Simple connection between conservation laws in the Korteweg–de Vries and sine-Gordon systems

Alan Chodos
Department of Physics, Yale University, New Haven, Connecticut 06520
(Received 7 August 1979)

An infinite sequence of conserved quantities follows from the Lax representation in both the Korteweg–de Vries and sine-Gordon systems. We show that these two sequences are related by a simple substitution. In an appendix, two different methods of deriving conservation laws from the Lax representation are presented.

I. INTRODUCTION

In this note we point out a connection between conserved quantities in a pair of two-dimensional field theories: sine-Gordon and Korteweg–de Vries. They are among the simplest of the completely integrable models that have been attracting a great deal of attention in the literature,[1-3] and from that point of view it is perhaps not surprising that they should somehow be connected.

Nevertheless, the structure of the two theories is quite different, and the existence of any direct connection is certainly not apparent upon casual inspection. The Korteweg–de Vries (KdV) equation arises in the study of shallow waves,[4] and is a nonrelativistic nonlinear evolution equation, with one time derivative and three space derivatives. The sine-Gordon equation, on the other hand, possesses relativistic invariance, and it first arose in a purely geometrical context.[5]

Thus, it may be of some interest to uncover a link between these two seemingly unrelated evolution equations. In Sec. II we briefly review the Lax form[6] of these equations, and in Sec. III we derive the connection between the respective conservation laws. In an appendix, somewhat off the main line of argument, we present two methods of proceeding from the Lax form to conservation laws. One, perhaps excessively naive but straightforward, we believe to be original; the other is a summary of some work that has appeared in the mathematical literature,[7] and is included with the intention of introducing the machinery of pseudo-differential operators into this branch of the physics literature.[8]

II. LAX FORM FOR SINE-GORDON AND KdV EQUATIONS

The Korteweg–de Vries equation is

$$\dot{u}(x,t)=6uu'-u''' ,$$

where $\dot{u}=\partial u/\partial t$, $u'=\partial u/\partial x$. It can be recast in the Lax form

$$\dot{L}=[L,B], \tag{1}$$

where

$$L=-\frac{\partial^2}{\partial x^2}+u(x,t),$$
$$B=4\frac{\partial^3}{\partial x^3}-3\left(u\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u\right).$$

The advantage in so doing is that the solution to Eq. (1), viz.,

$$L(t)=S(t)L(0)S^{-1}(t),\quad \dot{S}=-BS \tag{2}$$

immediately implies that the spectrum of L is conserved. That is, consider the eigenvalue problem

$$L(t)\phi(t)=\lambda\phi(t).$$

Clearly, if $\phi(0)$ is an eigenfunction of $L(0)$ with eigenvalue λ then

$$\phi(t)=S(t)\phi(0)$$

is an eigenfunction of $L(t)$ with the same value of λ. The conservation of the spectrum of L is the origin of an infinite sequence of conserved quantities in the KdV system. These may be viewed as the traces of powers of L:

$$C_p=\mathrm{Tr}(L^p),\quad p=1,2,\ldots \tag{3}$$

where Tr denotes the operator trace taken in the space on which L acts. Since this operation is not well-defined, a little care must be used in extracting meaningful functionals from Eq. (3). This is treated more fully in the Appendix.

The sine-Gordon equation, written in light-cone variables,

$$\dot{v}'=-\sin v$$

[here $\dot{f}=\partial f/\partial x_+$, $f'=\partial f/\partial x_-$, where $x_\pm=\frac{1}{2}(x\pm t)$] also possesses a Lax representation, Eq. (1), where now the operators L and B, in addition to acting on a suitable function space, are endowed with a 2×2 matrix structure[3]:

$$L=2\sigma_3\frac{\partial}{\partial x}+\sigma_2 v',$$
$$B=\tfrac{1}{2}[\sigma_3\cos v+\sigma_2\sin v)]L^{-1}.$$

 © 1980 The American Physical Society

Once again, the Lax form implies an infinite sequence of conservation laws. (Actually, since we are using light-cone variables, the conserved quantities will be integrals of functions of v over the variable x_-, which will then be independent of x_+.) What we shall see in the next section is that this sequence can be obtained at once from the corresponding KdV sequence by means of a simple substitution.

