Supersymmetric Dirac particles in external fields

F. Ravndal
Institute of Physics, University of Oslo, Oslo 3, Norway
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A classical Lagrangian is proposed for a relativistic particle with spin. It is supersymmetric under transformations between position and spin variables. The theory can be quantized and becomes identical with conventional Dirac theory. This correspondence continues to be valid when the particle interacts with external electromagnetic or gravitational fields as long as its coupling to these fields conserves the supersymmetry.

I. INTRODUCTION

In classical physics essentially all dynamical variables are bosonic since they can be represented by ordinary commuting numbers. On the other hand, in modern theories of elementary particles one is tempted to speculate that all the really fundamental particles in nature are fermions and all bosons are composite. Anticommuting quantum fields do not have a classical limit in terms of ordinary numbers but must be represented by anticommuting Grassmann numbers.

Supersymmetry strikes a middle road between these two extreme situations in that it demands complete equivalence between bosonic and fermionic variables. This requires that a supersymmetric theory has operators which connect these two types of variables and hence are themselves fermionic. A supersymmetric quantum field theory describing particles with different spin must have the same number of fermionic and bosonic degrees of freedom. Transition operators therefore have half-integer spin and the simplest ones have spin $\frac{1}{2}$.

One can also imagine a different kind of supersymmetric theory where one does not have transitions between integer- and half-integer-spin particle operators with the corresponding Bose or Fermi statistics, but where one has transitions between operators with the same spin but opposite statistics. This could, for example, be the case in gauge theories where the ghosts resulting from quantization have the wrong statistics. We will see that the supersymmetry we shall discuss here is of this latter type.

Spinor operators are the basic building blocks in most supersymmetric theories. This intimate relationship between spin $\frac{1}{2}$ and supersymmetry makes it natural to look for such a symmetry in ordinary Dirac theory. In contrast to the Klein-Gordon theory for spin-0 particles, the Dirac theory exists only as a quantum theory. So far it has not been obtained by quantization of a corresponding classical theory, as can easily be done for the Klein-Gordon theory. This has to do with the problem of giving a consistent description of the spin degrees of freedom for a classical point particle in terms of ordinary bosonic variables.

A way out of this dilemma was originally suggested by Martin[1] who showed that anticommuting Grassmann variables can be used for this purpose. This gave rise to the so-called pseudoclassical mechanics which has many interesting properties and has been studied by Casalbuoni and his collaborators.[2] It can be generalized to relativistic mechanics as shown by Berezin and Marinov.[3] When these theories are quantized the Grassmann spin variables become operators which can be represented by ordinary spin matrices.

A most important observation was made by Brink *et al.*[4] who demonstrated that one could construct such a pseudoclassical theory for a spinning particle which had a new type of symmetry. It involved transformations relating ordinary position variables to the new spin variables and was a realization of supersymmetry.[5] Unfortunately, this formulation of the theory turned out to be very complicated for particles with mass and one had to invoke a fifth dimension in order to describe them properly. This complication has recently been circumvented with the proposal of a very simple Lagrangian for a massive, spinning relativistic particle which is also supersymmetric.[6]

Here we will discuss the supersymmetric aspects of this new theory and how it ties up with the well-known properties of quantized Dirac particles. This becomes especially illuminating when we consider the particle moving in an external electromagnetic field. We then know what results our new theory has to reproduce. It can then be used to investigate the motion of the particle in an external gravitational field of which less is known.

In Sec. II we give a short summary of the relativistic mechanics of a spinless particle and how the proper-time formulation can be quantized to give the ordinary Klein-Gordon equation. The scalar propagator can be obtained in this first-quantized theory by using a Feynman path integral[7] or the equivalent Schwinger operator method.[8] This is done in the Appendix.

In Sec. III we present the corresponding Lagrangian description of a relativistic particle with spin and demonstrate that it is supersymmetric. Introducing a new position variable for the particle which we call its superposition, we can write the Lagrangian in a very compact and explicitly supersymmetric form.

The more interesting case of a spinning particle in an external electromagnetic field is considered in Sec. IV. Again we find that the particle has an additional constant of motion which is due to the inherent supersymmetry of its dynamics and we find the equations of motion. The theory is quantized in Sec. V using proper-time methods. We find the Feynman propagator for the particle and derive its quantum-mechanical equations of motion in an electromagnetic field. These agree with Schwinger's results.[8] In Sec. VI we consider the particle in a gravitational field. This interaction can again be made in a supersymmetric way and gives directly Papapetrou's equation of motion.[9] It is also pointed out how one can obtain a quantum-mechanical description of a Dirac particle in curved spacetime from this supersymmetric formulation. Finally, we show that there are good reasons for interpreting the Grassmann spin variables as classical coordinates of the quantum-mechanical *Zitterbewegung*.

II. PROPER-TIME METHODS FOR SPINLESS PARTICLES

The dynamics of a relativistic particle is determined by the action

$$S=\int d\lambda\, L(x,\dot{x}), \tag{2.1}$$

where λ is a time evolution parameter. Inspired by the geometric content of the theory of relativity, it is usual to choose the Lagrangian

$$L=(-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} \tag{2.2}$$

for a free, spinless particle where $\eta_{\mu\nu}$ is the Minkowski metric with positive signature. Since the four-velocity is $\dot{x}^\mu=dx^\mu/d\lambda$, the action now becomes

$$S=\int ds \tag{2.3}$$

and the particle moves so that its proper time s is maximal.

