Effective Lagrangian for an SU(N) gauge theory with scalar fields

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We compute the one-loop contribution to the effective Lagrangian of an SU(N) gauge theory coupled to scalar fields in the adjoint representation. We study the limit of small and large (constant) $F_{\mu\nu}{}^2$ and discuss the minima of the resulting effective Lagrangian.

I. INTRODUCTION

There have been recent suggestions on the possibility that the minima of the classical action may have little to do with the quantum ground state of different field theories. In particular, Witten[1] has discussed, in the context of a class of nonlinear [CP(N)-invariant] σ models, an effective Lagrangian, obtained by means of the $1/N$ expansion, which depicts some correct features of the quantum theory, but which does not exhibit instanton solutions (that are present at the classical level).

Concerning gauge theories, Pagels and Tomboulis[2] showed the nonexistence of finite-action solutions to the field equations of an effective quantum action that they proposed for Yang-Mills theory. Although it has no instantons, their model presents extremely interesting vacuums and the possibility of confinement.

It is the purpose of this paper to compute the effective action of an SU(N) gauge theory which includes Higgs scalars, in order to study the minima of the resulting effective Lagrangian. We are able to do the calculations expanding the action around covariantly constant gauge fields[3-6] and constant scalar fields. We find that the one-loop effective action changes sufficiently as to lose, in some cases, the possibility of topological solutions, which were present at the classical level. We also get interesting changes in the structure of the vacuum.

The plan of the paper is as follows: In Sec. II we set up the general formalism, establish our conventions, and compute the one-loop correction to the classical Lagrangian of an SU(N) gauge theory coupled to scalar fields (in the adjoint representation). We then discuss in Sec. III the limit of small $F_{\mu\nu}{}^2$ which is related to the computation of the effective potential for the Higgs field. We study the minima of this potential and discuss the Prasad-Sommerfield limit.[7]

We also study the limit of large $F_{\mu\nu}{}^2$ which is related to the already computed effective action of a pure Yang-Mills theory.[3-6,8] In our calculations we are able to include the contribution of scalars in the adjoint representation. We then discuss the changes in the minima of the Lagrangian and how this affects the structure of the vacuum. Finally, in Sec. IV we give a brief discussion of the results obtained in the previous sections.

II. THE EFFECTIVE LAGRANGIAN

Let us consider the Lagrangian density for an SU(N) gauge theory which includes scalar fields in d space-time dimensions:

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}(D_\mu\phi D^\mu\phi) - \frac{\lambda}{8}(\phi^a\phi^a - a^2)^2 - \frac{1}{\alpha}F[A] + \mathcal{L}_{\text{ghost}}, \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$
$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi], \tag{2.2}$$

$$A_\mu = A_\mu^a T_a,$$
$$\phi = \phi^a T_a, \tag{2.3}$$

and T_a form some matrix representation of the generators of the SU(N) group, satisfying

$$[T_a, T_b] = iC_{abc}T_c, \quad \mathrm{Tr}(T_aT_b) = \tfrac{1}{2}\delta_{ab}. \tag{2.4}$$

We have chosen the potential $V(\phi)$,

$$V(\phi) = \tfrac{1}{8}\lambda(\phi^a\phi^a - a^2)^2, \quad \lambda > 0 \tag{2.5}$$

in such a way that if we set $a^2 > 0$, the symmetry is spontaneously broken.

We will work in the background gauge[10]

$$F[A] = D_\mu^B A^\mu = 0, \tag{2.6}$$

where

$$D_\mu^B = \partial_\mu I - ig[A_\mu^B, \ ] \tag{2.7}$$

and A_μ^B is a configuration which will be chosen later. In this gauge the ghost term takes the form

$$\mathcal{L}_{\text{ghost}} = -2\,\mathrm{Tr}(\bar{c}D_\mu^B D^\mu c), \tag{2.8}$$

where

$$c = c^a T_a\,, \quad \bar{c} = \bar{c}^a T_a$$

are the ghost fields.

As usual, the effective action is defined as[11]

$$\Gamma[A;\phi] = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots\,, \tag{2.9}$$

where

$$\Gamma_0 = S_{cl} = \int dx\, \mathcal{L}\,,$$

and Γ_n is the nth-order correction in the loop expansion of the effective action.

