Symmetry breaking and mass generation by space-time topology

David J. Toms

Department of Physics and Scarborough College, University of Toronto, West Hill, Ontario, M1C 1A4 Canada

(Received 13 November 1979; revised manuscript received 1 February 1980)

Scalar-field theory with a $\lambda\phi^4$ self-interaction is studied in space-times with a non-Minkowskian topology. An application of the effective potential to symmetry breaking is discussed. In addition, expressions for the vacuum energy density and topological mass to order λ are obtained for a theory which is massless at the tree-graph level. One of the cases which is considered is the scalar-field version of the Casimir effect. In this case we also obtain the one-loop vacuum energy density for the massive scalar field and discuss its renormalization.

I. INTRODUCTION

In the past few years there has been a large number of papers dealing with quantum field theory on a fixed, curved background space-time.[1] Until quite recently only free quantum fields have been considered, although the effects of interactions have now begun to be examined.[2] In addition to the effects of curvature, the role of the topological structure of the space-time manifold has received some attention.

Ford[3] and Ford and Yoshimura[4] have examined the effect of a $\lambda\phi^4$ self-interaction on the vacuum energy and the self-energy of a scalar field in several space-times which are topologically distinct from Minkowski space-time. It was found that as a consequence of both the $\lambda\phi^4$ self-interaction and the nontrivial topology, a field which was massless at the tree-graph level could develop a mass at the one-loop level which depended upon the topology. This phenomenon may be called topological mass generation. The method which was used in the calculations of Refs. 3 and 4 was first-order perturbation theory, where any divergences were regularized away using ζ-function regularization. No renormalization was performed using this procedure. It has been emphasized by Kay[5] that this will not work if the fields are massive. In Ref. 5, Kay proposed a new method for regularizing and renormalizing the energy-momentum tensor which can deal with the massive case. In a previous paper,[6] the author discussed the analogy between field theory in a flat space-time which is given the topology of $S^1\times R^3$ by making a periodic identification in one of the spatial coordinates, and field theory at a finite temperature; this analogy, which holds regardless of whether the fields are massive or not, was then used to deduce the Casimir effect[7] and to derive the topological mass. The method described in the present paper could also be applied to the massive case and is perhaps a more familiar approach than that described in Ref. 5.

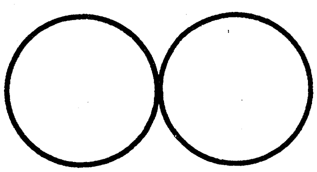
Another role of topology in quantum field theory has been discussed by Isham,[8] where a non-Minkowskian topology can lead to what he refers to as twisted fields. These fields are realized as cross sections of nontrivial (i.e., nonproduct) vector bundles, where the number of such twisted fields is determined by the space-time topology. Ford[9] has considered the vacuum polarization for quantum electrodynamics in a flat space-time with the topology $R^3\times S^1$. Twisted as well as untwisted spinor fields are allowed, and Ford finds that untwisted (i.e., standard) spinor fields lead to noncausal effects, whereas twisted spinor fields give rise to causal effects. These results indicate that twisted fields are of more than just academic interest. In Ref. 6 the renormalization of a twisted scalar field with a $\lambda\phi^4$ self-interaction in a flat space-time with the topology $R^3\times S^1$ was discussed. The renormalization counterterms in the Lagrangian were found to be identical to those for an ordinary scalar field in Minkowski space-time. The regularization of the energy-momentum tensor for free twisted fields in various cases has been discussed by DeWitt, Hart, and Isham.[10]

The effective potential has received a great deal of attention after being brought to prominence by several authors,[11-13] mainly because it allows one to study spontaneous symmetry breaking beyond the tree-graph level. The minima of the effective potential give the ground states of the theory.[14] Another feature of the effective potential $V(\phi)$ which we shall find useful is that $V(\phi)$ gives the energy density of the state for which the expectation value of the field[15] is ϕ.

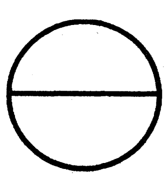
Since one cannot evaluate the effective potential exactly, the usual approach is to resort to a loop expansion.[11-13] We prefer the functional methods described by Jackiw[13] because all of the graphs contributing to the one-loop effective potential are automatically summed, in contrast to Ref. 11. Two- and higher-loop contributions to the effective potential are reduced to an evaluation of a finite number of graphs. For example, Fig. 1 depicts the two graphs which contribute to the two-loop effective potential.

Because the one- and higher-loop contributions to the effective potential are divergent, one must adopt a regularization procedure and then remove the divergences by renormalization. Hawking's[16] version of ζ-function regularization seems to be particularly well suited to the regularization of the determinant which arises from performing a Gaussian functional integration to obtain the one-loop effect.[17] This has been used by Ghika and Visinescu[18] to obtain the one-loop effective potential at a finite temperature for a few models of interest. For the evaluation of the higher-loop contributions dimensional regularization[19] is probably the most convenient. One could of course use it for the one-loop contribution as well. The contribution of the two-loop effects to the vacuum energy density may be obtained without a full calculation of the two-loop effective potential since it arises solely from Fig. 1(a). The complete expression for the two-loop effective potential appears to be difficult to obtain, and requires an evaluation of Fig. 1(b).

For notational purposes, as well as for completeness, in Sec. II we repeat some of the formalism leading to the effective potential[20] and present a short summary of Hawking's[16] version of ζ-function regularization. In Sec. III we examine scalar fields with a $\lambda\phi^4$ self-interaction in space-times which are flat, but which have a non-Minkowskian topology. As an example where the manifold has a nonzero curvature as well as a non-Minkowskian topology, the one-loop effective potential in the static Einstein universe is obtained in Sec. IV. In Sec. V the main results of the paper are summarized and some comments are made. An appendix to the paper is provided where some of the properties of the series which arise in the calculations are presented.



(a)



(b)

FIG. 1. The graphs which contribute to the two-loop effective potential.

II. THE EFFECTIVE POTENTIAL AND ζ-FUNCTION REGULARIZATION

Since we shall use a functional integral approach to discuss the effective potential,[20] it proves convenient to work with Euclidean field theory. The Euclidean manifold, M is obtained by making the $t\to -it$ substitution.[21] If the manifold has an infinite volume it is convenient to first compactify it by adding appropriate boundaries, and then to take the infinite-volume limit after the effective potential has been found. In particular, it is convenient in the functional integral to sum over fields which are periodic in Euclidean time with period β. If this calculation could be fully carried out, we would have results which were valid at a finite temperature[22-24]; however, we shall take the limit $\beta\to\infty$ and so our results will be those of quantum field theory at zero temperature.