The Lagrangian (2.2) has the useful property that it makes the action invariant under reparametrization of the evolution parameter λ. On the other hand, because of the square root it can only describe particles moving along timelike trajectories. It can therefore not be used for processes where a particle turns around, moves backwards in real time, and effectively becomes an antiparticle moving forward in time. This was first emphasized by Stückelberg[10] who proposed instead to use the Lagrangian

$$L=\tfrac{1}{4}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=\tfrac{1}{4}\dot{x}^2. \tag{2.4}$$

The time evolution parameter $\lambda=s$ can no longer be freely reparametrized but is proportional to the proper time of a massive particle. This is the Lagrangian we will use. From the canonical momentum

$$p_\mu=\frac{\partial L}{\partial\dot{x}^\mu}=\tfrac{1}{2}\dot{x}_\mu \tag{2.5}$$

we find the equation of motion

$$\ddot{x}_\mu=0, \tag{2.6}$$

which says that the particle moves with constant velocity.

Constants of motion can in general be found from Noether's theorem. If the Lagrangian changes by a derivative $\delta L=\dot{\Lambda}$ under the infinitesimal coordinate transformation $x^\mu\to x^\mu+\delta x^\mu$, then

$$K=\delta x^\mu p_\mu-\Lambda \tag{2.7}$$

is a conserved quantity, $\dot{K}=0$. Since L in Eq. (2.4) has no explicit dependence on the proper time s, the action is invariant under the translations

$$s\to s+a. \tag{2.8}$$

The corresponding conserved quantity is then the Hamiltonian

$$H=\dot{x}^\mu p_\mu-L=p^\mu p_\mu. \tag{2.9}$$

Its constant value is given by the mass m of the particle:

$$H=p^2=-m^2. \tag{2.10}$$

Notice that the Lagrangian (2.4) is valid both for massive and massless particles.

In an external electromagnetic potential $A_\mu(x)$ the interaction is obtained from letting one of the $\dot{x}$'s in (2.4) be changed into

$$\dot{x}_\mu\to\dot{x}_\mu+4eA_\mu(x). \tag{2.11}$$

This leads to the Lorentz equation of motion

$$\ddot{x}_\mu=2eF_{\mu\nu}\dot{x}^\nu, \tag{2.12}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{2.13}$$

Similarly, in a gravitational field $g_{\mu\nu}(x)$ the Lagrangian (2.4) becomes

$$L=\tfrac{1}{4}g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}. \tag{2.14}$$

This gives the geodetic equation of motion

$$\ddot{x}^{\mu}+\Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}=0, \tag{2.15}$$

where

$$\Gamma^{\mu}{}_{\alpha\beta}=g^{\mu\nu}\Gamma_{\nu\alpha\beta}$$

and

$$\Gamma_{\nu\alpha\beta}=\tfrac{1}{2}(\partial_{\alpha}g_{\nu\beta}+\partial_{\beta}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\beta}) \tag{2.16}$$

are Christoffel symbols.

In the quantized theory x_{μ} and p_{ν} become operators satisfying the canonical commutator

$$[x_{\mu},p_{\nu}]=i\eta_{\mu\nu}, \tag{2.17}$$

which is satisfied with $p_{\mu}=-i\partial_{\mu}$. The relativistic particle is described by a state vector $|\Phi(s)\rangle$ which is governed by a covariant Schrödinger equation[7]:

$$i\frac{\partial}{\partial s}|\Phi(s)\rangle=H|\Phi(s)\rangle. \tag{2.18}$$

When H is constant, it can be integrated to give

$$|\Phi(s)\rangle=e^{-iHs}|\Phi(0)\rangle. \tag{2.19}$$

An eigenstate $|\phi(0)\rangle$ of the Hamiltonian with eigenvalue $-m^2$ will satisfy

$$(p^2+m^2)|\phi(0)\rangle=0, \tag{2.20}$$

which is just the Klein-Gordon equation.

The scalar boson propagator is given by the amplitude to find the particle at point x_B at proper time s when it was at x_A when $s=0$:

$$\Delta_F(x,s)=\langle x_B(s)|x_A(0)\rangle \tag{2.21}$$

$$=\langle x_B|e^{-iHs}|x_A\rangle. \tag{2.22}$$

Inserting complete sets of states, one can rewrite this as a Feynman path integral[7]

$$\Delta_F(x,s)=\int \mathfrak{D}x\, e^{iS[x]}, \tag{2.23}$$

where S is the classical action. It is seen that with the Lagrangian (2.2) the functional integral is very difficult to do, if not impossible. In particular, it is not clear what to do with those paths which make the argument in the square root in (2.2) negative. This difficulty is related to the lack of particle-antiparticle symmetry in this Lagrangian.

Using instead Stückelberg's Lagrangian (2.4), the functional integral is much simpler, being essentially a product of Gaussian integrals. Following Feynman[7] we then project out the contribution to a particle with mass m and obtain the standard result

$$\Delta_F(x,m)=\int_0^{\infty} ds\, e^{-ism^2}\Delta_F(x,s) \tag{2.24}$$

$$=\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2}e^{ik\cdot x}. \tag{2.25}$$

This can also be obtained directly from the matrix element (2.22) of the time evolution operator as shown in the Appendix.

III. FREE PARTICLE WITH SPIN

The geodetic equation (2.15) is usually taken to describe the motion of all particles, independently of their spin. But we know that spinning macroscopic bodies feel an extra force due to their rotation in a gravitational field.[11] It is natural to expect the same to hold for a point particle with spin. Such a modified equation of motion could be obtained if we had a Lagrangian for the spinning particle. The same Lagrangian could then be used in the Feynman path integral (2.23) to give the quantum-mechanical particle propagator.

A Lagrangian for a spinning particle has been constructed by Berezin and Marinov[3] and Brink *et al.*[4] using anticommuting Grassmann variables for the spin degrees of freedom. They chose the parametrization-invariant form (2.2) for the orbital part of the Lagrangian. This was necessary for the latter authors since they were really interested in a one-dimensional supergravity theory where this kind of local gauge invariance is essential. Unfortunately, this choice makes it very difficult to describe massive particles. One is forced to invoke a fifth dimension and the formalism becomes somewhat cumbersome.