We are interested in the one-loop correction, Γ_1, given by the functional integral

$$\hbar\Gamma_1 = -i\hbar \ln \int \mathfrak{D}\varphi\, \mathfrak{D}a\, \mathfrak{D}\bar{c}\, \mathfrak{D}c \cdot \times \exp\left\{\frac{i}{\hbar} S^{\rm II}[A_{cl}, \phi_{cl}; a, \varphi, \bar{c}, c]\right\}, \tag{2.10}$$

where $S^{\rm II}$ is the quadratic term in the expansion of S_{cl} around fixed classical configurations A_{cl}, ϕ_{cl}. This expression is obtained writing

$$A^\mu = A^\mu_{cl} + a^\mu\,, \qquad \phi = \phi_{cl} + \varphi\,, \tag{2.11}$$

and expanding the action in powers of the quantum fluctuations a, φ:

$$S[\phi_{cl}+\varphi, A_{cl}+a, \bar{c}, c] - S[\phi_{cl}, A_{cl}, 0, 0] - \int dx\, \varphi(x)\frac{\delta S}{\delta\phi_{cl}(x)}[\phi_{cl}, A_{cl}, 0, 0] - \int dx\, a^\mu(x)\frac{\delta S}{\delta A^\mu_{cl}(x)}[\phi_{cl}, A_{cl}, 0, 0]$$
$$= S^{\rm II}[A_{cl}, \phi_{cl}; a, \varphi, \bar{c}, c] + S^{\rm int}[A_{cl}, \phi_{cl}; a, \varphi, \bar{c}, c]\,, \tag{2.12}$$

where $S^{\rm int}$ contains the interaction (i.e., nonquadratic) terms in the shifted fields.

For simplicity, we will choose the background field A^B_μ equal to A^{cl}_μ. Then, if the fluctuations satisfy $D^{cl}_\mu a^\mu = 0$, the classical field obeys the condition $\partial_\mu A^\mu_{cl} = 0$. Now, inserting the explicit form of $S^{\rm II}$ in expression (2.10) and performing the path integration, we obtain

$$\hbar\Gamma_1[\phi_{cl}, A_{cl}] = -i\hbar \ln {\rm Det}[-iD^{cl}_\mu D^\mu_{cl}] + \tfrac{1}{2}i\hbar \ln {\rm Det}[i(D^{cl}_\mu D^\mu_{cl} + \tfrac{1}{2}\lambda(\phi^a_{cl}\phi^a_{cl} - a^2) + \lambda\phi_{cl}\otimes\phi_{cl})]$$
$$+\frac{i\hbar}{2}\ln {\rm Det}(-i\{D^{cl}_\alpha D^\alpha_{cl}\eta_{\mu\nu} + (1/\alpha - 1)D^{cl}_\mu D^{cl}_\nu + 2igF^{cl}_{\mu\nu} + g^2\phi_{cl}^2\eta_{\mu\nu}$$
$$+g^2[2(D^{cl}_\mu\phi_{cl}) - D^{cl}_\mu\phi_{cl}][D^{cl}_\alpha D^\alpha_{cl} + \tfrac{1}{2}\lambda(\phi^a_{cl}\phi^a_{cl} - a^2) + \lambda\phi_{cl}\otimes\phi_{cl}]^{-1}[2(D^{cl}_\nu\phi_{cl}) + \phi_{cl}D^{cl}_\nu]\})\,, \tag{2.13}$$

where

$$(\phi\otimes\phi)^{ab} = \phi^a\phi^b \tag{2.14}$$

is an $(N^2-1)\times(N^2-1)$ matrix, $\eta_{\mu\nu}$ is the metric tensor, and all the fields are in the adjoint representation. We henceforth suppress the "cl" designation.

Expression (2.13) is rather complicated. We will simplify the computations by considering covariantly constant fields[3-6] defined by the equations

$$D_\rho F_{\mu\nu} = 0\,, \tag{2.15}$$

$$D_\mu\phi = 0\,. \tag{2.16}$$

The general gauge-invariant solution of Eq. (2.15) is[3]

$$A_\mu = -\tfrac{1}{2}F_{\mu\nu}x^\nu + ig^{-1}U^{-1}\partial_\mu U\,. \tag{2.17}$$

If one assumes $U \to I$ as $|x| \to \infty$, then the condition $\partial_\mu A^\mu = 0$ implies $U = I$. In this case $F_{\mu\nu}$ is constant and it can be written as

$$F_{\mu\nu} = f_{\mu\nu}n\,, \tag{2.18}$$

where n is a constant element of the adjoint representation of the Lie algebra satisfying

$$n^a n^a = 1\,, \tag{2.19}$$

and $f_{\mu\nu}$ is a constant antisymmetric tensor. For simplicity we will take a pure "magnetic" field in the 3rd direction:

$$f_{\mu\nu} = H\epsilon_{0\mu\nu3}\,,$$

and we will discuss other possibilities later.

Furthermore, we will choose ϕ to be parallel to n,

$$\phi = \varphi n\,. \tag{2.20}$$

Then condition (2.16) reduces to

$$\partial_\mu\varphi = 0\,. \tag{2.21}$$

Conditions (2.18) and (2.20) will allow us to obtain the first term of the expansion in powers of external momenta of the effective action. For example, for the Higgs scalar we will obtain the effective potential but not the correction $Z(\phi)$ to the kinetic term. In order to obtain $Z(\phi)$ one must use a less restrictive condition than expression (2.21).

