{2}\tau_m^a$
$(0i_1i_2\cdots i_m)$	$\rightarrow 2^{m-1}(Q_{i_m}\times Q_{i_{m-1}}\times\cdots\times Q_{i_1})^a$
$(i_1i_2\cdots i_m)$	$\rightarrow 2^m\delta^{a0}(Q_{i_m}\times Q_{i_{m-1}}\times\cdots\times Q_{i_1})^0$,

(39)

with the indices $i_1\cdots i_m$ all distinct and with $\times$ denoting the outer product[14]

$$(u\times v)^a = q^{abc}u^b v^c , \tag{40}$$

which is associative by virtue of the identity

$$q^{abe}q^{cde} = q^{dae}q^{bce} . \tag{41}$$

As a result of this isomorphism, the problem of finding the diagonalizing bases for the $(N,0)$ color charge algebra is reduced to the problem of diagonalizing the group algebra of the permutation group S_{N+1}, which is explicitly soluble using classical group-theory methods.[15] The results of this calculation for the case of the (3,0) algebra are given[16] in Tables III, IV, and V. Note that although S_4 contains 24 elements, only 12 of these are needed to provide a basis for the (3,0) algebra.

According to Tables III–V and Eq. (31), the bound-state dynamics of three massive spinor sources is described by three independent overlying classical field theories: a classical U(1) (Abelian) gauge theory, a classical SU(2) Yang-Mills theory, and a classical SU(3) Yang-Mills theory, with c-number effective charges given by

TABLE III. The 12 elements of S_4, and the corresponding charge matrix tensor products, which form a basis for the (3, 0) color charge algebra.

$Q_1=(01)$	$\longleftrightarrow \frac{1}{2}\tau_1^a$
$Q_2=(02)$	$\longleftrightarrow \frac{1}{2}\tau_2^a$
$Q_3=(03)$	$\longleftrightarrow \frac{1}{2}\tau_3^a$
$r_1=(23)$	$\longleftrightarrow \frac{1}{2}\delta^{a0}\tau_2^b\tau_3^b$
$r_2=(31)$	$\longleftrightarrow \frac{1}{2}\delta^{a0}\tau_3^b\tau_1^b$
$r_3=(12)$	$\longleftrightarrow \frac{1}{2}\delta^{a0}\tau_1^b\tau_2^b$
$s_0=(321)-(123)$	$\longleftrightarrow \frac{1}{2}\delta^{a0}(q^{bcd}-q^{bdc})\tau_1^b\tau_2^c\tau_3^d$
$s_1=(023)-(032)$	$\longleftrightarrow \frac{1}{2}(q^{acb}-q^{abc})\tau_2^b\tau_3^c$
$s_2=(031)-(013)$	$\longleftrightarrow \frac{1}{2}(q^{acb}-q^{abc})\tau_3^b\tau_1^c$
$s_3=(012)-(021)$	$\longleftrightarrow \frac{1}{2}(q^{acb}-q^{abc})\tau_1^b\tau_2^c$
$t_1=(0312)+(0213)-(0123)-(0321)$	$\longleftrightarrow \frac{1}{2}(q^{abe}-q^{aeb})(q^{cde}-q^{dce})\tau_3^b\tau_1^c\tau_2^d$
$t_2=(0123)+(0321)-(0231)-(0132)$	$\longleftrightarrow \frac{1}{2}(q^{abe}-q^{aeb})(q^{cde}-q^{dce})\tau_1^b\tau_2^c\tau_3^d$

the projections of the charge matrices $Q_{1,2,3}$ along the diagonalizing bases. I have developed elsewhere[17] detailed ideas on how a confining potential may appear in such a classical system, as

TABLE IV. A basis which diagonalizes the (3, 0) algebra.

U(1):
$$x=Q_1+Q_2+Q_3+r_1+r_2+r_3$$

SU(2):
$$y_1=\frac{1}{8\sqrt{3}}(Q_1-Q_2+r_1-r_2+t_1)$$
$$y_2=-\frac{i}{8\sqrt{3}}(s_0+s_1+s_2+s_3)$$
$$y_3=\tfrac{1}{24}(Q_1+Q_2-2Q_3+r_1+r_2-2r_3+t_1+2t_2)$$
$$[y_a,y_b]=i\epsilon_{abc}y_c$$

SU(3):
$$z_1=-\tfrac{1}{4}(Q_3-r_3)$$
$$z_2=-\frac{i}{8}(s_0-s_1-s_2+s_3)$$
$$z_3=\tfrac{1}{8}(Q_1-Q_2+r_1-r_2-t_1)$$
$$z_4=\tfrac{1}{4}(Q_1-r_1)$$
$$z_5=-\frac{i}{8}(s_0+s_1-s_2-s_3)$$
$$z_6=-\tfrac{1}{4}(Q_2-r_2)$$
$$z_7=-\frac{i}{8}(s_0-s_1+s_2-s_3)$$
$$z_8=\frac{1}{8\sqrt{3}}(-Q_1-Q_2+2Q_3-r_1-r_2+2r_3+t_1+2t_2)$$
$$[z_a,a_b]=if_{(3)}^{abc}z_c$$

TABLE V. Initial charges $Q_{1,2,3}$ expressed in terms of the diagonalizing basis.

$$Q_1=\tfrac{1}{6}x+y_3+\sqrt{3}y_1+z_3+2z_4-\frac{1}{\sqrt{3}}z_8$$
$$Q_2=\tfrac{1}{6}x+y_3-\sqrt{3}y_1-z_3-2z_6-\frac{1}{\sqrt{3}}z_8$$
$$Q_3=\tfrac{1}{6}x-2y_3-2z_1+\frac{2}{\sqrt{3}}z_8$$
$$Q_1+Q_2+Q_3=\tfrac{1}{2}x-2(z_1-z_4+z_6)$$

a result of the action of monopolelike background-field solutions associated with the SU(2) subgroups of the overlying classical fields. I will not pursue these ideas further here, beyond remarking that the postulated role of SU(2) monopoles in confinement gives an automatic mechanism for the production of several quark-lepton generations, if the quarks and leptons are three-spinor composites. The reason is that according to Wilkinson and Bais[18] and Weinberg,[19] a classical SU(N) Yang-Mills theory contains at most $N-1$ fundamental SU(2) monopole solutions; hence, the U(1) × SU(2) × SU(3) overlying classical field theory describing the internal dynamics of three-spinor composites can contain 0 + 1 + 2 = 3 fundamental monopole background solutions, permitting the presence of three or more topologically inequivalent quark-lepton generations as different combinations of these backgrounds are excited.[20]