For our purpose here it is natural to take the form (2.4) for the orbital part. The proposed Lagrangian for a free spinning particle is then

$$L=\tfrac{1}{4}(\dot{x}^{\mu}\dot{x}_{\mu}-i\xi^{\mu}\dot{\xi}_{\mu}). \tag{3.1}$$

We have here introduced the four Grassmann variables $\xi^{\mu}=\xi^{\mu}(s)$ which anticommute

$$\{\xi_{\mu},\xi_{\nu}\}=0, \tag{3.2}$$

so that the square of each equals zero.

From the canonical momenta

$$p_{\mu}=\frac{\partial L}{\partial\dot{x}^{\mu}}=\tfrac{1}{2}\dot{x}_{\mu} \tag{3.3a}$$

and

$$\zeta_{\mu}=\frac{\partial L}{\partial\dot{\xi}^{\mu}}=\tfrac{1}{4}i\xi_{\mu}, \tag{3.3b}$$

we find the Hamiltonian

$$H=\dot{x}^{\mu}p_{\mu}+\dot{\xi}^{\mu}\zeta_{\mu}-L$$

$$=p^2 \tag{3.4}$$

and the equations of motion

$$\ddot{x}_\mu = 0\,, \tag{3.5a}$$

$$\dot{\xi}_\mu = 0\,. \tag{3.5b}$$

The Hamiltonian is seen to be independent of the spin of the particle and it is again a constant of motion, $p^2 = -m^2$.

By construction the Lagrangian is also invariant under Lorentz transformations

$$\delta x_\mu = \epsilon_{\mu\nu} x^\nu\,, \tag{3.6a}$$

$$\delta \xi_\mu = \epsilon_{\mu\nu} \xi^\nu\,, \tag{3.6b}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ are intinitesimal parameters. Noether's theorem (2.7) states that there is a corresponding conserved quantity

$$\begin{aligned} J &= \delta x^\mu p_\mu + \delta \xi^\mu \zeta_\mu \\ &= \epsilon^{\mu\nu}(x_\nu p_\mu + \tfrac{1}{4} i \xi_\nu \xi_\mu)\,. \end{aligned} \tag{3.7}$$

This gives the six constant components of the angular momentum

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}\,, \tag{3.8}$$

where

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{3.9}$$

is the orbital angular momentum and

$$S_{\mu\nu} = \tfrac{1}{2} i \xi_\mu \xi_\nu = \tfrac{1}{4} i [\xi_\mu, \xi_\nu] \tag{3.10}$$

is the spin angular momentum. This is really a pseudoclassical quantity with no definite size.

The Lagrangian (3.1) gives rise to one additional conserved quantity. From the equations of motion (3.5) we see that

$$Q = \xi^\mu p_\mu \tag{3.11}$$

is constant along the trajectory of the particle. Since $Q^2 = 0$, we can choose this constant to be zero for all classical particles. As a consequence, we than have

$$p_\mu S^{\mu\nu} = 0\,. \tag{3.12}$$

Thus, in the rest frame of the particle, the spin tensor has only spatial components. In Sec. V we will see that in the quantized theory $Q \neq 0$. Equation (3.12) then no longer holds.

This new conserved quantity (3.11) results from a corresponding symmetry in the Lagrangian (3.1). By inspection, we find that under the transformation

$$\delta x_\mu = i\epsilon \xi_\mu\,, \tag{3.13a}$$

$$\delta \xi_\mu = \epsilon \dot{x}_\mu \tag{3.13b}$$

it changes by a divergence. The parameter ϵ is an anticommuting number so that $\epsilon^2 = 0$. This transformation which mixes the spin variables with the position variables is a supersymmetry transformation.[5] Mathematically it means that we must accept that the position variables of a spinning particle can contain a component which is an even Grassmann variable. This generalization of the coordinate concept is typical of all supersymmetric theories. What it means physically is not clear. In Sec. VII we will argue that the Grassmann part of the position variable has to do with the quantum-mechanical *Zitterbewegung*.

This new symmetry in the problem makes it possible to write the Lagrangian (3.1) in a more compact form which clearly exhibits the supersymmetry. For this purpose we introduce the superposition $X_\mu(s,\theta)$ where θ is a new Grassmann parameter which is needed to describe the full dynamical development of the system. X_μ contains the ordinary position and spin variables in its Taylor expansion:

$$X_\mu(s,\theta) = x_\mu(s) + i\theta \xi_\mu(s)\,. \tag{3.14}$$

The supersymmetry transformation (3.13) is now induced by the parameter transformation

$$\theta \to \theta + \epsilon\,, \tag{3.15a}$$

$$s \to s - i\epsilon\theta\,. \tag{3.15b}$$

It leaves invariant the differential form $ds + i\theta d\theta$ which defines a proper supertime for the particle. Under this transformation the superposition is changed to

$$\begin{aligned} \delta X_\mu &= \delta x_\mu + i\theta \delta \xi_\mu \\ &= -i\epsilon \bar{D} X_\mu\,, \end{aligned}$$

where

$$\bar{D} = \theta \frac{\partial}{\partial s} + i \frac{\partial}{\partial \theta}\,. \tag{3.16}$$

This suggests the introduction of the covariant derivative

$$D = \theta \frac{\partial}{\partial s} - i \frac{\partial}{\partial \theta}\,, \tag{3.17}$$

which anticommutes with the supersymmetry generator $\bar{D}$:

$$\{D, \bar{D}\} = 0\,. \tag{3.18}$$

On the other hand, we see that the anticommutator of two supersymmetry generators is simply the generator of proper-time translations:

$$\{\bar{D}, \bar{D}\} = 2i \frac{\partial}{\partial s}\,. \tag{3.19}$$

We will see the significance of the relation in Sec. V where we will quantize the theory.