The classical Euclidean action is taken to be

$$I_E[\phi]=\int_M \mathcal{L}_E(x)dv_x\,, \tag{1}$$

where $dv_x=\sqrt{g}\,d^4x$ is the invariant volume element on M, and

$$\mathcal{L}_E(x)=-\tfrac{1}{2}(\partial_a\phi)(\partial^a\phi)-\tfrac{1}{12}R\phi^2-U(\phi) \tag{2}$$

is the Lagrangian density. The subscript E is to remind us that this is Euclidean field theory. The term in $\frac{1}{12}R\phi^2$, where R is the scalar curvature, is the usual one for a conformally coupled scalar field; with this term, in the case $U(\phi)=0$, the theory is conformally invariant. In the examples which are discussed in Sec. III, the metric is the Minkowski metric, so that the conformal term in Eq. (2) vanishes identically. In the static Einstein universe which is discussed in Sec. IV, $R=6/a^2$, where a gives the radius of the spatial sections and so the conformal term in Eq. (2) is just like having a massive scalar field.[25] From now on we shall treat R as a constant, although if this is not true, then it may not be possible in general to obtain explicitly some of the results described below.

We are interested in massless $\lambda\phi^4$ theory and so we shall take $U(\phi)$ to be given by

$$U(\phi)=\frac{\lambda}{4!}\phi^4+\frac{\delta\lambda}{4!}\phi^4\,. \tag{3}$$

There is nothing to stop us from adding a mass term to Eq. (3), although the calculation would prove slightly more difficult. In principle, there could also be a counter-term for the conformal term in Eq. (2), although in the examples discussed below we shall see that this is not necessary.

The generating functional expressed as a functional integral is

$$Z_E[J] = \int d[\phi] \exp\left\{\frac{1}{\hbar} I_E[\phi] + \frac{1}{\hbar}\int_M J(x)\phi(x)dv_x\right\}, \tag{4}$$

where $J(x)$ is a scalar source which is kept in so that the Green's functions may be obtained by functional differentiation of Eq. (4) with respect to $J(x)$. In Eq. (4) we shall sum over all fields which are periodic in Euclidean time with period β, and which satisfy certain other boundary conditions which depend upon the manifold M, which is chosen as well as the nature of the field. $Z_E[J=0]$ is also called the partition function. The generating functional for connected graphs, $W_E[J]$, is given in terms of $Z_E[J]$ by

$$W_E[J] = \hbar \ln Z_E[J]\,. \tag{5}$$

Functional differentiation of Eq. (5) with respect to J gives rise to the connected Green's functions.

The Legendre transform of $W_E[J]$, called the effective action, is defined by

$$\Gamma_E[\Phi] = W_E[J] - \int_M J(x)\Phi(x)dv_x\,, \tag{6}$$

where

$$\frac{\delta W_E[J]}{\delta J(x)} = \Phi(x)\,. \tag{7}$$

Expansion of $\Gamma_E[\Phi]$ in position space about Φ = constant gives

$$\Gamma_E[\Phi] = \int_M dv_x[-V(\Phi) + \text{terms in } \partial\Phi]\,. \tag{8}$$

By setting $\Phi = \hat{\phi}$, where $\hat{\phi}$ is a constant field, we then select out

$$V(\hat{\phi}) = \frac{-1}{\mathrm{vol}(M)}\Gamma_E[\hat{\phi}]\,, \tag{9}$$

which is called the effective potential. Here, $\mathrm{vol}(M) = \int_M dv_x$ denotes the volume of the manifold M.

In order to evaluate $\Gamma_E[\hat{\phi}]$, and hence $V(\hat{\phi})$, it is necessary to resort to the loop expansion. The loop expansion may be shown[11,26] to be equivalent to an expansion in powers of $\hbar$. We may define

$$V(\hat{\phi}) = V^{(0)}(\hat{\phi}) + \hbar V^{(1)}(\hat{\phi}) + \hbar^2 V^{(2)}(\hat{\phi}) + O(\hbar^3)\,. \tag{10}$$

The first term in the expansion is just the tree-graph contribution to the effective potential, which in our case is

$$V^{(0)}(\hat{\phi}) = \tfrac{1}{12}R\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4\,. \tag{11}$$

If we define an operator $A(x,y)$ by

$$A(x,y) = -\frac{\delta^2 I_E[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\hat{\phi}}\,, \tag{12}$$

then the one-loop contribution to the effective potential arises from performing the following Gaussian functional integral:

$$V^{(1)}(\hat{\phi}) = \frac{-1}{\mathrm{vol}(M)} \ln \int d[\phi] \exp\left[-\frac{1}{2}\int_M dv_x \int_M dv_y \phi(x)A(x,y)\phi(y)\right]. \tag{13}$$

In our case, Eq. (12) leads to

$$A(x,y) = \left(\frac{\lambda}{2}\hat{\phi}^2 + \tfrac{1}{6}R - \Box_x\right)\delta(x-y)\,, \tag{14}$$

and one can then do the dv_y integration which appears in Eq. (13) leaving

$$V^{(1)}(\hat{\phi}) = \frac{-1}{\mathrm{vol}(M)} \ln \int d[\phi] \exp\left[-\frac{1}{2}\int_M dv_x \phi(x)\left(\frac{\lambda}{2}\hat{\phi}^2 + \tfrac{1}{6}R - \Box_x\right)\phi(x)\right]. \tag{15}$$

The operator $\frac{1}{2}\lambda\hat{\phi}^2 + \frac{1}{6}R - \Box_x$ will be elliptic and self-adjoint with an unbounded, positive spectrum of eigenvalues in the examples dealt with below. We may evaluate the functional integration in Eq. (15) following Hawking.[16]