Without doing any of the detailed bound-state dynamics, what can one say about the form of the residual interactions acting between a pair of three-spinor composites? Because the direct product of three SU(2) doublet representations does not contain any singlet states, that is, because

$$\underline{2}\times\underline{2}\times\underline{2}=\underline{4}+\underline{2}+\underline{2}\not\supset\underline{1}\,, \tag{42}$$

it is plausible to postulate that the matrix-valued components of the gauge field will not in general be screened to zero at large distances from a three-spinor composite. Motivated by this remark, I will further postulate that the leading residual interaction between a three-spinor composite (1) and a three-spinor composite (2) has the phenomenological charge × propagator × charge form

$$A^{ab}x_{(1)}^a x_{(2)}^b+B_{ij}^{ab}y_{(1)i}^a y_{(2)j}^b+C_{ij}^{ab}z_{(1)i}^a z_{(2)j}^b\,, \tag{43}$$

with $A^{ab}, B_{ij}^{ab}, C_{ij}^{ab}$ functions of the momenta of the composites. In the absence of symmetry breaking, one expects the coefficient tensors to be diagonal in their indices,

TABLE VI. Action of the $a=0+3$ component of the elements of the $(3,0)$ color charge algebra on the basis states, with $\alpha=1/(4\sqrt{3})$, $\beta=\frac{1}{12}$, $\gamma=\frac{1}{4}$. Defining $\mathrm{Tr}=\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_3$, the matrices x^a, y_i^a, and z_i^a tabulated in Tables VI–IX have the following trace norms: $\mathrm{Tr}(x^a x^a)=108$, $\mathrm{Tr}(y_i^a y_j^a)=\frac{1}{4}\delta_{ij}$, $\mathrm{Tr}(z_i^a z_j^a)=\frac{3}{4}\delta_{ij}$, $\mathrm{Tr}(x^a y_i^a)=\mathrm{Tr}(x^a z_i^a)=\mathrm{Tr}(y_i^a z_j^a)=0$.

	e^+	u_1	u_2	u_3	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	ν_e
x^{0+3}	$6e^+$	$3u_1+u_2+u_3$	$3u_2+u_1+u_3$	$3u_3+u_1+u_2$	$2\bar{d}_1+\bar{d}_2+\bar{d}_3$	$2\bar{d}_2+\bar{d}_1+\bar{d}_3$	$2\bar{d}_3+\bar{d}_1+\bar{d}_2$	$3\nu_e$
y_1^{0+3}	0	0	0	0	$\alpha(\bar{d}_1-\bar{d}_3)$	$\alpha(\bar{d}_3-\bar{d}_2)$	$\alpha(\bar{d}_2-\bar{d}_1)$	0
y_2^{0+3}	0	0	0	0	$i\alpha(\bar{d}_2-\bar{d}_3)$	$i\alpha(\bar{d}_3-\bar{d}_1)$	$i\alpha(\bar{d}_1-\bar{d}_2)$	0
y_3^{0+3}	0	0	0	0	$\beta(\bar{d}_1+\bar{d}_3-2\bar{d}_2)$	$\beta(\bar{d}_2+\bar{d}_3-2\bar{d}_1)$	$\beta(\bar{d}_1+\bar{d}_2-2\bar{d}_3)$	0
z_1^{0+3}	0	$\gamma(u_2-u_1)$	$\gamma(u_1-u_2)$	γu_3	$\gamma\bar{d}_2$	$\gamma\bar{d}_1$	0	$\gamma\nu_e$
z_2^{0+3}	0	$-i\gamma u_3$	$i\gamma u_3$	$i\gamma(u_1-u_2)$	$i\gamma\bar{d}_2$	$-i\gamma\bar{d}_1$	0	0
z_3^{0+3}	0	$-\gamma u_3$	γu_3	$\gamma(u_2-u_1)$	$\gamma\bar{d}_1$	$-\gamma\bar{d}_2$	0	0
z_4^{0+3}	0	$-\gamma u_1$	$\gamma(u_2-u_3)$	$\gamma(u_3-u_2)$	0	$-\gamma\bar{d}_3$	$-\gamma\bar{d}_2$	$-\gamma\nu_e$
z_5^{0+3}	0	$i\gamma(u_2-u_3)$	$-i\gamma u_1$	$i\gamma u_1$	0	$i\gamma\bar{d}_3$	$-i\gamma\bar{d}_2$	0
z_6^{0+3}	0	$\gamma(u_3-u_1)$	γu_2	$\gamma(u_1-u_3)$	$\gamma\bar{d}_3$	0	$\gamma\bar{d}_1$	$\gamma\nu_e$
z_7^{0+3}	0	$i\gamma u_2$	$i\gamma(u_3-u_1)$	$-i\gamma u_2$	$-i\gamma\bar{d}_3$	0	$i\gamma\bar{d}_1$	0
z_8^{0+3}	0	$\alpha(2u_2-u_3)$	$\alpha(2u_1-u_3)$	$-\alpha(u_1+u_2)$	$-\alpha\bar{d}_1$	$-\alpha\bar{d}_2$	$2\alpha\bar{d}_3$	0

$$A^{ab}_{\text{symmetric limit}}=A\delta^{ab},$$
$$B^{ab}_{ij\text{ symmetric limit}}=B\delta_{ij}\delta^{ab},\tag{44}$$
$$C^{ab}_{ij\text{ symmetric limit}}=C\delta_{ij}\delta^{ab},$$

while when dynamical symmetry breaking is included, their structure will probably be more complicated. According to the above-stated assumptions, the internal-symmetry character of the residual interactions acting between composites can be inferred by computing the action of the diagonalizing bases x^a, y_i^a, z_i^a of Tables III and IV on the basis states of Eq. (30).