The Lagrangian (3.1) can now be written as

$$L=\tfrac{1}{4}\int d\theta\,\eta_{\mu\nu}\dot{X}^{\mu}DX^{\nu}\,, \tag{3.20}$$

when we use the standard integrals

$$\int d\theta=0,\quad \int d\theta\,\theta=1\,. \tag{3.21}$$

Having a product of only supervariables in the Lagrangian we are guaranteed that the action is invariant under supersymmetry transformations.[5]

IV. SPINNING PARTICLE IN AN ELECTROMAGNETIC FIELD

When the spinning particle moves in an electromagnetic potential $A_\mu(x)$ we could get an interaction from the Lagrangian (3.1) by the same minimal substitution (2.11) as we used for the spinless particle. But it is seen that such a coupling would not give any interaction with the spin degrees of freedom and the particle would be without a magnetic moment. In addition, this coupling would break the supersymmetry of the theory.

A satisfactory interaction with the external field can instead be obtained from the very similar minimal substitution

$$\dot{X}_\mu\to\dot{X}_\mu+4eA_\mu(X) \tag{4.1}$$

in the Lagrangian (3.20). Here

$$A_\mu(X)=A_\mu(x)+i\theta\xi^\nu A_{\mu,\nu} \tag{4.2}$$

is a superpotential and the interacting theory will remain supersymmetric. More specifically, the interaction is given by

$$\begin{aligned}L_{\rm em}&=e\int d\theta\,A_\mu(X)DX^\mu\\&=e(A_\mu\dot{x}^\mu+i\xi^\mu\xi^\nu A_{\nu,\mu})\\&=e(A_\mu\dot{x}^\mu+S^{\mu\nu}F_{\mu\nu})\,.\end{aligned} \tag{4.3}$$

It modifies the canonical momentum conjugate to x^μ:

$$p_\mu=\tfrac{1}{2}\dot{x}_\mu+eA_\mu\,. \tag{4.4}$$

Defining a kinematical momentum

$$\Pi_\mu=\tfrac{1}{2}\dot{x}_\mu=p_\mu-eA_\mu\,, \tag{4.5}$$

the Hamiltonian becomes

$$H=\Pi^\mu\Pi_\mu-eS^{\mu\nu}F_{\mu\nu}\,. \tag{4.6}$$

The last term obviously represents the interaction with the magnetic moment of the particle.

We now find the equations of motion:

$$\dot{\xi}_\mu=2eF_{\mu\nu}\xi^\nu\,, \tag{4.7}$$

$$\ddot{x}_\mu=2eF_{\mu\nu}\dot{x}^\nu+2eS^{\lambda\nu}F_{\lambda\nu,\mu}\,. \tag{4.8}$$

Hence, the spin $S^{\lambda\nu}$ varies according to

$$\dot{S}^{\lambda\nu}=2e(F^{\lambda}{}_{\mu}S^{\mu\nu}-S^{\lambda}{}_{\mu}F^{\mu\nu})\,. \tag{4.9}$$

In the rest frame of the particle where the spin tensor has only spatial components $S_{ij}=\epsilon_{ijk}S_k$, this equation can be written as

$$\underline{\dot{S}}=2e\underline{S}\times\underline{B}. \tag{4.10}$$

The spinning particle has therefore a gyromagnetic ratio $g=2$. Notice that for a particle with mass m our dot derivative is with respect to $s=\tau/2m$ where τ is the more commonly used proper time.

We will now demonstrate that the interaction (4.3) is supersymmetric. Under the transformation (3.13) the full Lagrangian is changed by

$$\delta L=\dot{\Lambda}+\tfrac{1}{2}i\epsilon e\xi^\mu\xi^\nu\xi^\lambda F_{\mu\nu,\lambda}\,, \tag{4.11}$$

where

$$\Lambda=\tfrac{1}{4}i\epsilon(\dot{x}_\mu+4eA_\mu)\xi^\mu\,. \tag{4.12}$$

The last term in (4.11) is zero because of Maxwell's equation

$$\begin{aligned}F_{[\mu\nu,\lambda]}&\equiv F_{\mu\nu,\lambda}+F_{\lambda\mu,\nu}+F_{\nu\lambda,\mu}\\&=0\,.\end{aligned} \tag{4.13}$$

Thus, the change in L is a total divergence and the corresponding conserved quantity Q follows from Noether's theorem (2.7):

$$\begin{aligned}i\epsilon Q&=\delta x^\mu p_\mu+\delta\xi^\mu\zeta_\mu-\Lambda\\&=\tfrac{1}{2}i\epsilon\dot{x}_\mu\xi^\mu\,,\end{aligned}$$

i.e.,

$$Q=\xi^\mu\Pi_\mu\,. \tag{4.14}$$

The proper-time derivative

$$\dot{Q}=\dot{\xi}^\mu\Pi_\mu+\xi^\mu\dot{\Pi}_\mu$$

should be zero. This is verified from the equations of motion together with Maxwell's equation (4.13):

$$\dot{Q}=eS^{\lambda\nu}\xi^\mu F_{\lambda\nu,\mu}=0\,. \tag{4.15}$$

Surprisingly enough, this new constant of motion is also conserved in the quantized theory as we will see in Sec. V.