Let $\psi_N(x)$ denote an eigenfunction of the operator $\frac{1}{2}\lambda\hat{\phi}^2 + \frac{1}{6}R - \Box_x$, with an eigenvalue of a_N. Assume that the eigenfunctions form a complete set and expand the field in terms of the $\psi_N(x)$ as

$$\phi(x) = \sum_N C_N \psi_N(x) \tag{16}$$

for some expansion coefficients C_N. Define

$$d[\phi] = \prod_N \frac{\mu}{(2\pi)^{1/2}}\, dC_N\,, \tag{17}$$

where μ represents some sort of measure on the space of functions. A straightforward calculation of Eq. (15) leads to

$$V^{(1)}(\hat{\phi}) = \frac{1}{2\,\mathrm{vol}(M)} \sum_N \ln\left(\frac{a_N}{\mu^2}\right) , \tag{18}$$

which is divergent. To regularize the expression in Eq. (18), define a generalized ζ function by

$$\zeta(s) = \sum_N a_N^{-s} , \tag{19}$$

which will converge in some region of the complex s plane. Since $\zeta(s)$ is analytic[27] at $s=0$, we may define the regularized expression in Eq. (18) by[18]

$$V^{(1)}(\hat{\phi}) = \frac{-1}{2\,\mathrm{vol}(M)} [\zeta'(0) + \zeta(0)\ln\mu^2] . \tag{20}$$

The term in $\ln\mu^2$ is to be removed by renormalization.

The renormalization counterterms may also be expanded in powers of $\hbar$. The complete, unrenormalized, one-loop effective potential follows from Eqs. (10), (11), and (20) as

$$V(\hat{\phi}) = \tfrac{1}{12} R\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\delta\lambda}{4!}\hat{\phi}^4 - \frac{1}{2\,\mathrm{vol}(M)}[\zeta'(0) + \zeta(0)\ln\mu^2] . \tag{21}$$

We shall see in Secs. III A and IV that the divergent term $[-1/2\,\mathrm{vol}(M)]\zeta(0)\ln\mu^2$ is proportional to $\hat{\phi}^4$ and is independent of the parameters which are associated with the nontrivial topology of the manifold. As a result, the counterterm $\delta\lambda$ will be the same as that for the theory in Minkoswski space-time. We may then impose a renormalization condition in Minkowski space-time by adopting

$$\left.\frac{d^4V(\hat{\phi})}{d\hat{\phi}^4}\right|_{\hat{\phi}=M} = \lambda , \tag{22}$$

where M is an arbitrary number with the dimensions of mass.[11] This condition fixes $\delta\lambda$ in terms of M and μ, and we shall see that the $\ln\mu^2$ term in Eq. (21) disappears.

Once we have calculated the effective potential to a given order in the loop expansion, we can examine the stability of the state $\hat{\phi}=0$. To see if symmetry breaking occurs, one may examine

$$\left.\frac{dV}{d\hat{\phi}}\right|_{\hat{\phi}=v} = 0$$

for any nonzero values of v which minimize $V(\hat{\phi})$. To see if the fields develop a topological mass we just need to calculate

$$m^2 = \left.\frac{d^2V(\hat{\phi})}{d\phi^2}\right|_{\hat{\phi}=v} . \tag{23}$$

Higher-loop effects in the effective potential may be obtained, as discussed by Jackiw.[13] For $\lambda\phi^4$ theory there is an effective interaction of $-(\lambda/3!)\hat{\phi}\phi^3 - (\lambda/4!)\phi^4$ in the Lagrangian. There are two graphs which contribute to the two-loop effective potential shown in Fig. 1. We have as the two-loop contribution to the effective potential

$$V^{(2)}(\hat{\phi}) = \frac{\lambda}{8} I_1(\hat{\phi}) - \frac{\lambda^2}{12}\hat{\phi}^2 I_2(\hat{\phi}) , \tag{24}$$

where $I_1(\hat{\phi})$ and $I_2(\hat{\phi})$ are expressions containing a summation over the propagators corresponding to internal lines arising from Figs. 1(a) and 1(b), respectively. The exact expressions for $I_1(\hat{\phi})$ and $I_2(\hat{\phi})$ depend upon the manifold. Because of the non-Minkowskian topology which is adopted, the Feynman rules are similar to those for field theory at a finite temperature.[22-24] The analogy between field theory at a finite temperature and the examples in Secs. III A and III B has been discussed previously.[6] The rules in Secs. III C and IV follow in a similar manner.

Although it is not possible to obtain a simple result for $I_2(\hat{\phi})$, and hence to find an analytic expression for $V^{(2)}(\hat{\phi})$, it is possible to calculate the two-loop contribution to the vacuum energy density. As a result of the renormalizability discussed in Ref. 6, the divergent parts of $V^{(2)}(\hat{\phi})$ can involve $\hat{\phi}$ only as $\hat{\phi}^4$. Assuming that $\hat{\phi}=0$ gives the vacuum state, the two-loop contribution to the vacuum energy density is finite and given by

$$V^{(2)}(\hat{\phi}=0) = \frac{\lambda}{8} I_1(\hat{\phi}=0) . \tag{25}$$

In order to calculate the two-loop contribution of the effective potential to the vacuum energy density it is not necessary to go through the complete renormalization calculation, which would prove to be extremely difficult.

III. APPLICATION TO FLAT SPACE-TIMES WITH A NON-MINKOWSKIAN TOPOLOGY

A. An untwisted scalar field in periodically identified flat space-time

Consider ordinary flat space-time which is given a topology of $S^1\times S^1\times S^1\times S^1$ by making a periodic identification in each of the coordinates. Take the periodicity in the time coordinate to be β, and in the spatial coordinates to be L_1, L_2, L_3; then the volume of the manifold is $\mathrm{vol}(M) = \beta L_1L_2L_3$. In the functional integral, Eq. (4), we sum over all fields with the above periods. The eigenvalues a_N which enter into the generalized ζ function in Eq. (19) are given by

$$a_N=\frac{\lambda}{2}\phi^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{L_3}\right)^2+\left(\frac{2\pi n_4}{\beta}\right)^2, \tag{26}$$

where N stands for the set (n_1, n_2, n_3, n_4), with each member of this set taking on all integral values. From now on we shall drop the caret which appears over ϕ in Sec. II.