This calculation is entirely straightforward, and the results are summarized in Tables VI–IX. From the entries in these tables, one can also infer the action of the diagonalizing bases on the conjugated states $\bar{e}$, $\bar{u}_{1,2,3}$, $d_{1,2,3}$, $\bar{\nu}_e$ by using the relation between antispinor and spinor charge matrices given in Eq. (25). According to Eq. (25), the change from spinors to antispinors is accomplished by changing the sign of the $a=1,3$ components of all operators in the color algebra, while leaving their $a=0,2$ components invariant, followed by a shift in the 0 components of the source charge matrices $Q^0_{1,2,3}$ by -1. Since from

TABLE VII. Action of the $a=0-3$ component of the elements of the $(3,0)$ color charge algebra on the basis states, with $\alpha=1/(4\sqrt{3})$, $\beta=\frac{1}{12}$, $\gamma=\frac{1}{4}$.

	e^+	u_1	u_2	u_3	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	ν_e
x^{0-3}	$3e^+$	$2u_1+u_2+u_3$	$2u_2+u_1+u_3$	$2u_3+u_1+u_2$	$3\bar{d}_1+\bar{d}_2+\bar{d}_3$	$3\bar{d}_2+\bar{d}_1+\bar{d}_3$	$3\bar{d}_3+\bar{d}_1+\bar{d}_2$	$6\nu_e$
y_1^{0-3}	0	$\alpha(u_1-u_3)$	$\alpha(u_3-u_2)$	$\alpha(u_2-u_1)$	0	0	0	0
y_2^{0-3}	0	$i\alpha(u_2-u_3)$	$i\alpha(u_3-u_1)$	$i\alpha(u_1-u_2)$	0	0	0	0
y_3^{0-3}	0	$\beta(u_1+u_3-2u_2)$	$\beta(u_2+u_3-2u_1)$	$\beta(u_1+u_2-2u_3)$	0	0	0	0
z_1^{0-3}	γe^+	γu_2	γu_1	0	$\gamma(\bar{d}_2-\bar{d}_1)$	$\gamma(\bar{d}_1-\bar{d}_2)$	$\gamma\bar{d}_3$	0
z_2^{0-3}	0	$i\gamma u_2$	$-i\gamma u_1$	0	$-i\gamma\bar{d}_3$	$i\gamma\bar{d}_3$	$i\gamma(\bar{d}_1-\bar{d}_2)$	0
z_3^{0-3}	0	γu_1	$-\gamma u_2$	0	$-\gamma\bar{d}_3$	$\gamma\bar{d}_3$	$\gamma(\bar{d}_2-\bar{d}_1)$	0
z_4^{0-3}	$-\gamma e^+$	0	$-\gamma u_3$	$-\gamma u_2$	$-\gamma\bar{d}_1$	$\gamma(\bar{d}_2-\bar{d}_3)$	$\gamma(\bar{d}_3-\bar{d}_2)$	0
z_5^{0-3}	0	0	$i\gamma u_3$	$-i\gamma u_2$	$i\gamma(\bar{d}_2-\bar{d}_3)$	$-i\gamma\bar{d}_1$	$i\gamma\bar{d}_1$	0
z_6^{0-3}	γe^+	γu_3	0	γu_1	$\gamma(\bar{d}_3-\bar{d}_1)$	$\gamma\bar{d}_2$	$\gamma(\bar{d}_1-\bar{d}_3)$	0
z_7^{0-3}	0	$-i\gamma u_3$	0	$i\gamma u_1$	$i\gamma\bar{d}_2$	$i\gamma(\bar{d}_3-\bar{d}_1)$	$-i\gamma\bar{d}_2$	0
z_8^{0-3}	0	$-\alpha u_1$	$-\alpha u_2$	$2\alpha u_3$	$\alpha(2\bar{d}_2-\bar{d}_3)$	$\alpha(2\bar{d}_1-\bar{d}_3)$	$-\alpha(\bar{d}_1+\bar{d}_2)$	0

TABLE VIII. Action of the $a=1+i2$ component of the elements of the (3, 0) color charge algebra on the basis states, with $\alpha=1/(4\sqrt{3})$, $\beta=\frac{1}{12}$, $\gamma=\frac{1}{4}$.

	e^+	u_1	u_2	u_3	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	ν_e
x^{1+i2}	0	e^+	e^+	e^+	u_2+u_3	u_1+u_3	u_1+u_2	$\bar{d}_1+\bar{d}_2+\bar{d}_3$
y_1^{1+i2}	0	0	0	0	$\alpha(u_1-u_3)$	$\alpha(u_3-u_2)$	$\alpha(u_2-u_1)$	0
y_2^{1+i2}	0	0	0	0	$i\alpha(u_2-u_3)$	$i\alpha(u_3-u_1)$	$i\alpha(u_1-u_2)$	0
y_3^{1+i2}	0	0	0	0	$\beta(u_1+u_3-2u_2)$	$\beta(u_2+u_3-2u_1)$	$\beta(u_1+u_2-2u_3)$	0
z_1^{1+i2}	0	0	0	$-\gamma e^+$	$-\gamma u_2$	$-\gamma u_1$	0	$-\gamma\bar{d}_3$
z_2^{1+i2}	0	$i\gamma e^+$	$-i\gamma e^+$	0	$-i\gamma u_2$	$i\gamma u_1$	0	$i\gamma(\bar{d}_2-\bar{d}_1)$
z_3^{1+i2}	0	γe^+	$-\gamma e^+$	0	$-\gamma u_1$	γu_2	0	$\gamma(\bar{d}_1-\bar{d}_2)$
z_4^{1+i2}	0	γe^+	0	0	0	γu_3	γu_2	$\gamma\bar{d}_1$
z_5^{1+i2}	0	0	$i\gamma e^+$	$-i\gamma e^+$	0	$-i\gamma u_3$	$i\gamma u_2$	$i\gamma(\bar{d}_3-\bar{d}_2)$
z_6^{1+i2}	0	0	$-\gamma e^+$	0	$-\gamma u_3$	0	$-\gamma u_1$	$-\gamma\bar{d}_2$
z_7^{1+i2}	0	$-i\gamma e^+$	0	$i\gamma e^+$	$i\gamma u_3$	0	$-i\gamma u_1$	$i\gamma(\bar{d}_1-\bar{d}_3)$
z_8^{1+i2}	0	$-\alpha e^+$	$-\alpha e^+$	$2\alpha e^+$	αu_1	αu_2	$-2\alpha u_3$	$\alpha(2\bar{d}_3-\bar{d}_1-\bar{d}_2)$