III. THE CONNECTION

Let us look at the eigenvalue problem for the sine-Gordon case

$$L_{SG}\,\psi=\lambda\psi\,. \tag{4}$$

Here ψ is a two-component object

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

and Eq. (4) becomes, explicitly,

$$2\psi_1'-iv'\psi_2=\lambda\psi_1\,,$$

$$-2\psi_2'+iv'\psi_1=\lambda\psi_2\,.$$

Let

$$\phi\equiv\psi_1+\psi_2,\quad \chi\equiv\psi_1-\psi_2\,.$$

Then

$$2\chi'+iv'\chi=\lambda\phi\,,$$

$$2\phi'-iv'\phi=\lambda\chi\,.$$

From these equations we eliminate χ to obtain a single second-order equation for ϕ. It is

$$-\phi''-\tfrac{1}{4}[v'^2-2iv'']\phi=-\tfrac{1}{4}\lambda^2\phi\,.$$

This is identical in form to the corresponding KdV eigenvalue problem

$$L_{\rm KdV}\phi=\hat{\lambda}\phi\,,$$

provided we let

$$u(x,t)\to-\tfrac{1}{4}[(v')^2-2iv''] \tag{5}$$

and set

$$\hat{\lambda}=-\tfrac{1}{4}\lambda^2\,.$$

This does not imply that if v satisfies the sine-Gordon equation then $-\frac{1}{4}[(v')^2-2iv'']$ will satisfy the KdV equation. But it *does* imply that if v satisfies the sine-Gordon equation, then $Q_p\equiv{\rm Tr}\mathcal{L}^p$ will be conserved, where

$$\mathcal{L}=-\frac{\partial^2}{\partial x^2}-\tfrac{1}{4}[(v')^2-2iv'']\,.$$

However, $\mathcal{L}$ is the KdV Lax operator with the replacement given in Eq. (5). Thus, these traces are exactly the conserved quantities associated with the KdV equation, with the indicated substitution.

For example, the first few KdV conserved quantities are[2]

$$C_1=\int_{-\infty}^{\infty}u\,dx\,,$$

$$C_2=\int_{-\infty}^{\infty}u^2\,dx\,,$$

$$C_3=\int_{-\infty}^{\infty}[u^3+\tfrac{1}{2}(u')^2]dx\,,$$

and

$$C_4=\int_{-\infty}^{\infty}[u^4+2u(u')^2+\tfrac{1}{5}(u'')^2]dx\,. \tag{6}$$

Letting

$$u=-\tfrac{1}{4}[(v')^2-2iv'']\,,$$

we obtain a corresponding sequence

$$Q_1=-\tfrac{1}{4}\int_{-\infty}^{\infty}(v')^2dx_-\,,$$

$$Q_2=\tfrac{1}{16}\int_{-\infty}^{\infty}[(v')^4-4(v'')^2]dx_-\,,$$

$$Q_3=-(\tfrac{1}{4})^3\int_{-\infty}^{\infty}[(v')^6-20(v')^2(v'')^2+8(v''')^2]dx_-\,,$$

$$Q_4=(\tfrac{1}{4})^4\int_{-\infty}^{\infty}[(v')^8-\tfrac{112}{5}(v'')^4-56(v')^4(v'')^2+\tfrac{224}{5}(v')^2(v''')^2-\tfrac{64}{5}(v'''')^2]dx_-\,.$$

It is straightforward, although tedious, to check explicitly that $Q_1,\ldots,Q_4$ are conserved, provided v obeys the sine-Gordon equation.

At first glance, it might seem that each C_p gives rise to a pair of conserved quantities in the sine-Gordon system since the argument of the functional, Eq. (5), is complex, and thus

$$Q_p=Q_p^{(r)}+iQ_p^{(i)}\,.$$

However, it is a general property that $Q_p^{(i)}$ vanishes identically for all p. The reason for this is given in the Appendix.

In another sense, though, there *are* a pair of conservation laws for each Q_p. This is because Q_p satisfies

$$\frac{dQ_p}{dx_+}=0\,,$$

and there are two inequivalent ways of going to ordinary space and time. For example,

$$Q_1=\int dx_-\mathcal{Q}_1,\quad \mathcal{Q}_1=-\tfrac{1}{4}(v')^2\,,$$

$$\dot{\mathcal{Q}}_1=-\tfrac{1}{2}\dot{v}'v'=\tfrac{1}{2}v'\sin v=-\tfrac{1}{2}\frac{\partial}{\partial x_-}(\cos v)\,.$$

Thus,

$$\frac{\partial}{\partial x_+}\left[-\frac{1}{4}\left(\frac{\partial v}{\partial x_-}\right)^2\right]=\frac{\partial}{\partial x_-}(-\tfrac{1}{2}\cos v).$$

If we identify, as above,

$$\frac{\partial}{\partial x_+}=\frac{\partial}{\partial x}+\frac{\partial}{\partial t},\quad \frac{\partial}{\partial x_-}=\frac{\partial}{\partial x}-\frac{\partial}{\partial t},$$

this then reads

$$\frac{\partial}{\partial t}\left[-\frac{1}{4}\left(\frac{\partial v}{\partial t}\right)^2-\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{2}\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}-\tfrac{1}{2}\cos v\right]$$
$$=\frac{\partial}{\partial x}\left[\frac{1}{4}\left(\frac{\partial v}{\partial t}\right)^2+\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2-\frac{1}{2}\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}-\tfrac{1}{2}\cos v\right].$$