In order to find an explicit result for the ζ function, we shall assume that $\beta, L_2, L_3 \gg L_1$. After the effective potential has been found, we may let $\beta, L_2, L_3 \to \infty$ so that we are working in a flat space-time where one of the spatial coordinates has been given a periodic identification; the relevant topology is then $R^3\times S^1$. With this assumption the ζ function in Eq. (19) is

$$\zeta(s)=\left(\frac{L_2}{2\pi}\right)\left(\frac{L_3}{2\pi}\right)\left(\frac{\beta}{2\pi}\right)\times\int d^3k\sum_{n=-\infty}^{+\infty}\left[\frac{\lambda}{2}\phi^2+\left(\frac{2\pi n}{L_1}\right)^2+k^2\right]^{-s}.$$

This expression may be evaluated to give

$$\zeta(s)=\frac{\beta L_2L_3}{4\pi^2}\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}\left(\frac{2\pi}{L_1}\right)^{3-2s}D(s-\tfrac{3}{2};\nu), \tag{27}$$

where $\nu^2=(\lambda L_1^2/8\pi^2)\phi^2$, and the function $D(s-\frac{3}{2};\nu)$, given in Ref. 18, is discussed in the Appendix. Utilizing results from the Appendix, from Eq. (27) we have

$$\zeta(0)=\frac{\beta L_1L_2L_3}{128\pi^2}\lambda^2\phi^4, \tag{28}$$

$$\zeta'(0)=\frac{4\pi^2\beta L_2L_3}{3L_1^3}\left[D_0(2;\nu)+\left(\frac{8}{3}-\ln\frac{16\pi^2}{L_1^2}\right)\frac{3\lambda^2L_1^4}{512\pi^4}\phi^4\right]. \tag{29}$$

From Eq. (21), the unrenormalized one-loop effective potential is, using Eqs. (28) and (29),

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\frac{\delta\lambda}{4!}\phi^4-\frac{\lambda^2}{256\pi^2}\phi^4\ln\mu^2-\frac{\lambda^2}{96\pi^2}\phi^4+\frac{\lambda^2}{256\pi^2}\phi^4\ln\left(\frac{16\pi^2}{L_1^2}\right)-\frac{2\pi^2}{3L_1^4}D_0(2;\nu). \tag{30}$$

The only divergent term in Eq. (30) is proportional to ϕ^4 and is independent of L_1; thus, we may impose the renormalization condition, Eq. (22), in Minkowski space-time. Letting $L_1\to\infty$ in Eq. (30) and using Eqs. (A13) and (A19), we have

$$V(\phi)\to\frac{\lambda}{4!}\phi^4+\frac{\delta\lambda}{4!}\phi^4-\frac{3\lambda^2}{512\pi^2}\phi^4-\frac{\lambda^2}{256\pi^2}\phi^4\ln\mu^2+\frac{\lambda^2}{256\pi^2}\phi^4\ln\left(\frac{\lambda}{2}\phi^2\right). \tag{31}$$

This is exactly the result which would have been obtained by applying the methods of Sec. II to $\lambda\phi^4$ theory in Minkowski space-time from the start. The renormalization condition, Eq. (22), fixes $\delta\lambda$ by

$$\frac{\delta\lambda}{4!}=\frac{3\lambda^2}{512\pi^2}+\frac{\lambda^2}{256\pi^2}\ln\mu^2-\frac{\lambda^2}{256\pi^2}\left[\ln\left(\frac{\lambda}{2}M^2\right)+\tfrac{25}{6}\right]. \tag{32}$$

Substitution of Eq. (32) back into Eq. (30) gives the renormalized one-loop effective potential to be

$$V(\phi)=\frac{\lambda}{4!}\phi^4-\frac{\lambda^2}{48\pi^2}\phi^4-\frac{2\pi^2}{3L_1^4}D_0(2;\nu)-\frac{\lambda^2}{256\pi^2}\phi^4\ln\left(\frac{\lambda L_1^2M^2}{32\pi^2}\right). \tag{33}$$

By letting $L_1\to\infty$ in Eq. (33) [or else by substituting Eq. (32) into Eq. (31)], we obtain the Coleman-Weinberg[11] result.

We next wish to examine the stability of the state $\phi=0$, which is the ground state at the tree-graph level. The energy density of the state $\phi=0$ may be computed from Eq. (33) using Eqs. (A12) and (A19) to be given by

$$V(\phi=0)=-\frac{\pi^2}{90L_1^4}. \tag{34}$$

This result may be recognized as just the Casimir result[7] for a real scalar field which satisfies periodic boundary conditions on two parallel plates, rather than vanishing boundary conditions.[3,10] One may easily show that $\phi=0$ is a solution to $dV/d\phi=0$. A computation of the mass term in Eq. (23) will tell us whether $\phi=0$ is a local maximum or a local minimum. Using Eqs. (A19) and (A12) one has

$$m^2=\frac{\lambda}{24L_1^2}, \tag{35}$$

so that $\phi=0$ is a local minimum. The topological mass derived in Eq. (35) agrees with the result of Ford and Yoshimura.[4] Because of the analogy between this case and field theory at a finite temperature, which was discussed in Ref. 6, we could also have obtained Eq. (35) from Ref. 28. In order to determine whether or not the state $\phi=0$ remains the ground state of the theory, we must know whether or not it is a global minimum of the effective potential. Because of the complicated form of $V(\phi)$, this is not a straightforward question to answer and we shall return to it after a discussion of the two-loop effects.

The contribution of the two-loop graphs to the effective potential is given in Eq. (24), where $I_1(\phi)$ is computed from Fig. 1(a), and $I_2(\phi)$ is computed from Fig. 1(b). As a result of the discussion of the Feynman rules in Ref. 6, we have

$$I_1(\phi)=\left\{\sum_{n=-\infty}^{+\infty}\frac{1}{L_1}\int\frac{d^3k}{(2\pi)^3}\left[k^2+\left(\frac{2\pi n}{L_1}\right)^2+\frac{\lambda}{2}\phi^2\right]^{-1}\right\}^2 \tag{36}$$

and a similar expression for $I_2(\phi)$ which need not be given here since it is very difficult to evaluate, and it is not needed in order to compute the vacuum energy density. It would, however, be needed in order to evaluate the two-loop contribution to the topological mass.