Using conditions (2.15)–(2.21) one can now compute the different contributions to the one-loop term of the effective action, Γ, Eq. (2.13).

As an example we will describe in detail the computation of the second term in the right-hand side of Eq. (2.13):

$$\ln \operatorname{Det}(i\mathfrak{D}^{-1}) = \ln \operatorname{Det}[i(D_\mu D^\mu + \tfrac{1}{2}\lambda(\varphi^2 - a^2) + \lambda\varphi^2 n\otimes n)]$$

$$= -\operatorname{Tr}\int_0^\infty \frac{ds}{s}\exp\{-is[D_\mu D^\mu + \tfrac{1}{2}\lambda(\varphi^2 - a^2) + \lambda\varphi^2 n\otimes n]\} + K, \tag{2.22}$$

where K is an irrelevant constant and Tr means a sum over group (b) and space-time (x) indices. Now since

$$[n, n\otimes n] = 0 \tag{2.23}$$

we will take both n and $n\otimes n$ in their diagonal form. Then, expression (2.22) reduces to

$$\ln \operatorname{Det}(i\mathfrak{D}^{-1}) = -\sum_{b=1}^{N^2-1}\int_0^\infty \frac{ds}{s}\exp\{-is\lambda[\tfrac{1}{2}(\varphi^2 - a^2)I + \varphi^2 n\otimes n]_{bb}\}\int d^dx\,\langle x|U_{bb}(s)|x\rangle\,, \tag{2.24}$$

where $U(s)$ is the proper-time evolution operator of a particle in the presence of a constant magnetic field H[3-12]:

$$U(s) = \exp[is(P^\mu + gnA^\mu)(P_\mu + gnA_\mu)]\,. \tag{2.25}$$

It is now easy to compute $\langle x|U(s)|x\rangle$, using the well-known result for the matrix elements of the evolution operator of the harmonic oscillator and performing the appropriate analytic continuation in the metric:

$$\int d^dx\,\langle x|U_{aa}(s)|x\rangle = \int d^dx\,\langle x|\exp[is(P_\mu + gn_aA_\mu)(P^\mu + gn_aA^\mu)]|x\rangle$$

$$= \frac{i^{d/2}s^{1-d/2}}{(4\pi)^{d/2}(\det\eta)^{1/2}}\int d^dx\,\frac{\eta^{11}|gn_af_{12}|}{\sin(s\eta^{11}|gn_af_{12}|)}\,. \tag{2.26}$$

Now we can perform the remaining integration over s getting

$$\ln \operatorname{Det} i\mathfrak{D}^{-1} = \frac{-i}{(2\pi)^{d/2}}\Gamma(1-d/2)\int d^dx\,\operatorname{Tr}\{(g^2H^2n^2)^{d/4}\zeta(1-d/2;\tfrac{1}{2}+\tfrac{1}{2}\lambda[\tfrac{1}{2}(\varphi^2-a^2)+\varphi^2 n\otimes n][g^2H^2n^2]^{-1/2})\}\,, \tag{2.27}$$

where ζ is the Riemann zeta function defined as a matrix through the formula[13]

$$\zeta(z,B) = \sum_{k=0}^{\infty}(kI+B)^{-z}\,,\quad \operatorname{Re} z > 1 \tag{2.28}$$

for a nonsingular matrix B.

In a completely analogous way we can compute the contribution of the first term to Γ_1 in expression (2.13). The result is

$$\ln \operatorname{Det}[iD_\mu D^\mu] = \frac{-i}{(2\pi)^{d/2}}\Gamma(1-d/2)\int d^dx\,\zeta(1-d/2;\tfrac{1}{2})\operatorname{Tr}\{[g^2H^2n^2]^{d/4}\}\,. \tag{2.29}$$

Finally, using conditions (2.15)–(2.21), the third term in expression (2.13) reduces to

$$\ln \operatorname{Det}[-i(D_\alpha D^\alpha\eta_{\mu\nu} + (1/\alpha - 1)D_\mu D_\nu + 2igF_{\mu\nu} + g^2\phi^2\eta_{\mu\nu}$$
$$+ g^2[2(D_\mu\phi) - D_\mu\phi][D_\beta D^\beta + \tfrac{1}{2}\lambda(\phi^a\phi^a - a^2) + \lambda\phi\otimes\phi]^{-1}[2(D_\nu\phi) + \phi D_\nu])]$$
$$= \ln \operatorname{Det}[-i(D_\alpha D^\alpha\eta_{\mu\nu} + (1/\alpha - 1)D_\mu D_\nu + 2igH\epsilon_{0\mu\nu3}n$$
$$+ g^2\varphi^2\eta_{\mu\nu}n^2 + g^2\varphi^2n^2D_\mu[D_\beta D^\beta + \tfrac{1}{2}\lambda(\varphi^2 - a^2) + \lambda\varphi^2 n\otimes n]^{-1}D_\nu)]\,. \tag{2.30}$$