However, it is equally permissible to reverse the roles of x_+ and x_-, since the sine-Gordon equation treats them symmetrically. We then obtain

$$\frac{\partial}{\partial t}\left[-\frac{1}{4}\left(\frac{\partial v}{\partial t}\right)^2-\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2-\frac{1}{2}\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}-\tfrac{1}{2}\cos v\right]$$
$$=\frac{\partial}{\partial x}\left[-\frac{1}{4}\left(\frac{\partial v}{\partial t}\right)^2-\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2-\frac{1}{2}\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}+\tfrac{1}{2}\cos v\right].$$

Thus, Q_1 leads to two conservation laws in ordinary spacetime which turn out in this case to be proportional to the sum and difference of the energy and momentum. A similar result will hold for Q_p, $p>1$.

In this paper, by means of rather straightforward arguments, we have provided a direct correspondence between infinite sequences of conservation laws in the Korteweg–de Vries and sine-Gordon systems.[9] Whether this is merely fortuitous, or whether it bespeaks a more profound connection between the two theories remains an open question.

ACKNOWLEDGMENTS

I wish to thank Vincent Moncrief and Charles Sommerfield for helpful conversations. This research was supported in part by the U. S. Department of Energy under Contract No. EY-76-C-02-3075.

APPENDIX

In this appendix we discuss two methods of deriving the conservation laws [Eqs. (6), etc.] from the Lax representation of the KdV equation [Eq. (1)]. The first method demonstrates how to go directly from the expression $\mathrm{Tr}(L^p)$ to C_p. The second is more rigorous, and introduces another "trace" operation which is better defined.

1. Naive method

We have

$$L(x,y)\equiv-\frac{\partial^2}{\partial x^2}\delta(x-y)+u(x)\delta(x-y),$$

$$L^2(x,y)=\int dz\,L(x,z)L(z,y)$$
$$=\frac{\partial^4}{\partial x^4}\delta(x-y)$$
$$-[u(x)+u(y)]\frac{\partial^2}{\partial x^2}\delta(x-y)+u^2\delta(x-y),$$

etc. The traces are then defined as

$$\mathrm{Tr}L=\int dx\,dy\,\delta(x-y)L(x,y)$$
$$=-V\delta''(0)+\delta(0)\int dx\,u(x)$$

and

$$\mathrm{Tr}L^2=V\delta''''(0)-2\delta''(0)\int dx\,u(x)$$
$$+\delta(0)\int dx\,u^2(x),$$

where

$$V\equiv\int_{-\infty}^{\infty}dx.$$

Clearly these expressions are ill-defined, and succeeding traces will contain even worse singularities. With a little imagination and the willingness to throw away irrelevant additive and multiplicative infinities, one can extract the well-defined conserved quantities

$$C_1=\int dx\,u(x),$$

$$C_2=\int dx\,u^2(x)$$

from $\mathrm{Tr}L$ and $\mathrm{Tr}L^2$, respectively. But as soon as the conserved functional contains more than one term, as it does in the case of C_3,

$$C_3=\int dx[u^3+\tfrac{1}{2}(u')^2],$$

it becomes impossible to compute the relative weights of the terms because of the infinities. However, from the structure of L we can extract the following rule: Each power of u is equivalent to two derivatives. So we start with, say, u^4 and add all independent equivalent terms, each term carrying an arbitrary coefficient

$$C_4=\int dx[u^4+\alpha u(u')^2+\beta(u'')^2],$$

the meaning of "independent" is that in each term, the highest derivative must occur to a power greater than one; all other terms can then be obtained from these through integration by parts. For example,

$$\int u^2 u'' dx = -2 \int u(u')^2 dx$$

and

$$\int u'u''' dx = -\int (u'')^2 dx .$$

The coefficients α and β are then determined to be $\alpha = 2$, $\beta = \frac{1}{5}$ by calculating $\dot{C}_4 = 0$ with the help of the KdV equation. The higher conservation laws can be obtained analogously at the cost of increased computational labor.

2. Method using pseudo-differential operators

Consider objects $\phi(x, \xi)$ of the form

$$\phi(x, \xi) = \sum_{j=-\infty}^{N} a_j(x)\xi^j . \quad \text{(A1)}$$

Here x may be thought of as a real variable defined over an appropriate domain, and the $a_j(x)$ are suitable well-behaved functions. As we shall see, ξ will behave like $-i\partial/\partial x$. The value of N may change from ϕ to ϕ. We define the product of two ϕ's as

$$\phi_1 \cdot \phi_2 \equiv \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \frac{\partial^\nu \phi_1}{\partial \xi^\nu} \frac{\partial^\nu \phi_2}{\partial x^\nu} .$$

This can be proved to be associative and distributive, but is not commutative. In fact, if we consider first $\phi_1 = \xi$, $\phi_2 = a(x)$, and then $\phi_1 = a(x)$, $\phi_2 = \xi$, we find

$$\xi \cdot a - a \cdot \xi = -ia' ,$$

which is consistent with our interpretation of ξ.