$I_1(\phi)$ is divergent and needs to be regularized. We shall choose dimensional regularization[19] and turn $\int d^3k/(2\pi)^3$ into $\int d^\omega k/(2\pi)^\omega$. $I_1(\phi)$ will then be defined by analytic continuation back to $\omega=3$. Performing this procedure leads to

$$I_1(\phi=0)=\frac{1}{144L_1^4}\,,$$

where Eqs. (A14) and (A16) have been used. The contribution of the two-loop effects to the energy density of the state $\phi=0$ follows from Eq. (25) as

$$V^{(2)}(\phi=0)=\frac{\lambda}{1152L_1^4}\,.$$

Combining this result with Eq. (34), we see that the total energy density of the state $\phi=0$ to order λ is

$$V(\phi=0)=-\frac{\pi^2}{90L_1^4}+\frac{\lambda}{1152L_1^4}\,, \tag{37}$$

which is in agreement with Ford.[3]

In order to calculate the energy density to the next order in λ, we would require an evaluation of some of the three-loop vacuum bubbles which contribute to the effective potential. They would be expected to make contributions of order λ^2 and higher to the effective potential. This is seen to be quite unlike the situation in Minkoswski space-time,[11] where the nth loop makes a contribution to the effective potential which is of order λ^{n+1}. The argument given in Ref. 11 does not work here because one of the components of the momentum has discrete values and is summed over rather than integrated over.

If we now expand the effective potential in powers of λ, which is equivalent to an expansion in powers of ϕ about $\phi=0$, using results from the Appendix we obtain

$$V(\phi)=-\frac{\pi^2}{90L_1^4}+\frac{\lambda}{1152L_1^4}+\frac{\lambda}{4!}\phi^4+\frac{\lambda}{48L_1^2}\phi^2-\frac{\lambda^{3/2}}{24\sqrt{2}\,\pi L_1}|\phi|^3+O(\lambda^2)\,. \tag{38}$$

The terms which are designated by $O(\lambda^2)$ arise both from further terms in the expansion of Eq. (33) and higher-loop contributions which we have not evaluated. We are justified in neglecting these terms provided that $\lambda\ll1$, and provided that we restrict ourselves to a region where the neglected terms remain smaller than those which have been retained.[29] We may easily verify from Eq. (38) that $\phi=0$ gives the only solution to $dV/d\phi=0$. Thus to the order in which we may consistently work, we may conclude that $\phi=0$ remains as the stable ground state. The result in Eq. (37) then represents the vacuum energy density to order λ.

B. A twisted scalar field in periodically identified flat space-time

A manifold with a topology of $R^3\times S^1$ admits a twisted scalar field[8,10] in addition to an untwisted field. For the twisted field one sums in the functional integral, Eq. (4), over fields which are antiperiodic in the x^1 coordinate, where this coordinate has been periodically identified with period L_1.

An interesting problem now arises if one attempts to apply the formalism of Sec. II to the twisted field. In order to select out the effective potential from the effective action, one relies on the device of expanding the effective action about a constant field; however, in the case of a twisted scalar field the only constant field which is allowed is $\phi=0$. As a result, the twisted scalar field can never develop a vacuum expectation value. One can use the effective potential to give only the energy density of the state $\phi=0$, and one cannot calculate the topological mass as in Sec. III A or discuss the stability of the state $\phi=0$ by use of the effective potential.

Keeping in mind the above remarks, the generalized ζ function in this case is

$$\zeta(s)=\left(\frac{\beta}{2\pi}\right)\left(\frac{L_2}{2\pi}\right)\left(\frac{L_3}{2\pi}\right)\times\sum_{n=-\infty}^{+\infty}\int d^3k\left\{k^2+\left[\frac{\pi}{L_1}(2n+1)\right]^2\right\}^{-s}.$$

Using Eq. (A21), this may be evaluated to give

$$\zeta(s)=\frac{\beta L_2L_3}{4\pi^2}\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}\left(\frac{\pi}{L_1}\right)^{3-2s}G(s-\tfrac{3}{2};0)\,. \tag{39}$$

It is not actually necessary to use results given in the Appendix, since for the massless case considered here, the resulting series may be expressed in terms of Riemann ζ functions. From Eq. (39), one has

$$\zeta(0)=0\,, \tag{40a}$$

$$\zeta'(0)=-\frac{7\pi^2\beta L_2L_3}{360L_1^{\,3}}\,. \tag{40b}$$

Using Eqs. (40a) and (40b) in Eq. (21), the vacuum energy density to the one-loop level is

$$V^{(1)}(\phi=0)=+\frac{7\pi^2}{720L_1^{\,4}}\,. \tag{41}$$

This result agrees with that of Ref. 10, where it is remarked that the twist has increased the energy of the vacuum.

Note that since $\zeta(0)=0$, no renormalization needs to be performed. We shall comment on this fact after discussing the two-loop contribution to the vacuum energy density. This contribution follows from Eq. (25), where

$$I_1(\phi=0)=\left\{\sum_{n=-\infty}^{+\infty}\frac{1}{L_1}\int\frac{d^3k}{(2\pi)^3}\left[k^2+\left(\frac{\pi}{L_1}(2n+1)\right)^2\right]^{-1}\right\}^2. \tag{42}$$

Using dimensional regularization, and results contained in the Appendix, we have after a short calculation

$$I_1(\phi=0)=\frac{1}{576L_1^{\,4}}\,.$$

The two-loop contribution to the energy density of the state $\phi=0$ is then

$$V^{(2)}(\phi=0)=\frac{\lambda}{4608L_1^{\,4}}\,. \tag{43}$$

Combining results in Eqs. (41) and (43), the energy density of the state $\phi=0$ to order λ is then

$$V(\phi=0)=\frac{7\pi^2}{720L_1^{\,4}}+\frac{\lambda}{4608L_1^{\,4}}\,. \tag{44}$$

Again this result has been obtained without renormalization.