V. QUANTUM MECHANICS FOR SUPERSYMMETRIC DIRAC PARTICLES

One can easily quantize this supersymmetric theory for spinning particles by canonical methods. The operators x_μ and p_ν have the commutator (2.17) which can be written as

$$[\Pi_\mu,x_\nu]=-i\eta_{\mu\nu} \tag{5.1}$$

when the particle is in an external electromagnetic field. This spin variable ξ_μ and its canonical momentum ζ_μ (3.3b) are similarly quantized by the canonical anticommutator

$$\{\zeta_\mu, \xi_\nu\} = -\tfrac{1}{2} i \eta_{\mu\nu}, \tag{5.2}$$

so that

$$\{\xi_\mu, \xi_\nu\} = -2\eta_{\mu\nu}. \tag{5.3}$$

Thus it is possible to represent the quantized spin variables by ordinary Dirac matrices:

$$\xi_\mu = \gamma_\mu . \tag{5.4}$$

Since the spin (3.10) of the particle now becomes

$$S_{\mu\nu} = \tfrac{1}{4} i [\gamma_\mu, \gamma_\nu] = \tfrac{1}{2}\sigma_{\mu\nu}, \tag{5.5}$$

we see that we are actually dealing with a spin-$\frac{1}{2}$ particle.

In an external electromagnetic field the Hamiltonian describing the time evolution of the particle is given by Eq. (4.6) which now can be written as

$$H = -\not{\Pi}^2, \tag{5.6}$$

where $\not{\Pi} = \gamma^\mu \Pi_\mu$. Here we have used the commutator

$$[\Pi_\mu, \Pi_\nu] = ieF_{\mu\nu}. \tag{5.7}$$

We see that the Hamiltonian is simply just minus the square of the new conserved quantity $Q = \not{\Pi}$. Another way of expressing this property is by the anticommutator

$$\{Q, Q\} = -2H, \tag{5.8}$$

which is so characteristic for a supersymmetric theory.[5]

The quantum-mechanical equations of motion for the particle now follow from the fundamental operator equation

$$\dot{A} = i[H, A]. \tag{5.9}$$

We find

$$\dot{x}_\mu = 2\Pi_\mu , \tag{5.10}$$

$$\dot{\Pi}_\mu = e(F_{\mu\nu}\Pi^\nu + \Pi^\nu F_{\mu\nu}) + eS^{\lambda\nu}F_{\lambda\nu,\mu}$$
$$= 2eF_{\mu\nu}\Pi^\nu - ieF_\mu{}^\nu{}_{,\nu} + eS^{\lambda\nu}F_{\lambda\nu,\mu},$$

$$\dot{\xi}_\mu = 2eF_{\mu\nu}\xi^\nu. \tag{5.11}$$

Except for the divergence of the field tensor in Eq. (5.10), these operator equations are identical to the classical equations of motion we found in Sec. IV. Equation (5.10) is identical to the equation obtained by Schwinger.[8] He also derived it from the covariant Hamiltonian (5.6) which can be obtained from conventional Dirac theory.

We can now check whether $Q = \not{\Pi}$ is still a constant of motion in this quantized theory:

$$\dot{Q} = \dot{\gamma}^\mu \Pi_\mu + \gamma^\mu \dot{\Pi}_\mu$$
$$= ie\gamma^\mu(-F_\mu{}^\nu{}_{,\nu} + \tfrac{1}{2}\gamma^\lambda\gamma^\nu F_{\lambda\nu,\mu}). \tag{5.12}$$

In order to calculate the last term on the right-hand side we use

$$\gamma^\mu\gamma^\nu\gamma^\rho = s^{\mu\nu\rho\sigma}\gamma_\sigma + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5, \tag{5.13}$$

where

$$s_{\mu\nu\rho\sigma} = -\tfrac{1}{4}Sp\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$
$$= -(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{5.14}$$

Using this relation together with Maxwell's equation (4.13) one is left with the identity

$$\gamma^\mu\gamma^\lambda\gamma^\nu F_{\lambda\nu,\mu} = 2\gamma^\mu F_\mu{}^\nu{}_{,\nu}, \tag{5.15}$$

which in Eq. (5.12) makes $\dot{Q} = 0$. This result could obviously be obtained much more directly from

$$\dot{Q} = -i[Q^2, Q] = 0, \tag{5.16}$$

but that would be less illuminating compared with the above analysis which has a close classical analog.

The state vector $|\Psi(s,\theta)\rangle$ of the system now depends on two evolution parameters, s and θ. Translations in proper time are generated by the Hamiltonian H which gives the Schrödinger equation (2.18):

$$i\frac{\partial}{\partial s}|\Psi(s,\theta)\rangle = H|\Psi(s,\theta)\rangle. \tag{5.17}$$

From Sec. III we know that supersymmetry transformations which are generated by the operator Q will change the state vector by an amount determined by the operator $\overline{D}$ in (3.16):

$$i\overline{D}|\Psi(s,\theta)\rangle = Q|\Psi(s,\theta)\rangle. \tag{5.18}$$

Combining this with the Schrödinger equation (5.17) we find the differential equation

$$\frac{\partial}{\partial\theta}|\Psi(s,\theta)\rangle = (-Q + \theta H)|\Psi(s,\theta)\rangle. \tag{5.19}$$

It has the solution

$$|\Psi(s,\theta)\rangle = e^{-Q\theta}|\Psi(s,0)\rangle. \tag{5.20}$$

The full development of the state vector is now given by

$$|\Psi(s,\theta)\rangle = e^{-iHs-Q\theta}|\Psi(0,0)\rangle, \tag{5.21}$$

which replaces Eq. (2.19) for a scalar particle.

When the particle is in an eigenstate $|\psi(0,0)\rangle$ of the operator $Q = \not{\Pi}$ with eigenvalue $-m$, we have

$$(\not{\Pi} + m)|\psi(0,0)\rangle = 0, \tag{5.22}$$

which is just the Dirac equation. This state will now vary according to Eq. (5.21):

$$|\psi(s,\theta)\rangle = e^{im^2 s - m\theta}|\psi(0,0)\rangle. \tag{5.23}$$

The Schrödinger equation (5.17) then gives simply the mass-shell condition

$$(-\not{\Pi}^2+m^2)\,|\psi(0,0)\rangle=0 \tag{5.24}$$

in the form of a Klein-Gordon equation.