We can now compute expression (2.30) following the procedure developed in Ref. 3. The final expression for Γ_1 is

$$\Gamma_1=\tfrac{1}{2}[\Gamma(1-d/2)/(2\pi)^{d/2}]$$
$$\times\int d^dx\,\mathrm{Tr}\{(g^2H^2n^2)^{d/4}[-2\zeta(1-d/2;\tfrac{1}{2})+\zeta(1-d/2;\tfrac{1}{2}+\tfrac{1}{2}\lambda[\tfrac{1}{2}(\varphi^2-a^2)+\varphi^2n\otimes n](g^2H^2n^2)^{-1/2})$$
$$+(d-1)\zeta(1-d/2;\tfrac{1}{2}+\tfrac{1}{2}g^2\varphi^2n^2(g^2H^2n^2)^{-1/2})$$
$$-4(2i)^{-d/2}[1-g^4\varphi^4n^4(g^2H^2n^2)^{-1}]^{(d-2)/4}$$
$$\times\sin\{(d/2-1)\arctan[i(g^2H^2n^2)^{1/2}(g^2\varphi^2n^2)^{-1}(1+i\epsilon)]\}]\}$$
$$+\tfrac{1}{2}i\ln\mathrm{Det}[i(D_\mu D^\mu+\alpha g^2\varphi^2n^2\{1-D_\nu D^\nu[D_\alpha D^\alpha+\tfrac{1}{2}\lambda(\varphi^2-a^2)+\lambda\varphi^2n\otimes n]^{-1}\})]+\text{irrelevant constant terms}. \quad (2.31)$$

The last term in Eq. (2.31) is the only one which depends on the gauge parameter α. This reflects the well-known dependence of the effective action on the gauge choice.[11] Setting $\alpha=0$ (the background gauge), this last term simplifies considerably; defining the effective potential as

$$\Gamma=\int d^dx\,\mathcal{L}_{\mathrm{eff}}=\int d^dx(\mathcal{L}+\hbar\mathcal{L}_1+\cdots),$$

we finally have

$$\mathcal{L}_1(\varphi,H)=\frac{1}{2}\,\frac{\Gamma(1-d/2)}{(2\pi)^{d/2}}\,\mathrm{Tr}\{(g^2H^2n^2)^{d/4}[\zeta(1-d/2;\tfrac{1}{2}+\tfrac{1}{2}\lambda[\tfrac{1}{2}(\varphi^2-a^2)+\varphi^2n\otimes n](g^2H^2n^2)^{-1/2})$$
$$+(d-1)\zeta(1-d/2;\tfrac{1}{2}+\tfrac{1}{2}g^2\varphi^2n^2(g^2H^2n^2)^{-1/2})$$
$$-\zeta(1-d/2;\tfrac{1}{2})-4(2i)^{-d/2}[1-g^4\varphi^4n^4(g^2H^2n^2)^{-1}]^{(d-2)/4}$$
$$\times\sin\{(\tfrac{1}{2}d-1)\arctan[i(g^2H^2n^2)^{1/2}(g^2\varphi^2n^2)^{-1}(1+i\epsilon)]\}]\}. \quad (2.32)$$

Inspection of Eq. (2.32) shows that the only divergences in $\mathcal{L}_1$ come from the Γ function for even d. It is easy to compute the divergent part of the effective Lagrangian by performing an ϵ expansion ($\epsilon=4-d$). It takes the form

$$\mathcal{L}_1^{\mathrm{div}}=\frac{1}{\epsilon}\,\frac{C_2(G)}{16\pi^2}\left(\frac{22-1}{6}\right)g^2H^2+\frac{1}{\epsilon}\,\frac{1}{32\pi^2}\,\mathrm{Tr}\{\lambda^2[\tfrac{1}{2}(\varphi^2-a^2)+\varphi^2n\otimes n]^2+3g^4\varphi^4n^4\}. \quad (2.33)$$

The first term in (2.33) allows us to identify the divergent part of the renormalization constant Z_A of the field-strength renormalization. Then we can compute the coupling-constant renormalization

$$Z_g-1=\frac{-1}{\epsilon}\,\frac{C_2(G)}{16\pi^2}(\tfrac{11}{3}-\tfrac{1}{6})g^2. \quad (2.34)$$

From this expression one can easily obtain the β function[14] from the residue of the simple pole of Z_g:

$$\beta(g)=-\frac{C_2(G)}{16\pi^2}(\tfrac{11}{3}-\tfrac{1}{6})g^3+O(g^5)$$
$$=-\tfrac{1}{2}b_0g^3+O(g^5). \quad (2.35)$$

We have explicitly separated the $\frac{1}{6}$ factor which gives the contribution of the scalars (in the adjoint representation) to the β function.