The crucial step is now to define the trace of ϕ as

$$\mathrm{Tr}\phi \equiv \int_{-\infty}^{\infty} dx\, a_{-1}(x) . \quad \text{(A2)}$$

It can be proved[7] that this trace has the essential cyclic property

$$\mathrm{Tr}\phi_1 \cdot \phi_2 = \mathrm{Tr}\phi_2 \cdot \phi_1 . \quad \text{(A3)}$$

This is all that is necessary to generate conserved quantities; we let

$$L(x, \xi) = \xi^2 + u(x) .$$

Because of Eqs. (A3) and (2), the trace of L to any power will be conserved. Of course, L to an integral power will have vanishing trace by the definition Eq. (A2). However, each of the objects

$$\phi_N(x, \xi) \equiv [\xi^2 + u(x)]^{(2N-1)/2}, \quad N = 1, 2, \ldots$$

has a power-series expansion of the defining form Eq. (A1), and the quantities

$$C_N \equiv \mathrm{Tr}\phi_N$$

will yield well-defined nontrivial conservation laws. In fact, they are exactly the conserved quantities derived by method 1, the first four of which are listed in Eqs. (6). In practice, the amount of labor needed to calculate C_N explicitly by this method exceeds that of method 1, but it has the advantage of being well-defined at every step.

We now show, as promised in Sec. III, that the Q_N are always real. We consider

$$\phi(x, \xi) = \xi^2 - \tfrac{1}{4}(v')^2 + \tfrac{1}{2}iv''$$

and observe that it can be written

$$\phi = (\xi + \tfrac{1}{2}v') \cdot (\xi - \tfrac{1}{2}v') .$$

The Q_N are given by

$$Q_N = \mathrm{Tr}(\phi)^{(2N-1)/2} .$$

Their complex conjugates Q_N^* are obtained by replacing ϕ with

$$\chi \equiv \xi^2 - \tfrac{1}{4}v'^2 - \tfrac{1}{2}iv''$$
$$= (\xi - \tfrac{1}{2}v') \cdot (\xi + \tfrac{1}{2}v') .$$

However,

$$[(\xi - \tfrac{1}{2}v') \cdot (\xi + \tfrac{1}{2}v')]^{(2N-1)/2} = (\xi - \tfrac{1}{2}v') \cdot [(\xi + \tfrac{1}{2}v') \cdot (\xi - \tfrac{1}{2}v')]^{(2N-1)/2} \cdot (\xi - \tfrac{1}{2}v')^{-1} ,$$

as can easily be checked by squaring both sides. Hence, using the cyclic property of the trace, we have immediately

$$Q_N^* = Q_N$$

as required.

[1]A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973); R. M. Miura, J. Math. Phys. 9, 1202 (1968); K. Pohlmeyer, Commun. Math. Phys. 46, 207 (1976); B. Yoon, Phys. Rev. D 13, 3440 (1976); A. M. Polyakov, Phys. Lett. 72B, 224 (1977); P. Kulish and E. R. Nisimov, Pis'ma Zh. Eksp. Teor. Fiz. 24, 247 (1976) [JETP Lett. 24, 220 (1976)]; B. Hasslacher and A. Neveu, Nucl. Phys. B151, 1 (1979); M. Lüscher,

ibid. B135, 1 (1978).
[2]R. M. Miura, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. 9, 1204 (1968).
[3]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973); 31, 125 (1973).
[4]D. J. Korteweg and G. de Vries, Philos. Mag. 39, 442 (1895).
[5]F. Lund and T. Regge, Phys. Rev. D 14, 1524 (1976), especially Sec. III.
[6]P. Lax, Commun. Pure Appl. Math. 21, 467 (1968).
[7]M. Adler, Inventiones Math. 50, 219 (1979).
[8]These techniques have also been used by W. Symes, J. Math. Phys. 20, 721 (1979), in a related context.
[9]Since the literature associated with completely integrable systems is by now so vast, one might wonder how (or indeed whether) this very simple result has escaped notice so far. I have been able to find two papers containing equations that are essentially the same as my equation for ϕ in Sec. III. They are M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. 53, 249 (1974), Eq. (7.6); and G. L. Lamb, Jr., Phys. Rev. A 9, 422 (1974), Eq. (2.16b). However, in both these cases, the authors refrain from drawing the same conclusions that I do from this equation.