Eq. (34) we see that the 0 components of the operators u, v enter the outer product $P(u,v)$ only through commutators, but never through anticommutators, we conclude that shifting the 0 components of the charge matrices by a c number cannot change the abstract structure of the color algebra. In particular, such a shift can have no effect on the non-Abelian diagonalizing bases y_i^a, z_i^a, and therefore must result in only a shift in the 0 component of the Abelian element x. Referring to Table V, we see that

$$Q^0_{1,2,3} \to Q^0_{1,2,3} - 1 \Rightarrow x^0 \to x^0 - 6\,, \tag{45}$$

and so the rule for conjugating Tables VI–IX is as follows:

$$\begin{aligned} &e^+ \to e^-\,,\\ &u_{1,2,3} \to \bar{u}_{1,2,3}\,,\\ &\bar{d}_{1,2,3} \to d_{1,2,3}\,,\\ &\nu_e \to \bar{\nu}_e\,,\\ &y_i^{0\pm 3} \to y_i^{0\mp 3}\,,\\ &z_i^{0\pm 3} \to z_i^{0\mp 3}\,,\\ &x^{0\pm 3} \to x^{0\mp 3} - 6\,,\\ &w^{1\pm i2} \to -w^{1\mp i2}\,,\quad w = x, y_i, z_i\,. \end{aligned} \tag{46}$$

TABLE IX. Action of the $a=1-i2$ component of the elements of the (3, 0) color charge algebra on the basis states, with $\alpha=1/(4\sqrt{3})$, $\beta=\frac{1}{12}$, $\gamma=\frac{1}{4}$.

	e^+	u_1	u_2	u_3	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	ν_e
x^{1-i2}	$u_1+u_2+u_3$	$\bar{d}_2+\bar{d}_3$	$\bar{d}_1+\bar{d}_3$	$\bar{d}_1+\bar{d}_2$	ν_e	ν_e	ν_e	0
y_1^{1-i2}	0	$\alpha(\bar{d}_1-\bar{d}_3)$	$\alpha(\bar{d}_3-\bar{d}_2)$	$\alpha(\bar{d}_2-\bar{d}_1)$	0	0	0	0
y_2^{1-i2}	0	$i\alpha(\bar{d}_2-\bar{d}_3)$	$i\alpha(\bar{d}_3-\bar{d}_1)$	$i\alpha(\bar{d}_1-\bar{d}_2)$	0	0	0	0
y_3^{1-i2}	0	$\beta(\bar{d}_1+\bar{d}_3-2\bar{d}_2)$	$\beta(\bar{d}_2+\bar{d}_3-2\bar{d}_1)$	$\beta(\bar{d}_1+\bar{d}_2-2\bar{d}_3)$	0	0	0	0
z_1^{1-i2}	$-\gamma u_3$	$-\gamma\bar{d}_2$	$-\gamma\bar{d}_1$	0	0	0	$-\gamma\nu_e$	0
z_2^{1-i2}	$i\gamma(u_2-u_1)$	$-i\gamma\bar{d}_2$	$i\gamma\bar{d}_1$	0	$i\gamma\nu_e$	$-i\gamma\nu_e$	0	0
z_3^{1-i2}	$\gamma(u_1-u_2)$	$-\gamma\bar{d}_1$	$\gamma\bar{d}_2$	0	$\gamma\nu_e$	$-\gamma\nu_e$	0	0
z_4^{1-i2}	γu_1	0	$\gamma\bar{d}_3$	$\gamma\bar{d}_2$	$\gamma\nu_e$	0	0	0
z_5^{1-i2}	$i\gamma(u_3-u_2)$	0	$-i\gamma\bar{d}_3$	$i\gamma\bar{d}_2$	0	$i\gamma\nu_e$	$-i\gamma\nu_e$	0
z_6^{1-i2}	$-\gamma u_2$	$-\gamma\bar{d}_3$	0	$-\gamma\bar{d}_1$	0	$-\gamma\nu_e$	0	0
z_7^{1-i2}	$i\gamma(u_1-u_3)$	$i\gamma\bar{d}_3$	0	$-i\gamma\bar{d}_1$	$-i\gamma\nu_e$	0	$i\gamma\nu_e$	0
z_8^{1-i2}	$\alpha(2u_3-u_1-u_2)$	$\alpha\bar{d}_1$	$\alpha\bar{d}_2$	$-2\alpha\bar{d}_3$	$-\alpha\nu_e$	$-\alpha\nu_e$	$2\alpha\nu_e$	0

TABLE X. Operator matrix elements requiring a shift under charge conjugation, as indicated in Eq. (46) of the text. The operator $[\frac{1}{2}x-2(z_1-z_4+z_6)]^{0+3}$ plays the role of the conventional electric charge, and has the correct charge-conjugation properties.