III. LIMIT OF SMALL AND LARGE H

We have obtained in the preceding section the one-loop effective Lagrangian for an SU(N) gauge theory with scalar fields in the adjoint representation in d space-time dimensions. We have computed its divergent part (in $d=4$), Eq. (2.33), obtaining a very simple expression which shows that the only needed subtractions in order to render $\mathcal{L}_{\mathrm{eff}}$ finite correspond to coupling-constant, mass, and field-strength renormalizations.

We will proceed in this section to perform the renormalization of the effective Lagrangian in the two limits we are interested in. We will also discuss the resulting expressions which can be considered as defining a classical model which incorporates contributions of the full quantum problem.

(i) Limit of small H: the effective potential. In the limit $H\to 0$ only the first two terms in Eq. (2.32) contribute to $\mathcal{L}_1$. Actually, in this limit one is computing the one-loop correction to the effective potential, V_1,

$$V_1=-\lim_{H\to 0}\mathcal{L}_1. \quad (3.1)$$

Using the asymptotic form of the Riemann zeta function

$$\zeta(z,q)\sim\frac{q^{1-z}}{z-1},\quad \text{for } |q|\gg 1 \quad (3.2)$$

we obtain

$$V_1(\varphi)=\frac{\Gamma(1-d/2)}{d(4\pi)^{d/2}}\{(\tfrac{1}{2}\lambda)^{d/2}[(N^2-2)(\varphi^2-a^2)^{d/2}+(3\varphi^2-a^2)^{d/2}]+(d-1)(g^2\varphi^2)^{d/2}\mathrm{Tr}[(n^2)^{d/2}]\}. \tag{3.3}$$

As we stated above, for $d=4$ we have to renormalize expression (3.3). We will use as normalization conditions

$$\left.\frac{\partial^2 V_{\rm eff}}{\partial\varphi^2}\right|_{\varphi=0}=-\lambda a^2/2\,,\quad \left.\frac{\partial^4 V_{\rm eff}}{\partial\varphi^4}\right|_{\varphi=\varphi_0}=3\lambda\,, \tag{3.4}$$

where φ_0 is an arbitrary subtraction point. With these conditions, the renormalized effective potential results:

$$\begin{aligned}V_{\rm eff}(\varphi)=&\tfrac{1}{8}\lambda(\varphi^2-a^2)^2+\frac{3\,\mathrm{Tr}(n^4)}{2^6\pi^2}g^4\varphi^4[\ln(\varphi^2/\varphi_0^2)-\tfrac{25}{6}]\\&+\frac{\lambda}{2^8\pi^2}\Bigg\{\varphi^4\Bigg[(N^2-2)\left(\ln\left(\frac{\varphi^2-a^2}{\varphi_0^2-a^2}\right)-\frac{3}{2}-\frac{4\varphi_0^2}{\varphi_0^2-a^2}+\frac{4}{3}\frac{\varphi_0^4}{(\varphi_0^2-a^2)^2}\right)\\&\qquad+9\left(\ln\left(\frac{3\varphi^2-a^2}{3\varphi_0^2-a^2}\right)-\frac{3}{2}-\frac{12\varphi_0^2}{3\varphi_0^2-a^2}+\frac{12\varphi_0^4}{(3\varphi_0^2-a^2)^2}\right)\Bigg]\\&\qquad-2a^2\varphi^2\left[(N^2-2)\left(\ln\left(\frac{\varphi^2-a^2}{a^2}\right)-\tfrac{1}{2}\right)+3\left(\ln\left(\frac{3\varphi^2-a^2}{a^2}\right)-\tfrac{1}{2}\right)\right]\\&\qquad+a^4\left[(N^2-2)\ln\left(\frac{\varphi^2-a^2}{\varphi_0^2-a^2}\right)+\ln\left(\frac{3\varphi^2-a^2}{3\varphi_0^2-a^2}\right)\right]\Bigg\}+\text{constant term}.\end{aligned} \tag{3.5}$$