It is important to notice that in this quantized theory $Q=\not{\Pi}$ and θ continue to anticommute with each other. When the variables ξ_μ are represented by Dirac matrices, θ must be represented by a matrix which anticommutes with all these matrices.

In the Heisenberg picture Eq. (5.21) can be used to construct a superposition operator

$$X(s,\theta)=e^{iHs+Q\theta}X(0,0)e^{-iHs-Q\theta} \tag{5.25}$$

with corresponding eigenstates

$$|X(s,\theta)\rangle=e^{iHs+Q\theta}|X(0,0)\rangle. \tag{5.26}$$

Under a supersymmetry transformation this position eigenstate is changed into

$$e^{Q\epsilon}|X(s,\theta)\rangle=|X(s-i\epsilon\theta,\theta+\epsilon)\rangle \tag{5.27}$$

when we use

$$e^{Q\epsilon}e^{Q\theta}=e^{Q(\epsilon+\theta)}e^{[Q\epsilon,\,Q\theta]/2}$$

and (5.28)

$$[Q\epsilon,Q\theta]=-\{Q,Q\}\epsilon\theta$$
$$=2H\epsilon\theta\,.$$

The corresponding change in the dynamical variables is given by

$$\begin{aligned}\delta X_\mu&=\delta x_\mu+i\theta\delta\xi_\mu\\&=[Q\epsilon,X_\mu]\\&=i\epsilon\,\xi_\mu+i\theta\epsilon\dot{x}_\mu\,.\end{aligned} \tag{5.29}$$

Both this result and (5.27) are in full agreement with the supersymmetry transformations discussed in Sec. III.

This quantum-mechanical framework can be used to calculate the Feynman propagator of a Dirac particle. In analogy with Eq. (2.21) for a scalar particle, it is now given by

$$\begin{aligned}S_F(x,s,\theta)&=\langle X_B(s,\theta)\,|X_A(0,0)\rangle\\&=\langle x_B\,|e^{-iHs-Q\theta}\,|x_A\rangle.\end{aligned} \tag{5.30}$$

From this we project the contribution to a particle with a definite value $Q=-m$ using the wave function (5.23):

$$\begin{aligned}S_F(x,m)&=\int_0^\infty ds\,e^{-im^2s}\int d\theta\,e^{-m\theta}S_F(x,s,\theta)\\&=\int_0^\infty ds\,e^{-im^2s}\langle x_B\,|e^{-iHs}(Q-m)\,|x_A\rangle.\end{aligned} \tag{5.31}$$

In the absence of external fields this can be reduced to give the standard result

$$S_F(x,m)=-(i\not{\partial}+m)\Delta_F(x,m)\,, \tag{5.32}$$

where the scalar propagator is given by Eq. (2.25). The actual calculation is very similar to that originally done by Schwinger[8] and can be found in the Appendix.

VI. SPINNING PARTICLE IN A GRAVITATIONAL FIELD

The coupling of a spinless particle to an external gravitational field was obtained by the minimal substitution $\eta_{\mu\nu}\to g_{\mu\nu}(x)$ in the Lagrangian (2.4). Similarly, for a particle with spin we can find an interaction from the minimal substitution

$$\eta_{\mu\nu}\to g_{\mu\nu}(X) \tag{6.1}$$

in the Lagrangian (3.20). As in the electromagnetic case it will conserve the supersymmetry.

Making the Taylor expansion

$$g_{\mu\nu}(X)=g_{\mu\nu}(x)+i\theta\xi^\lambda g_{\mu\nu,\lambda} \tag{6.2}$$

and doing the θ integration in

$$L=\tfrac{1}{4}\int d\theta g_{\mu\nu}(X)\dot{X}^\mu DX^\nu\,, \tag{6.3}$$

we find the new Lagrangian

$$L=\tfrac{1}{4}g_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu-i\xi^\mu\dot{\xi}^\nu)-\tfrac{1}{4}ig_{\mu\nu,\lambda}\dot{x}^\mu\xi^\nu\xi^\lambda\,, \tag{6.4}$$

The canonical momenta become

$$\zeta_\mu=\frac{i}{4}\,\xi_\mu\,, \tag{6.5a}$$

$$p_\mu=\tfrac{1}{2}\dot{x}_\mu-\tfrac{1}{2}S^{\lambda\nu}\Gamma_{\lambda\nu\mu}\,, \tag{6.5b}$$

where the Christoffel symbol is given in Eq. (2.16).

It is now pretty straightforward to find the equations of motion. The spin variables change according to

$$\dot{\xi}^\mu+\Gamma^\mu{}_{\alpha\beta}\xi^\alpha\dot{x}^\beta=0\,, \tag{6.6}$$

which means that they are parallel transported along the trajectory of the particle. The position coordinates satisfy the modified geodesic equation

$$\ddot{x}^\mu+\Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=ig^{\mu\lambda}[\Gamma_{\sigma\lambda\alpha}\xi^\alpha\dot{\xi}^\sigma-\tfrac{1}{2}(g_{\lambda\beta,\nu\alpha}-g_{\nu\beta,\lambda\alpha})\dot{x}^\nu\xi^\alpha\xi^\beta]\,, \tag{6.7}$$

which is seen to depend on the spin of the particle. When using Eq. (6.6) it simplifies to

$$\ddot{x}^\mu+\Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=-R^\mu{}_{\nu\alpha\beta}\dot{x}^\nu S^{\alpha\beta}\,, \tag{6.8}$$

where $S^{\alpha\beta}$ is the spin tensor and $R^\mu{}_{\nu\alpha\beta}$ the Riemann curvature tensor. It is given by

$$R_{\lambda\nu\alpha\beta}S^{\alpha\beta}=\frac{i}{2}\xi^\alpha\xi^\beta(g_{\lambda\beta,\nu\alpha}-g_{\nu\beta,\lambda\alpha}-2g_{\rho\sigma}\Gamma^\rho{}_{\lambda\alpha}\Gamma^\sigma{}_{\nu\beta})\,. \tag{6.9}$$