To understand why we have obtained finite results without the necessity of renormalization, consider the case of a massive twisted field with a $\lambda\phi^4$ self-interaction. If one proceeds to calculate the one- and two-loop contributions to the energy density as above it will be found that there are divergences, and so a renormalization is necessary. The mass and coupling-constant counterterms, and the field renormalization factor were discussed for this theory in Ref. 6. In addition, in order to renormalize the vacuum energy density one must include a constant counterterm in the Lagrangian. Divergences are found to be independent of L_1, and so one may impose the renormalization in Minkowski space-time by taking the limit $L_1\to\infty$. In particular, the constant counterterm is fixed by demanding that as $L_1\to\infty$ the renormalized vacuum energy density should vanish. On dimensional grounds this constant counterterm is proportional to $m_R^{\,4}$, where m_R is the renormalized mass. (This may also be verified by a direct calculation similar to that which we show in Sec. III C.) If we now go to the massless limit, both this counterterm and the mass counterterm will vanish. Furthermore, because we are only working above to order λ, the field renormalization and the coupling-constant counterterm may be ignored since they are at least of order λ^2. With these remarks in mind, it is certainly no surprise that we have obtained results which are finite without performing renormalization. We emphasize, in agreement with Kay,[5] that this would no longer be true if the field theory were massive. Also, we remark that it would not be true if we were working to order λ^2 even with a massless theory, since graphs with coupling-constant counterterms would have to be included in order to cancel L_1-dependent divergences in the energy density. Finally, one should note that there is no problem in taking the massless limit of a massive theory since twisted scalar field theory is not infrared divergent.

The topological mass has already been calculated in Ref. 6, and here we merely quote the result of

$$m^2=-\frac{\lambda}{48L_1^{\,2}}\,. \tag{45}$$

Since this result is negative it would be tempting to conclude that the state $\phi=0$ is unstable; however, we must be careful since it does not make sense to talk about the effective potential away from $\phi=0$ for this theory. We are presently investigating this problem and we hope to report on it at a later time.

C. A scalar field in flat space-time with parallel conducting plates

Consider a box whose sides are of lengths L_1, L_2, L_3 and are made of a perfectly conducting material. In the functional integral, Eq. (4), we will sum over all fields which are periodic in Euclidean time with period β, and which vanish on the walls of the box. This is to be regarded as the scalar-field version of the Casimir effect.

The effective potential can be used in this case, as in Sec. III B, only to give us the vacuum energy density, since the only scalar field which is constant and which vanishes on the boundaries is $\phi=0$. We cannot define the effective potential away from $\phi=0$. Just as with the twisted scalar field, we cannot expect the scalar field in the Casimir case to develop a vacuum expectation value, since this would not be consistent with the boundary conditions imposed on the field. We shall use the methods of Sec. II to obtain the vacuum energy density, and then calculate the topological mass as in Ref. 6.

In order to show how the vacuum energy density calculation proceeds in a case where renormalization needs to be performed, we shall calculate the energy density for a massive theory at the one-loop level. [We just add a mass term, its counterterm, and a constant counterterm in Eq. (3), and proceed as before.] The eigenvalues a_N, which enter into the generalized ζ function in Eq. (19), are given by

$$a_N=\left(\frac{\pi n_1}{L_1}\right)^2+\left(\frac{\pi n_2}{L_2}\right)^2+\left(\frac{\pi n_3}{L_3}\right)^2+\left(\frac{2\pi n_4}{\beta}\right)^2+m_R^{\,2}\,, \tag{46}$$

where N stands for the set (n_1, n_2, n_3, n_4), with $n_1, n_2, n_3 = 1, 2, 3, \ldots$ and $n_4 = 0, \pm 1, \pm 2, \ldots$. Here m_R denotes the renormalized mass. We shall take the limit of $\beta, L_2, L_3 \gg L_1$ so that our results will hold for infinite parallel conducting plates. In this limit the generalized ζ function is

$$\zeta(s)=\left(\frac{\beta}{2\pi}\right)\left(\frac{L_2}{2\pi}\right)\left(\frac{L_3}{2\pi}\right)\int d^3k\sum_{n=1}^{\infty}\left[k^2+\left(\frac{\pi n}{L_1}\right)^2+m_R^{\,2}\right]^{-s}.$$

Using Eq. (A1) this may be evaluated to give

$$\zeta(s)=\frac{\beta L_2L_3}{4\pi^2}\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}\left(\frac{\pi}{L_1}\right)^{3-2s}F\left(s-\tfrac{3}{2};\,\frac{L_1m_R}{\pi}\right). \tag{47}$$

From results contained in the Appendix one obtains

$$\zeta(0)=\frac{\beta L_1L_2L_3}{32\pi^2}\,m_R^{\,4}\,, \tag{48a}$$

$$\zeta'(0)=\frac{\pi^2\beta L_2L_3}{6L_1^{\,3}}\,F_0\left(2;\,\frac{L_1m_R}{\pi}\right)+\frac{\beta L_1L_2L_3}{32\pi^2}\,m_R^{\,4}\left(\frac{8}{3}-\ln\frac{4\pi^2}{L_1^{\,2}}\right). \tag{48b}$$

In the massless limit, these two equations reduce to

$$\zeta(0)=0\,, \tag{49a}$$

$$\zeta'(0)=\frac{\pi^2\beta L_2L_3}{720L_1^{\,3}}\,, \tag{49b}$$

which has been given in Ref. 16 for the free, charged scalar field. Using Eq. (21), the vacuum energy density at the one-loop level for the massless theory is

$$V^{(1)}(\phi=0)=-\frac{\pi^2}{1440L_1^{\,4}}\,, \tag{50}$$

which is the well-known Casimir result for a real scalar field.[30] This result has followed without renormalization for the same reasons as discussed in Sec. III B.

Returning now to the massive case, from Eq. (21) with the addition of a constant counterterm δC, the energy density of the state $\phi=0$ at the one-loop level follows as

$$V^{(1)}(\phi=0)=\delta C-\frac{m_R^{\,4}}{64\pi^2}\left(\frac{8}{3}-\ln\frac{4\pi^2}{\mu^2L_1^2}\right)-\frac{\pi^2}{12L_1^{\,4}}F_0\left(2;\,\frac{L_1m_R}{\pi}\right), \tag{51}$$

using Eqs. (48a) and (48b). Since the coefficient of the divergent term is independent of L_1, we may fix δC by taking the limit $L_1\to\infty$ in Eq. (51), and then impose the usual requirement that $V^{(1)}(\phi=0)$ vanishes in Minkowski space-time. Using Eq. (A13), we find that this fixes δC as

$$\delta C=\frac{3m_R^{\,4}}{128\pi^2}+\frac{m_R^{\,4}}{64\pi^2}\ln\left(\frac{\mu^2}{m_R^{\,2}}\right); \tag{52}$$

and hence, the renormalized energy density follows from Eq. (51) as

$$V_R^{(1)}(\phi=0)=-\frac{\pi^2}{12L_1^{\,4}}F_0\left(2;\,\frac{L_1m_R}{\pi}\right)-\frac{7m_R^{\,4}}{384\pi^2}+\frac{m_R^{\,4}}{64\pi^2}\ln\left(\frac{4\pi^2}{L_1^{\,2}m_R^{\,2}}\right). \tag{53}$$

One can easily verify that by letting $m_R\to 0$ in this expression, the previous result found for the massless theory in Eq. (50), which was obtained without renormalization, is obtained.