In order to analyze expression (3.5) we write

$$a^2=2\mu^2/\lambda\,. \tag{3.6}$$

We then have, at the classical level, the usual expression for the potential. Now, following Coleman and Weinberg,[15] we will consider

$$\lambda\sim g^4\,. \tag{3.7}$$

Then Eq. (3.5) reduces to

$$\begin{aligned}V_{\rm eff}=&\tfrac{1}{8}\lambda\varphi^4-\tfrac{1}{2}\mu^2\varphi^2+\frac{3\,\mathrm{Tr}(n^4)}{2^6\pi^2}g^4\varphi^4[\ln(\varphi^2/\varphi_0^2)-\tfrac{25}{6}]\\&+\frac{\mu^4}{2^6\pi^2}\Bigg\{-\frac{\lambda\varphi^2}{\mu^2}\left[(N^2-2)\left(\ln\left(\frac{\lambda\varphi^2}{2\mu^2}-1\right)-\tfrac{1}{2}\right)+3\left(\ln\left(3\frac{\lambda\varphi^2}{2\mu^2}-1\right)-\tfrac{1}{2}\right)\right]\\&\qquad+(N^2-2)\ln\left(\frac{\lambda\varphi^2/2\mu^2-1}{\lambda\varphi_0^2/2\mu^2-1}\right)+\ln\left(\frac{3\lambda\varphi^2/2\mu^2-1}{3\lambda\varphi_0^2/2\mu^2-1}\right)\Bigg\}+\text{constant term}+O(\lambda^2)\,.\end{aligned} \tag{3.8}$$

The constant term in Eq. (3.8) can be eliminated by an appropriate renormalization of the zero-point energy if the normalization point is chosen satisfying

$$\varphi_0^2>2\mu^2/\lambda\,. \tag{3.9}$$

This relation forces the constant term to be real. Now it is easily seen that $V_{\rm eff}$ becomes complex whenever

$$\varphi^2<2\mu^2/\lambda\,. \tag{3.10}$$

Then, if we take $\mu^2>0$ in order to start with a theory with spontaneous symmetry breaking, we must impose conditions (3.4) only on the real part of $V_{\rm eff}$. Concerning the imaginary part, it is given by the expression

$$\begin{aligned}\mathrm{Im}V_{\rm eff}=&\frac{-\lambda^2}{2^8\pi}[(N^2-2)(\varphi^2-2\mu^2/\lambda)\theta(2\mu^2/\lambda-\varphi^2)\\&+(3\varphi^2-2\mu^2/\lambda)\theta(2\mu^2/\lambda-3\varphi^2)]\,.\end{aligned} \tag{3.11}$$

This imaginary part corresponds to an instability of the vacuum which has been already studied in different contexts.[15,8,9-16] We will return to it later. We finally note that the only group-dependent quantity in expression (3.8) is $\mathrm{Tr}(n^4)$ (apart from the trivial N^2-2 factors).

From Eqs. (2.18) and (2.19) we see that the computation of $\mathrm{Tr}(n^4)$ implies the evaluation of

$$C_{ai_1i_2}C_{bi_2i_3}C_{ci_3i_4}C_{di_4i_1}\,, \tag{3.12}$$

which is rather complicated for the general SU(N) group. As an example we give its value for SU(2):

$$\mathrm{Tr}(n^4)=2\,. \tag{3.13}$$

We can then study in this case the possible minima of the potential, given by Eq. (3.8) with Eq. (3.13). With φ_0 an arbitrary subtraction point, satisfying

$$\varphi_0{}^2>2\mu^2/\lambda\,, \tag{3.14}$$

we will choose it as the minimum of the potential:

$$\left.\frac{\partial V_{\rm eff}}{\partial\varphi^2}\right|_{\varphi^2=\varphi_0{}^2}=0\,. \tag{3.15}$$

This equation reduces to

$$Z(1-11g^4/4\pi^2\lambda)=1+(\lambda/32\pi^2)\left\{2\left[\ln(Z-1)+\frac{Z-\frac{1}{2}}{Z-1}-\tfrac{1}{2}\right]+3\left[\ln(3Z-1)+\frac{3Z-\frac{1}{2}}{3Z-1}-\tfrac{1}{2}\right]\right\}, \tag{3.16}$$

with

$$Z=\lambda\varphi_0{}^2/2\mu^2\,. \tag{3.17}$$

We have studied expressions (3.8)–(3.15) numerically. As expected,[15] radiative corrections change the location of the minimum. Writing $\alpha=11g^4/4\pi\lambda$, when α grows, we find that Z becomes larger and $\lim_{\alpha\to1^-}Z=\infty$. For $\alpha>1$ there is no real solution of Eq. (3.16).