	e^+	u_1	u_2	u_3	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	ν_e
x^{0+3}	$6e^+$	$3u_1+u_2+u_3$	$3u_2+u_1+u_3$	$3u_3+u_1+u_2$	$2\bar{d}_1+\bar{d}_2+\bar{d}_3$	$2\bar{d}_2+\bar{d}_1+\bar{d}_3$	$2\bar{d}_3+\bar{d}_1+\bar{d}_2$	$3\nu_e$
$[\frac{1}{2}x-2(z_1-z_4+z_6)]^{0+3}$	$3e^+$	$2u_1$	$2u_2$	$2u_3$	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	0
x^{0-3}	$3e^+$	$2u_1+u_2+u_3$	$2u_2+u_1+u_3$	$2u_3+u_1+u_2$	$3\bar{d}_1+\bar{d}_2+\bar{d}_3$	$3\bar{d}_2+\bar{d}_1+\bar{d}_3$	$3\bar{d}_3+\bar{d}_1+\bar{d}_2$	$6\nu_e$
$[\frac{1}{2}x-2(z_1-z_4+z_6)]^{0-3}$	0	u_1	u_2	u_3	$2\bar{d}_1$	$2\bar{d}_2$	$2\bar{d}_3$	$3\nu_e$
	e^-	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	d_1	d_2	d_3	$\bar{\nu}_e$
x^{0+3}	$-3e^-$	$-4\bar{u}_1+\bar{u}_2+\bar{u}_3$	$-4\bar{u}_2+\bar{u}_1+\bar{u}_3$	$-4\bar{u}_3+\bar{u}_1+\bar{u}_2$	$-3d_1+d_2+d_3$	$-3d_2+d_1+d_3$	$-3d_3+d_1+d_2$	0
$[\frac{1}{2}x-2(z_1-z_4+z_6)]^{0+3}$	$-3e^-$	$-2\bar{u}_1$	$-2\bar{u}_2$	$-2\bar{u}_3$	$-d_1$	$-d_2$	$-d_3$	0
x^{0-3}	0	$-3\bar{u}_1+\bar{u}_2+\bar{u}_3$	$-3\bar{u}_2+\bar{u}_1+\bar{u}_3$	$-3\bar{u}_3+\bar{u}_1+\bar{u}_2$	$-4d_1+d_2+d_3$	$-4d_2+d_1+d_3$	$-4d_3+d_1+d_2$	$-3\bar{\nu}_e$
$[\frac{1}{2}x-2(z_1-z_4+z_6)]^{0-3}$	0	$-\bar{u}_1$	$-\bar{u}_2$	$-\bar{u}_3$	$-2d_1$	$-2d_2$	$-2d_3$	$-3\bar{\nu}_e$

In Table X I have tabulated, for the three-spinor and three-antispinor cases, the operators $x^{0\pm3}$, as well as the charge operators

$$(Q_1+Q_2+Q_3)^{0\pm3}=[\tfrac{1}{2}x-2(z_1-z_4+z_6)]^{0\pm3}, \tag{47}$$

to which the shift of -6 in Eq. (46) contributes.

From Tables VI–X, the conjugation rule of Eq. (46) and the assumed propagation formula of Eq. (43), we can read off the following qualitative features of the leading residual forces acting between three-spinor and three-antispinor composites.

(i) There are two color-singlet, flavor diagonal interactions, one coupling to

$$Q\equiv[\tfrac{1}{2}x-2(z_1-z_4+z_6)]^{0+3}, \tag{48a}$$

and the second coupling to

$$Q'\equiv[\tfrac{1}{2}x-2(z_1-z_4+z_6)]^{0-3}. \tag{48b}$$

The charge Q has the correct eigenvalues, when acting on three-spinor and three-antispinor composites, to be the usual electric charge operator, with the usual charge-conjugation properties. The charge Q' is a "reversed" electric charge, coupling to the neutrino but not to the electron. Since a single linear combination of non-Abelian bases, such as $z_1-z_4+z_6$, cannot self-interact, $Q-Q$ and $Q'-Q'$ interactions will be mediated by massless Abelian gauge field propagators. So, in the symmetric limit, quaternionic chromodynamics contains a photon, and in addition, a "reversed" photon coupling to Q'.

(ii) The remaining 0 ± 3 components give flavor-diagonal, color-changing interactions. Since Table VI, giving the $0+3$ components, is related to Table VII, giving the $0-3$ components, by the transformation

$$e^+\longleftrightarrow\nu_e\,,\qquad u_i\longleftrightarrow\bar{d}_i\,,\quad i=1,2,3\qquad w^{0+3}\longleftrightarrow w^{0-3}\,,\quad w=x,y_i,z_i \tag{49}$$

it suffices to examine in detail only the structure of Table VI. Grouping the 8 basis states of Eq. (30) into an 8-component column vector,

$$\Psi=\begin{bmatrix} e^+ \\ \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix} \\ \begin{pmatrix} \bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\end{pmatrix} \\ \nu_e \end{bmatrix}, \tag{50}$$

the entries in Table VI can be rewritten as a set of 8×8 matrices acting on Ψ. Using the notations

$$\begin{aligned} &[0]=\text{null } 3\times3 \text{ matrix},\\ &\underset{\sim}{0}=\text{null } 3\times1 \text{ row vector},\\ &(0)=\text{null } 1\times3 \text{ column vector},\end{aligned} \tag{51}$$

the matrix representations of the $0+3$ operator bases take the form

$$\begin{aligned} y_i^{0+3}&=\begin{bmatrix} 0 & \underset{\sim}{0} & \underset{\sim}{0} & 0\\ (0) & [0] & [0] & (0)\\ (0) & [0] & [Y_i] & (0)\\ 0 & \underset{\sim}{0} & \underset{\sim}{0} & 0\end{bmatrix},\quad i=1,2,3\\ z_i^{0+3}&=\begin{bmatrix} 0 & \underset{\sim}{0} & \underset{\sim}{0} & 0\\ (0) & [U_i] & [0] & (0)\\ (0) & [0] & [D_i] & (0)\\ 0 & \underset{\sim}{0} & \underset{\sim}{0} & \Delta_i\end{bmatrix},\quad i=1,\ldots,8.\end{aligned} \tag{52}$$