We may continue to calculate the two-loop vacuum energy density for a massive theory, where now one needs to include a one-loop bubble with a mass counterterm as well as the constant counterterm δC. Because the finite part of the expression turns out to be rather intractable, we shall return again to the massless theory, where not surprisingly we obtain a result without renormalization.

The two-loop contribution follows from Eq. (25), where

$$I_1(\phi=0)=\left\{\sum_{n=1}^{\infty}\frac{1}{L_1}\int\frac{d^3k}{(2\pi)^3}\left[k^2+\left(\frac{\pi n}{L_1}\right)^2\right]^{-1}\right\}^2. \tag{54}$$

After dimensional regularization is performed, and use is made of Eq. (A14), we have

$$I_1(\phi=0)=\frac{1}{2304L_1^{\,4}}\,.$$

The two-loop contribution to the vacuum energy density is then

$$V^{(2)}(\phi=0)=\frac{\lambda}{18432L_1^{\,4}}\,. \tag{55}$$

This result is in disagreement with Ford,[3] who obtains an infinite result. The total vacuum energy density to order λ is then

$$V(\phi=0)=-\frac{\pi^2}{1440L_1^4}+\frac{\lambda}{18432L_1^4}. \qquad (56)$$

We may calculate the lowest-order contribution to the topological mass as discussed in Ref. 6 by evaluating the one-particle-irreducible self-energy part to order λ. This shall be done here for the massive theory to show that the mass counterterm which is required is the same as that for a scalar field theory in ordinary Minkowski space-time. The self-energy to order λ is

$$\Sigma^{(1)}=\tfrac{1}{2}(-\lambda)\mu^{4-\omega}\sum_{n=1}^{\infty}\frac{1}{L_1}\int\frac{d^{\omega-1}k}{(2\pi)^{\omega-1}}\left[k^2+\left(\frac{\pi n}{L_1}\right)^2+m_R^{\,2}\right]^{-1}$$
$$=-\frac{\pi\lambda}{2L_1^{\,2}}(4\pi)^{-3/2}\Gamma\left(\frac{3}{2}-\frac{\omega}{2}\right)\left(\frac{\pi}{4\mu^2L_1^{\,2}}\right)^{(\omega-4)/2}F\left(\frac{3}{2}-\frac{\omega}{2};\frac{L_1m_R}{\pi}\right), \qquad (57)$$

using Eq. (A1). Expanding about the simple pole at $\omega=4$, we have

$$\Sigma^{(1)}=-\frac{\lambda m_R^{\,2}}{16\pi^2}(\omega-4)^{-1}+\frac{\lambda}{8L_1^{\,2}}F_0\left(1;\frac{L_1m_R}{\pi}\right)+\frac{\lambda m_R^{\,2}}{32\pi^2}\left(2-\gamma-\ln\frac{\pi}{\mu^2L_1^{\,2}}\right). \qquad (58)$$

We then choose the mass counterterm to be given by

$$\delta m^2=-\frac{\lambda m_R^{\,2}}{16\pi^2}(\omega-4)^{-1} \qquad (59)$$

as in Ref. 6.

The topological mass follows for massless $\lambda\phi^4$ theory by taking the limit $m_R\to 0$ in the renormalized expression for $\Sigma^{(1)}$. Because of the way in which $\Sigma^{(1)}$ has been defined,[6] the topological $(\text{mass})^2$ is the negative of this limit. Thus, we have

$$m^2=\frac{\lambda}{96L_1^{\,2}} \qquad (60)$$

as the topological mass. This result is in disagreement with that obtained by Ford and Yoshimura[4] who claim a result which is both negative and spatially dependent.

IV. APPLICATION TO THE STATIC EINSTEIN UNIVERSE

As an example in which both the curvature and a non-Minkowskian topology are present, we shall examine the static Einstein universe. It was mentioned in Sec. II that because this is a manifold with a constant positive curvature, the conformal term behaves as a mass term in the Lagrangian. This makes it straightforward to calculate explicitly the eigenvalues which enter into Eq. (19), and then to proceed in a manner which is identical to that of Sec. III A.

The static Einstein universe has the line element

$$ds^2=dt^2+a^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\,d\phi^2)]. \qquad (61)$$

If we work initially at a finite temperature, then the manifold has a topology of $S^1\times S^3$, and a volume of

$$\text{vol}(M)=2\pi^2\beta a^3. \qquad (62)$$

The eigenfunctions of the operator $\frac{1}{2}\lambda\phi^2+1/a^2-\Box$ described in Sec. II may be chosen to be[31]

$$\psi_N(x)\propto e^{-(2\pi in/\beta)t}Y_{lm}(\theta,\phi)\sin^l\chi\, C_{k-l}^{l+1}(\cos\chi), \qquad (63)$$

where $C_\beta^\alpha(x)$ denotes a Gegenbauer polynomial, and N stands for the quantum numbers (n,k,l,m) which have the following ranges: $n=0,\pm1,\pm2,\ldots$; $k=0,1,2,\ldots$; $l=0,1,2,\ldots,k$; and $m=-l,-l+1,\ldots,l-1,l$. The eigenvalues a_N associated with the eigenfunctions appearing in Eq. (63) are

$$a_N=\frac{\lambda}{2}\phi^2+\frac{1}{a^2}+\left(\frac{2\pi n}{\beta}\right)^2+\frac{k(k+2)}{a^2}. \qquad (64)$$

In the large-β (i.e., low-temperature) limit, the generalized ζ function is

$$\zeta(s)=\left(\frac{\beta}{2\pi}\right)\int_{-\infty}^{+\infty}dk_4\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=-l}^{+l}\left[\frac{\lambda}{2}\phi^2+\frac{(k+1)^2}{a^2}+k_4^{\,2}\right]^{-s}.$$

This may be evaluated to give

$$\zeta(s)=\left(\frac{\beta}{2\pi}\right)\Gamma\left(\frac{1}{2}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}a^{2s-1}\times[F(s-\tfrac{3}{2};\nu)-\nu^2F(s-\tfrac{1}{2};\nu)], \qquad (65)$$

where $\nu^2=\frac{1}{2}\lambda a^2\phi^2$. From Eq. (65) and results in the Appendix it follows that $\zeta(0)=(\beta a^3\lambda^2/64)\phi^4$, and so no counterterm in ϕ^2 is necessary.