In the Prasad-Sommerfield limit,[7]

$$\lambda\to0\,,\quad \mu^2\to0\,, \tag{3.18}$$

with

$$a^2=2\mu^2/\lambda\to\text{constant}\,, \tag{3.19}$$

the minimum of the potential can be worked out analytically. The result is

$$\varphi_0{}^2=a^2/(1-11g^4/4\pi^2\lambda)\,. \tag{3.20}$$

Expression (3.20) shows the general behavior we discussed above. For

$$g^4>\frac{4\pi^2}{11}\lambda=3.59\lambda\,, \tag{3.21}$$

the effective potential has no stable minimum. This shows that one must be very careful in studying the Prasad-Sommerfield limit: In the domain (3.21), in agreement with the relation (3.7), there are no minima to ensure the existence of topological solitons (Bogomolny-Prasad-Sommerfield monopoles[7]) at the one-loop level. [The existence of finite-energy solutions requires that

$$\mathrm{Tr}(\phi^2)\underset{r\to\infty}{\longrightarrow}\tfrac{1}{2}\varphi_0{}^2$$

with $\varphi_0{}^2$ a nontrivial minimum of the potential.]

If condition (3.21) is not fulfilled, then the effective potential has a real minimum at $\varphi_0{}^2>a^2$ given by Eq. (3.20). The quantum fluctuations have only changed the position of the minimum. Of course, for $a^2<0$ a similar analysis leads to dynamical symmetry breaking for $g^4>(4\pi^2/11)\lambda$, etc.

Moreover, had we started with $a^2=0$, that is, no mass term, the result would have been that of Coleman and Weinberg[15]: dynamical symmetry breaking for the particular relation

$$11g^4/4\pi^2\lambda=1\,. \tag{3.22}$$

(ii) Limit of large $H(g\varphi^2/H\to0,\lambda\varphi^2/H\to0)$. From Eq. (2.32) it is easily seen that in this limit the effective Lagrangian reduces to

$$\mathcal{L}_1=\frac{1}{2}\frac{\Gamma(1-d/2)}{(2\pi)^{d/2}}\,\mathrm{Tr}\big((g^2H^2n^2)^{d/4}\{\zeta(1-d/2;\tfrac{1}{2})+(d-2)\zeta(1-d/2;\tfrac{1}{2})+4(2i)^{-d/2}\sin[(d/2-1)\pi/2]\}\big)\,. \tag{3.23}$$

The first term in Eq. (2.23) corresponds to the contribution of the scalar fields.

In order to renormalize this expression we make an ϵ expansion. We then obtain an imaginary, finite contribution coming from the last term in Eq. (3.23). This imaginary part takes the form

$$\mathrm{Im}\mathcal{L}_1=\frac{C_2(G)}{16\pi}g^2H^2\,. \tag{3.24}$$

This imaginary term has been already obtained for a pure Yang-Mills theory by Nielsen and Olesen.[8] They have associated it with an unstable mode, showing that its vacuum polarization corresponds to the self-energy of a tachyon in 1+1 dimensions. They have also discussed the relation of the unstable mode with asymptotic freedom and its dependence with the particular configuration, Eq. (2.18), adopted for the gauge field.[9] Because of this imaginary part, the possible minima of the effective action are not really associated with stable vacuums but with unstable ones.

Since the divergent part of expression (3.23) is real, it can be eliminated by counterterms. We will use the normalization condition

$$\mathrm{Re}\mathcal{L}_{\rm eff}|_{H^2=\Lambda^4/g^2}=-\Lambda^4/2g^2\,, \tag{3.25}$$

where $[\Lambda]=[\text{mass}]$. Then we get

$$\mathcal{L}_{\rm eff}^{(\rm ren)} = -\tfrac{1}{2}H^2\left[1+\frac{C_2(G)}{32\pi^2}\left(\tfrac{11}{3}-\tfrac{1}{6}\right)g^2\ln(g^2H^2/\Lambda^4)\right] + i\frac{C_2(G)}{16\pi}g^2H^2+O(g^4)\,. \qquad (3.26)$$

Defining

$$t=\ln(gH/\Lambda^2)\,, \qquad (3.27)$$

we can rewrite expression (3.26) in the form

$$\mathcal{L}_{\rm eff} = -\tfrac{1}{2}[g/\bar{g}(t)]^2H^2+i\frac{C_2(G)}{16\pi}g^2H^2\,, \qquad (3.28)$$

where $\bar{g}(t)$ is the running coupling constant given by

$$\bar{g}(t)=g-\frac{C_2(G)}{32\pi^2}\left(\tfrac{11}{3}-\tfrac{1}{6}\right)g^3t+O(g^5)\,. \qquad (3.29)$$

We then obtain the correct expression for the β function:

$$\beta(g)=\frac{d\bar{g}(t)}{dt}=-\tfrac{1}{2}b_0g^3+O(g^5)\,, \qquad (3.30)$$

with b_0 given as in Eq. (2.35).

Equation (3.29) gives the transformation law of the renormalization group at the one-loop level. Although we have obtained this result for (covariantly) constant fields, they are generally valid. We see from Eqs. (3.29) and (3.30) the relation between the effective Lagrangian for an intense field and the behavior of gauge theory at small distances, controlled by the β function. In fact, the derivation of asymptotic freedom by considerations of external fields has been discussed in detail in the literature (see Refs. 8, 17, and 18).