In these expressions, Δ_i is a number,

$$\Delta_1=-\Delta_4=\Delta_6=\tfrac{1}{4}\,,\quad \Delta_i=0\,,\quad i\neq1,4,6\,, \tag{53}$$

while Y_i, U_i, and D_i are 3×3 matrices, which when written in terms of the standard Gell-Mann λ-matrix basis[21] for SU(3) take the form

$$Y_1=\frac{1}{4\sqrt{3}}(\lambda_3-\lambda_4+\lambda_6)\,,\quad Y_2=\frac{1}{4\sqrt{3}}(\lambda_2-\lambda_5+\lambda_7)\,,\quad Y_3=\tfrac{1}{12}(-2\lambda_1+\lambda_4+\lambda_6+\sqrt{3}\lambda_8)\,,\tag{54a}$$

$$U_1=\tfrac{1}{4}\left(-\tfrac{1}{3}+\lambda_1-\frac{2}{\sqrt{3}}\lambda_8\right),\quad U_2=\tfrac{1}{4}(-\lambda_5+\lambda_7)\,,\quad U_3=\tfrac{1}{4}(-\lambda_4+\lambda_6)\,,$$

$$U_4=\tfrac{1}{4}\left(\tfrac{1}{3}-\lambda_3-\lambda_6-\frac{1}{\sqrt{3}}\lambda_8\right),\quad U_5=\tfrac{1}{4}(\lambda_2-\lambda_5)\,,\quad U_6=\tfrac{1}{4}\left(-\tfrac{1}{3}-\lambda_3+\lambda_4+\frac{1}{\sqrt{3}}\lambda_8\right),\tag{54b}$$

$$U_7=\tfrac{1}{4}(\lambda_2+\lambda_7)\,,\quad U_8=\frac{1}{4\sqrt{3}}(2\lambda_1-\lambda_4-\lambda_6)\,,$$

$$D_1=\tfrac{1}{4}\lambda_1\,,\quad D_2=\tfrac{1}{4}\lambda_2\,,\quad D_3=\tfrac{1}{4}\lambda_3\,,\quad D_4=-\tfrac{1}{4}\lambda_6\,,\quad D_5=\tfrac{1}{4}\lambda_7\,,\quad D_6=\tfrac{1}{4}\lambda_4\,,\quad D_7=-\tfrac{1}{4}\lambda_5\,,\quad D_8=-\tfrac{1}{4}\lambda_8\,.\tag{54c}$$

A simple calculation shows that

$$Y_lY_m=\tfrac{1}{4}iY_n\,,\quad l,m,n \text{ cyclic}\tag{55}$$

and so the y^{0+3} piece of Eq. (43) gives a color force acting on the three $\bar{d}$ states, coupling to an $SU(2)_{color}$ subgroup. Since the SU(3) structure constants $f^{abc}_{(3)}$ and $d^{abc}_{(3)}$ are, respectively, even and odd under the transformation

$$\begin{aligned}&i\to i\,,\quad i=1,2,3\\&4\to -6\,,\\&5\to 7\,,\\&6\to 4\,,\\&7\to -5,\\&8\to -8\,,\end{aligned}\tag{56}$$

the matrices D_i are unitarily equivalent to the matrices $-\frac{1}{4}\lambda_i^*$, and so the z^{0+3} piece of Eq. (43) gives a second color force acting on the three $\bar{d}$ states, coupling to the full set of $SU(3)_{color}$ charges. However, the matrices U_i of Eq. (54b) are *not* unitarily equivalent to λ or λ^* matrices [$\mathrm{tr}(U_1)\neq 0$, $\mathrm{tr}(U_2U_5)\neq 0$, etc.], and so the z^{0+3} piece of Eq. (43) gives a nonstandard color force acting on the u states, as well as a force acting between the ν_e and the u states. By the transformation of Eq. (49), the y^{0-3} piece of Eq. (43) gives a color force acting on the u states, coupling to an $SU(2)_{color}$ subgroup, while the z^{0-3} piece gives an $SU(3)_{color}$ force acting on u states, and in addition a nonstandard color force acting on $\bar{d}$ states, as well as an $e^+-\bar{d}$ interaction. To sum up, quaternionic chromodynamics generates $SU(3)_{color}$ forces for composites, without the appearance of SU(3) structure constants in the fundamental Lagrangian. However, there are also nonstandard color forces and nonstandard lepton-quark couplings, which computationally are associated with the appearance of the terms $u^{1-i2}v^{1+i2}$, etc., in the expression for $P^{0\pm 3}$ in Eq. (34). It is also easy to see, by consulting the tables and using Eq. (46), that the flavor-diagonal, color-changing interactions are not manifestly charge-conjugation-symmetric.

(iii) The $1\pm i2$ components produce a complicated pattern of flavor-changing, color-changing interactions. Since the $1\pm i2$ quanta have electric charges of magnitude $\frac{1}{3}$ in units of e, they cannot be exchanged singly between color-singlet states (which have integer charges in units of e). The exchange of three $1+i2$ quanta[22] can change a ν_e to an e^+, resembling (apart from the fact that a $\bar{\nu}_e$ should be involved) the effect of the exchange of a conventional weak boson, without necessarily implying the existence of a very massive, narrow state in this channel. In other words, the "intermediate bosons" in quaternionic chromodynamics may be broad, and possibly relatively low[23] mass, continuum states, associated with a complicated van der Waals-type of interaction. However, in the symmetric version of the theory computed in this section, even three-quantum exchanges cannot convert a u quark to a d quark, or a $\bar{\nu}_e$ to an e^+, since these transitions require the action of a charge conjugation.