Equation (6.8) has previously been obtained by Papapetrou[9] by considering the motion of a macroscopic body in the limit where its size went to zero for constant spin angular momentum. More recently it has also been derived by Barducci *et al.*[12] in a vierbein extension of their classical theory of spinning particles. In their more complicated way of obtaining this result no use has been made of the supersymmetry in the problem. Indeed, it is not at all obvious that there is such an extra symmetry in their formulation of the theory.

Only in very rare astrophysical situations involving extremely large curvatures will the spin-dependent term in Eq. (6.8) affect the motion of the particle. But in principle, a spinning particle will always move along a trajectory slightly different from a particle with no spin.

The supersymmetry of the Lagrangian (6.3) means again that the quantity

$$Q = \tfrac{1}{2}\xi^{\mu}\dot{x}_{\mu} \tag{6.10}$$

is a constant of motion. In fact, we find from the equations of motion

$$\dot{Q} = \frac{i}{4}\dot{x}^{\mu}R_{\mu\nu\alpha\beta}\xi^{\nu}\xi^{\alpha}\xi^{\beta},$$

which is zero from the Bianchi identity

$$R_{\mu[\nu\alpha\beta]} \equiv R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0. \tag{6.11}$$

Papapetrou's equation (6.8) would not result in this theory if the coupling to gravity was not made to be supersymmetric.

A quantum-mechanical description of the spinning particle in a gravitational field can be obtained from the Hamiltonian which follows from Eqs. (6.4) and (6.5):

$$H = \Pi^2. \tag{6.12}$$

The kinematical momentum is now

$$\Pi_{\mu} = \tfrac{1}{2}\dot{x}_{\mu} = p_{\mu} + \tfrac{1}{2}S^{\lambda\nu}\,\Gamma_{\lambda\nu\mu} \tag{6.13}$$

and the canonical commutators are

$$[\Pi_{\mu}, x_{\nu}] = -ig_{\mu\nu}, \tag{6.14}$$

$$\{\xi_{\mu}, \xi_{\nu}\} = -2g_{\mu\nu}, \tag{6.15}$$

where $g_{\mu\nu} = g_{\mu\nu}(x)$. Hence the quantized spin variables ξ_{μ} will vary with the position of the particle according to the operator version of Eq. (6.6). In the neighborhood of every point one can then find solutions satisfying the anticommutator (6.15) in terms of ordinary Dirac matrices. Equivalent solutions should be related by similarity transformations

$$\xi'_{\mu}(x) = S\xi_{\mu}(x)S^{-1}, \tag{6.16}$$

where $S = S(x)$ is a local transformation matrix. The anticommutator (6.15) is invariant under these transformations. But as pointed out by Schrödinger,[13] this gives well-defined solutions only when the curvature of spacetime vanishes. This is so because Eq. (6.6) is not invariant under these similarity transformations.

A consistent quantum theory can be obtained by observing that the local similarity transformation (6.16) is really a gauge transformation consisting of a local Lorentz transformation of an orthonormal vierbein which can be chosen freely at every point. It is then possible to couple the spinning particle to a gravitational field in both a supersymmetric and gauge-invariant way similarly to what we did when coupling to electromagnetism. This must be done within the vierbein formalism. One can then derive an operator equation of motion of the same form as Schwinger's equation (5.10). It is a quantum-mechanical generalization of Papapetrou's equation (6.8) involving additional couplings to the curvature of the gravitational field.[14]

VII. DISCUSSION AND CONCLUSION

It is of some interest to understand the origin of the simple supersymmetry we have found in this classical counterpart to Dirac's theory. A consistent description of the spin degrees of freedom can be obtained at least formally with the use of anticommuting Grassmann variables. The supersymmetry of the dynamics is an expression of the possibility for the position coordinates of the spinning particle to vary along these spin directions. This is reminiscent of the *Zitterbewegung* in ordinary quantized Dirac theory.[15] It leads to the introduction of the superposition of the particle in Eq. (3.14) where the last term then would be the classical representation of this *Zitterbewegung*.

The supersymmetric coupling to electromagnetism in Eq. (4.1) where we take the superposition as the argument in the electromagnetic potential is also very similar to what is done in order to calculate the energy shift of a Dirac particle in an electromagnetic field. It corresponds to a certain lack of localizability of the spinning particle and also gives rise to the spin term in Papapetrou's equation (6.8). This is perhaps not so surprising because this equation was originally derived from considerations of the motion of an extended object.

We can try to quantify these analogs by considering the ordinary Dirac equation. It can be written covariantly as

$$i\frac{\partial}{\partial\tau}\psi = H\psi, \tag{7.1}$$

where $\psi = \psi(x,\tau)$ and τ is ordinary proper time.