If we discard for a moment the imaginary part of $\mathcal{L}_{\rm eff}$ and seek for a minimum of the energy density we easily obtain

$$gH_{\rm min}=\Lambda^2\exp[-(1/b_0g^2)-\tfrac{1}{2}] =\bar{\Lambda}^2\exp(-1/b_0g^2)\,. \qquad (3.31)$$

We see that the effective action of an SU(N) gauge theory with scalar fields (in the adjoint representation) develops a minimum away from the classical one. This result has been already obtained for a pure Yang-Mills theory in Refs. 2–5, and 8. Because of the imaginary part, it corresponds to a metastable state. The value of the effective Lagrangian at this point results:

$$\mathrm{Re}\mathcal{L}_{\rm eff}(H_{\rm min})=\tfrac{1}{4}g^2b_0H_{\rm min}{}^2>0\,, \qquad (3.32)$$

thus showing that $\bar{g}(t)$ has become imaginary. This situation corresponds to "spontaneous magnetization" in the vacuum as discussed for pure Yang-Mills theory by Pagels and Tomboulis.[2] From our computations we see that the phenomenon also happens in the presence of scalar fields. The only change is given by the different value of b_0.

From the real part of our effective Lagrangian we can obtain the trace anomaly of the energy-momentum tensor. Using

$$\Theta_\mu{}^\mu=\eta^{\mu\nu}\Theta_{\mu\nu}=2\frac{\partial\mathcal{L}_{\rm eff}}{\partial\eta^{\mu\nu}}\eta^{\mu\nu}-d\,\mathcal{L}_{\rm eff} \qquad (3.33)$$

and

$$H^2=\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} \qquad (3.34)$$

we obtain in the limit we are considering

$$\Theta_\mu{}^\mu\simeq(\beta(g)/2\bar{g}(t))F_{\mu\nu}F^{\mu\nu} \text{ for } g\varphi^2/H\to 0\,. \qquad (3.35)$$

Of course, a complete result should take into account the contribution to $\Theta_{\mu\nu}$ of the Callan, Coleman, and Jackiw improvement term necessary[19] because of the presence of scalars. However, in the limit we are discussing, $\varphi=$ constant, $g\varphi^2/H\to 0$, there is no such contribution since the improvement term depends on derivatives of the scalar fields which vanish in our case.[20] We would like to note that the nonvanishing trace for the energy-momentum suggests that soliton solutions of several field theories which satisfy conditions of the form $\Theta_\mu{}^\mu=0$ (Ref. 21) disappear if one considers the contribution of quantum fluctuation to the effective action and then studies the resulting equations of motion.

IV. DISCUSSION

We have studied the effective action for an SU(N) gauge theory with scalar fields. The underlying viewpoint was to investigate if the minima of the classical action could be drastically altered because of quantum corrections.

This possibility was first considered by Coleman and Weinberg[15] in their study of the effective potential of several scalar and gauge theories. More recent investigations[2–9] discussed the effective Lagrangian of pure Yang-Mills theory, finding that this Lagrangian is likely to have a nontrivial minimum away from the "perturbative ground state."

For an SU(N) gauge theory which included scalar fields, our results point in the same direction: Both in the limit of small and large H, the one-loop effective Lagrangian differs qualitatively from the classical one.

In the $H\to 0$ limit, we have found that the effective potential could lose the minima that it had at the classical level. In particular, for the Prasad-Sommerfield limit this means that quantum corrections do not ensure the existence of monopoles.

In the $H\to\infty$ limit we have found the phenomenon of "spontaneous magnetization" in the vacuum, already discussed for pure Yang-Mills theory,[2] but our result incorporates the contribution of scalar fields. We have done all our computations for pure "magnetic" field H. If we include an electric field E, the results remain valid by replacing H^2 by (H^2-E^2) (in Minkowski space). Of course, for $E^2>H^2$ the effective Lagrangian develops an imaginary part, a situation discussed by Euler and Heisenberg[16] in QED.

Moreover, as was first pointed out by Nielsen and Olesen,[8] even for a pure magnetic field the effective Lagrangian has an imaginary part. This implies that the new minimum that one obtains corresponds to a metastable state.

Thus, we have seen that in various situations the effective action of the SU(N) theory with scalars differs sufficiently from the classical action as to change completely the nature of the minima, changing the structure of the vacuum.

A more complete analysis should be done in order to clarify different aspects of this investigation. In particular one should try to expand the action around nonconstant gauge and scalar fields. One should also study intermediate cases between the two limits that we have discussed. We shall return to these problems more thoroughly elsewhere.

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