III. DISCUSSION

The results of Sec. II suggest that the idea of taking quaternionic chromodynamics as the dynamics for quark-lepton constituents is on the right track. At the same time, there are clearly features of the U(2)-symmetric, massive spinor version of the theory which rule it out as a realistic model. I believe that a correct quark-lepton constituent theory may well be obtained by looking for a symmetry-breaking solution to the equations of Sec. I, and taking relativistic kinematics into account, with the following points in mind:

(i) First, one clearly wants to break the U(2) gauge symmetry down to a U(1) gauge symmetry in such a way that only the $a=0+3$ components of the gauge fields survive as massless excitations.

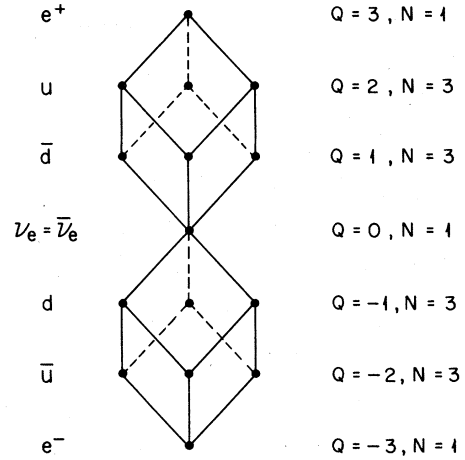

FIG. 3. Charge-multiplicity pattern in a generation including antiparticles, in a scheme with the D and the $\bar{D}$ identified. Each dot represents two helicity states.

This would give a single photon with the correct couplings and a single set of flavor-conserving color gluons.

(ii) Second, one wants the symmetry breaking to give the correct counting of neutrino states and to permit $d\rightarrow u$ or $\bar{\nu}_e \rightarrow e^+$ transitions to occur through three exchanges of the massive $1+i2$ component of the gluon field, while at the same time eliminating nonstandard and charge-conjugation asymmetric color couplings to the $0+3$ components. I believe that these objectives may be simultaneously accomplished by a symmetry-breaking scheme which treats the U and D components of the fundamental spinor asymmetrically, by keeping the U as a 4-component spinor, which acquires a mass, while identifying the D with the $\bar{D}$ state, so that the D becomes a 2 component, and very likely massless, Majorana spinor.[25] This gives precisely 15 three-spinor/three-antispinor states, as shown in Fig. 3 and Table XI. With $D-\bar{D}$ identification, exchanges of $1\pm i2$ quanta can now connect the states on the left- and right-hand sides of the center line in Table I, and in particular, three-quantum exchange processes can change a $\bar{\nu}_e$ to an e^+ and a d to a u, as required to give the weak interactions the correct structure. [Parity violation would have to appear as an appropriate selection rule governing the couplings of left- and right-handed helicity states, with the details of the relativistic bound-state dynamics very likely playing an important role here.[24] Similarly, the observed $\mathrm{SU}(2)\times \mathrm{U}(1)$ gauge symmetry of the weak interaction effective Lagrangian would have to be sought, in this scheme, as a dynamical consequence of the underlying U(2) operator gauge invariance.] Furthermore, with D and $\bar{D}$ identified, the anomalous couplings of the ν_e to the $z^{0+3}_{1,4,6}$ charges will be suppressed. Since these couplings appeared in association with the anomalous action of the z^{0+3}_i components on the u states, it is plausible that $D-\bar{D}$ identification will also eliminate the nonstandard and charge-conjugation asymmetric color couplings to the $0+3$ components. If all this works out as conjectured, it would amount to starting from a skewed (C-violating, CPT-violating) set of field equations, and looking for a skewed [U(2) gauge symmetry violating, $U-D$ asymmetric] solution, in such a way that the effective long-range components acting on composites are made fully C, P, and T invariant, with residual symmetry violations appearing only in the short-range, flavor-changing interactions.[26]

TABLE XI. Assignments for composite leptons and quarks, with D and $\bar{D}$ identified. The 15 states to the left of the solid line are obtained when one adopts the rule of Harari (Ref. 3) and Shupe (Ref. 4) that the states should be representable as either three-spinor or three-antispinor composites. If this rule is relaxed, mixings with the additional states on the right of the solid line are possible.

e^+	UUU		
$u_{1,2,3}$	$DUU,\ldots$		
$\bar{d}_{1,2,3}$	$UDD,\ldots$	$\bar{U}UU,\ldots$	(1 3-component state)
$\nu_e=\bar{\nu}_e$	$DDD=\bar{D}\bar{D}\bar{D}$	$\bar{U}UD,\ldots$	(6 1-component states)
$d_{1,2,3}$	$\bar{U}\bar{D}\bar{D},\ldots$	$U\bar{U}\bar{U},\ldots$	(1 3-component state)
$\bar{u}_{1,2,3}$	$\bar{D}\bar{U}\bar{U},\ldots$		
e^-	$\bar{U}\bar{U}\bar{U}$		

(iii) Since quaternionic chromodynamics can be formulated[5] using axial-vector couplings to the spinors as well as vector couplings, there are alternative ways of implementing the reduction from two four-component spinors to one four-component and one two-component spinor. For example, one could introduce parity violation directly into the kinematics, by building the model from chiral spinors as in the Weinberg-Salam model. Clearly, it would be very useful to have a general analysis of the various forms of the theory, obtained by coupling a U(2) connection through vector and axial-vector couplings to two-component spinors.

(iv) Since it is straightforward[27] to extend quaternionic chromodynamics so that the gauge connection is coupled to a complex scalar field ϕ_A with two internal-symmetry components, it should be possible to implement the various symmetry-breaking schemes described above by using the Higgs mechanism. It should be kept in mind, however, that symmetry breaking may not require the introduction of scalar fields, a point which I will discuss in more detail elsewhere.