The Hamiltonian is now

$$H = \not{p} . \tag{7.2}$$

For a particle with definite mass m the wave function will be

$$\psi = \psi(x) e^{i m\tau} \tag{7.3}$$

and the position operator will vary according to

$$\dot{x}_\mu = i[H, x_\mu] = \gamma_\mu . \tag{7.4}$$

Its momentum p_μ is a constant of motion while

$$\dot{\gamma}_\mu = 2i(m\gamma_\mu - p_\mu) . \tag{7.5}$$

This equation can now be integrated to give

$$\gamma_\mu(\tau) = \frac{p_\mu}{m} + \left(\gamma_\mu(0) - \frac{p_\mu}{m}\right) e^{2i m\tau} . \tag{7.6}$$

The full trajectory then follows from the integration of Eq. (7.4):

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu}{m}\tau + \frac{1}{2im}\left(\gamma_\mu(0) - \frac{p_\mu}{m}\right) e^{2i m\tau} . \tag{7.7}$$

The first two terms describe the smooth motion of the "center of mass" while the last term gives the rapidly oscillating *Zitterbewegung*. Its amplitude is of the same size as the Compton wave length of the particle. Averaged over time this term gives zero.

Let us compare these conventional results with what we find in the supersymmetric formulation. Equation (3.3a) gives by integration

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu}{m}\tau \tag{7.8}$$

since $\tau = 2ms$. This obviously represents the first smooth term in Eq. (7.7). When we now consider the superposition (3.14), the last term can be considered as a classical expression of the quantum mechanical *Zitterbewegung*. The fact that θ is an odd Grassmann variable with $\theta^2 = 0$ may be a reflection of the extremely rapid oscillations in the *Zitterbewegung* which makes it average out to zero over finite time intervals. The correspondence is even better in the quantized theory where ξ_μ can be represented by Dirac matrices. If one could attribute a size to the variable θ, it would have to be something of the order $1/m$ which is seen from the wave function (5.23). So this would also set the correct scale for the *Zitterbewegung*.

From Eq. (7.4) we see that a conventional Dirac particle moves with the velocity of light. Much of this velocity goes into the very rapid *Zitterbewegung* so that the particle on the average moves somewhat slower. The position variable $x_\mu(s)$ in the supersymmetric theory is therefore not really the position of the particle but rather of its "center of mass". The actual position of the spinning point particle is instead given by the superposition $X_\mu(s, \theta)$ which then includes the *Zitterbewegung*.

The supersymmetric theory we have presented here helps to complete the understanding we have of spinning, relativistic particles in that it gives a classical counterpart to the conventional Dirac theory and ties naturally up with it when quantized. It is certainly somewhat surprising that this formulation is endowed with the additional supersymmetry which is realized here in the most simple way. But traces of this extra symmetry could also have been seen in other formulations like Schwinger's proper-time description where $\not{\pi}$ is a constant of motion. After this realization it could then be natural to look for the underlying symmetry giving rise to such a conserved quantity. It would then have been not too difficult to unravel the supersymmetry of the problem which we found in the classical formulation of the theory.

The supersymmetry transformation (3.13) mixes the commuting position variables x_μ with the anticommuting spin variables ξ_μ. Both transform as four vectors under the Lorentz group. In a quantum field theory this would correspond to supersymmetry transformations between fields with opposite statistics but with the same spin. Exactly this kind of supersymmetry has recently been observed in gauge theories[16] and theories for spin glasses.[17] It represents a new and exciting application of supersymmetric ideas and concepts and could lead to new physical insight of fundamental significance.

APPENDIX

The Feynman propagator for a scalar particle is given by the matrix element in Eq. (2.22). It can be calculated by inserting a complete set of momentum eigenstates $|p\rangle$ normalized according to

$$\langle p|p'\rangle = (2\pi)^4 \delta(p - p') , \tag{A1}$$

so that plane waves are given by

$$\langle x|p\rangle = e^{ix\cdot p} . \tag{A2}$$

Making use of the integral

$$\int d^4x\, e^{iux^2} = \frac{i\pi^2}{u^2}\frac{u}{|u|} , \tag{A3}$$

we then find

$$\Delta_F(x, s) = \frac{-i}{(4\pi s)^2}\exp\left(\frac{ix^2}{4s}\right) , \tag{A4}$$

where $x = x_B - x_A$.

The propagator for a particle with a definite mass m is now obtained from Eq. (2.24). This integral is most easily done by making a Fourier transformation to momentum space:

$$\Delta_F(p,m)=\int d^4x\,\Delta_F(x,m)e^{-ip\cdot x}$$

$$=\frac{-i}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\int d^4x\,e^{-ip\cdot x-im^2s-x^2/4is}. \tag{A5}$$

Again using (A3), this gives the result

$$\Delta_F(p,m)=\int_0^\infty ds\,e^{-is(p^2+m^2)} \tag{A6}$$

$$=\frac{-i}{p^2+m^2-i\epsilon}, \tag{A7}$$

where the $i\epsilon$ is needed to make the integral well-behaved when $s\to\infty$.

Similarly, the propagator for a free Dirac particle is given by Eq. (5.31) which can be written as

$$S_F(x,m)=\int_0^\infty ds\,e^{-im^2s}\langle x_B(s)|\not{\Pi}-m|x_A(0)\rangle. \tag{A8}$$

This matrix element can be calculated from the equations of motion $\dot{\Pi}_\mu=0$ and $\dot{x}_\mu=2\Pi_\mu$. They give

$$\Pi_\mu=\frac{1}{2s}[x_\mu(s)-x_\mu(0)], \tag{A9}$$

so that

$$\langle x_B(s)|\not{\Pi}|x_A(0)\rangle=\frac{\not{x}}{2s}\langle x_B(s)|x_A(0)\rangle\ . \tag{A10}$$

Equation (A8) then becomes

$$S_F(x,m)=\int_0^\infty ds\,e^{-im^2s}\left(\frac{\not{x}}{2s}-m\right)\Delta_F(x,s) \tag{A11}$$

$$=-(i\not{\partial}+m)\Delta_F(x,m), \tag{A12}$$

which is the desired result.

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