(v) Finally, in order to thoroughly investigate the ideas sketched above, it will clearly be essential to understand the quantization rules for quaternionic chromodynamics, and in particular those aspects of the quantum field theory in which nonlocality plays a role. Among other things, such an analysis will indicate whether the form of the leading residual interaction postulated in Eq. (43) is correct, or whether additional terms, not included in the analysis of Sec. II, are present.

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[1]The three-generation picture extrapolates beyond the current status of experiment in assuming that the t exists, and that the ν_τ is a sequential neutrino carrying the τ-lepton quantum number.

[2]S. L. Glashow, Proceedings of the 1979 Cargèse Summer Institute (unpublished).

[3]H. Harari, Phys. Lett. 86B, 83 (1979). Harari associates the $D_1D_2D_3$ state with the ν_e, and I adopt this assignment in Table II.

[4]M. A. Shupe, Phys. Lett. 86B, 87 (1979). Shupe associates the $D_1D_2D_3$ state with the $\bar{\nu}_e$, rather than with the ν_e.

[5]S. L. Adler, Phys. Lett. 86B, 203 (1979); see also Phys. Rev. D 17, 3212 (1978).

[6]D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. 3, 207 (1962); 4, 788 (1963). I only recently learned of this reference from a report by U. Wolff [Max Planck Institute report (unpublished)]. As Wolff points out, the papers of Finkelstein *et al.* anticipate many later developments in gauge field theories, such as the Higgs mechanism and the Georgi-Glashow model, and contain the idea of a U(2) or quaternionic connection. The new ingredients added in Ref. 5 are the inclusion of spinor sources, the demonstration of general operator gauge covariance, and the use of the color charge algebras for studying composite systems.

[7]The definitions of J^ν and $j^{a\nu}$ given in Eqs. (10) and (11) differ from those of Ref. 5 by a minus sign.

[8]This assumption about the operator gauge is only needed for the reordering of χ_c and $\bar{\chi}_c$ in Eq. (17). Such reorderings, and the corresponding dropping of an infinite c-number piece, are really counter to the philosophy of operator gauge covariance, and should be avoided in a more careful treatment of charge conjugation, possibly by using a Dirac sea formulation of the theory. Such subtleties could make a difference in discussing symmetry breaking and mass renormalization.

[9]This transformation only works for the U(2) case because SU(2) is self-conjugate. The spinor and antispinor charges in U(n) algebraic chromodynamic (Ref. 5) with $n \geq 3$ cannot be related by an SU(n) rotation and a shift of the 0 component.

[10]The heuristic scheme of Shupe, Ref. 4, contains two photon states with these characteristics.

[11]P. Cvitanović, R. J. Gonsalves, and D. E. Neville, Phys. Rev. D 18, 3881 (1978); V. Rittenberg and D. Wyler, *ibid.* 18, 4806 (1978); S.-C. Lee, *ibid.* 20, 1951 (1979).

[12]Equation (28) represents the color charges in an operator gauge in which the fermion fields obey canonical commutation relations. I assume that such a gauge choice can be made at one time along the system world line.

[13]For general n, the factor $2^m\delta^{a0}$ in the third line of Eq. (39) is replaced by $2^{m-1}n\,\delta^{a0}$.

[14]In terms of the outer product $\times$, the outer product P is given by $P^a(u,v) = 2(u\times v - v\times u)^a$.

[15]S.-C. Lee, Ref. 11.

[16]The results are those given by P. Cvitanović, R. J. Gonsalves, and D. E. Neville, Ref. 11, with some corrections (missing factors of -1 or $-i$) included in Table IV. Table V was furnished by R. J. Gonsalves (unpublished).

[17]S. L. Adler, Phys. Rev. D 18, 411 (1978); 19, 1168 (1979); 20, 1386 (1979).

[18]D. Wilkinson and F. A. Bais, Phys. Rev. D 19, 2410 (1979).

[19]E. Weinberg, CERN Report No. CERN-TH-2779 (unpublished).

[20]According to the analysis of Ref. 17, the two-particle confinement problem involves a single monopole background. The three-particle confinement problem may involve more than one monopole background.

[21]See, for example, S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), pp. 261 and 262.

[22]In considering multiquantum exchange processes, one gets only part of the answer by iterating the single quantum effective interaction of Eq. (43). There will be additional contributions involving the full 6-charge overlying algebra, which are not obtainable from a computation of the 3-charge algebra.

[23]The fact that the dynamics is nonlocal may invalidate phenomenological estimates of mean intermediate boson masses based on dispersion relations, such as those given by J. D. Bjorken, Phys. Rev. D 19, 335 (1979).

[24]What is needed is for the D to couple as a Majorana spinor in flavor-conserving processes and as a Weyl spinor in flavor-changing processes. Since a massless two-component spinor can be represented in either Majorana or Weyl form, with the representations related by a unitary transformation, I believe that this behavior is plausible. I further suspect that the chiral (self-dual or anti-self-dual) nature of the classical background field needed for the confinement mechanism of Ref. 17 will also play a role in the helicity selection rule leading to parity violation.

[25]A possible mechanism leading to $D-\bar{D}$ identification is as follows: When the U becomes massive, chiral symmetry is broken and $\langle\bar{U}U\rangle \neq 0$. As a result, one may be able to make an invertible operator gauge transformation (in which $\langle\bar{U}U\rangle^{-1/2}$ appears) to remove two Majorana degrees of freedom from the four-component massless D state, leaving a two-component Majorana D. To pursue this issue further, a general analysis is needed of the extent to which operator gauge invariance allows degrees of freedom to be traded back and forth between the spinors and the gauge connection.

[26]If $D-\bar{D}$ identification does not eliminate all of the anomalous couplings, there are still the possibilities, which I have ignored in the discussion up to this point, that their effects may cancel when only overall color-singlet states are present, or that the assumed form of Eq. (43) is only partially correct, and that additional terms, which cancel the anomalous couplings, are present in the correct expression for the leading residual interaction.

[27]S. L. Adler, Phys. Rev. D 21, 550 (1980).