The unrenormalized one-loop effective potential follows from Eq. (21) as

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\frac{1}{2a^2}\phi^2+\frac{\delta\lambda}{4!}\phi^4 -\frac{\lambda^2}{256\pi^2}\phi^4\ln\mu^2-\frac{\lambda^2}{128\pi^2}\phi^4-\frac{\lambda^2}{256\pi^2}\phi^4\ln\left(\frac{a^2}{4}\right) +\frac{1}{4\pi^2a^4}[F_0(2;\nu)-\nu^2F_0(1;\nu)]\,. \quad (66)$$

The only divergent term is proportional to ϕ^4 and independent of the curvature. We may then impose the renormalization condition, Eq. (22), on $V(\phi)$ in Minkowski space-time, which gives

$$\frac{\delta\lambda}{4!}=\frac{3\lambda^2}{512\pi^2}+\frac{\lambda^2}{256\pi^2}\ln\mu^2-\frac{\lambda^2}{256\pi^2}\left[\ln\left(\frac{\lambda}{2}M^2\right)+\frac{25}{6}\right]. \quad (67)$$

The renormalized one-loop effective potential is then

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\frac{1}{2a^2}\phi^2-\frac{7\lambda^2}{384\pi^2}\phi^4 -\frac{\lambda^2}{256\pi^2}\phi^4\ln\left(\frac{\lambda}{8}a^2M^2\right) +\frac{1}{4\pi^2a^4}[F_0(2;\nu)-\nu^2F_0(1;\nu)]\,. \quad (68)$$

The energy density of the state $\phi=0$ follows upon using Eq. (A12) as

$$V(\phi=0)=\frac{1}{480\pi^2a^4}\,. \quad (69)$$

This result was originally given by Ford.[32] The state $\phi=0$ is a solution to $dV/d\phi=0$. The mass term from Eq. (25) is computed to be given by

$$m^2=\frac{1}{a^2}\left(1-\frac{\lambda}{96\pi^2}\right), \quad (70)$$

in agreement with Ford and Yoshimura.[4] Since λ is assumed to be small, we have $m^2>0$, and so $\phi=0$ remains a local minimum of the effective potential. By expanding Eq. (68) in powers of λ, and keeping only terms of order less than λ^2 as before, one can see that there are no other solutions to $dV/d\phi=0$ in this range; thus, $\phi=0$ remains as the vacuum state.

The two-loop contribution to the vacuum energy density follows from Eq. (25), where in this case $I_1(\phi)$ is given by

$$I_1(\phi)=\left\{\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=-l}^{+l}\frac{1}{2\pi^2a^3}\int_{-\infty}^{+\infty}\frac{dk_4}{2\pi}\left[\frac{\lambda}{2}\phi^2+\frac{(k+1)^2}{a^2}+k_4^2\right]^{-1}\right\}^2. \quad (71)$$

One may regularize this expression by replacing $\int_{-\infty}^{+\infty}dk_4/2\pi$ with $\int d^\omega k/(2\pi)^\omega$ and then define $I_1(\phi)$ by analytic continuation of the result to $\omega=1$. If this is done, then one obtains upon the use of Eq. (A14)

$$I_1(\phi=0)=\frac{1}{2304\pi^4a^4}\,. \quad (72)$$

The two-loop contribution to the vacuum energy density is then

$$V^{(2)}(\phi=0)=\frac{\lambda}{18\,432\pi^4a^4}\,. \quad (73)$$

Combining this result with Eq. (69), the vacuum energy density to order λ is

$$V(\phi=0)=\frac{1}{480\pi^2a^4}+\frac{\lambda}{18\,432\pi^4a^4}\,, \quad (74)$$

in agreement with Ford.[3]

V. SUMMARY AND CONCLUSIONS

In Secs. III A and IV, we have seen how one could use the effective potential to discuss symmetry breaking in space-times with a non-Minkowskian topology. As well, we used the effective potential to derive the energy density of the vacuum state and the associated topological mass. We feel that this procedure offers some advantages over other ones because we can obtain the above-mentioned results all from one object rather than having to perform separate calculations for each.

In Sec. III A we found the one-loop effective potential for a massless scalar field with a $\lambda\phi^4$ self-interaction in a flat space-time in which a periodic identification is made in one of the coordinates. The state $\phi=0$ was found to remain as the stable vacuum state to the order at which we were able to consistently work.

In Secs. III B and III C, we have discussed why one cannot use the effective potential to discuss symmetry breaking for the twisted field and in the Casimir case. We were only able to obtain the vacuum energy density. In the massless case it was found that results could be obtained without performing a renormalization, and it was discussed why this should be so. We emphasized that this would not be expected to be true if we were working to order λ^2. The one-loop Casimir result for a massive scalar field and its renormalization were also discussed.

Note added in proof. The discussion of the renormalization for both twisted and untwisted $\lambda\phi^4$ theory in flat space-time with the topology $S^1 \times R^3$ referred to in Ref. 6 has also been presented by N. D. Birrell and L. H. Ford [King's College report (unpublished)]. R. Banach [University of Manchester report (unpublished)] has shown generally that a field theory in a multiply connected space-time will be renormalizable, and with the same choice of counterterms, if it is renormalizable in the covering space.

ACKNOWLEDGMENTS

I would like to thank P. J. O'Donnell for many helpful discussions, and the Natural Sciences and Engineering Research Council of Canada for financial support.

APPENDIX

Define for $\mathrm{Re}(s) > \frac{1}{2}$ a series $F(s;\nu)$ by

$$F(s;\nu) = \sum_{n=1}^{\infty} (n^2+\nu^2)^{-s}. \tag{A1}$$

This series has been examined by Ghika and Visinescu[18] and independently by Ford.[9] It may be summed using a summation formula due to Plana.[33,34] The resulting expression, which is quoted below, may be analytically continued throughout the complex s plane to give a function which is analytic everywhere except at $s=\frac{1}{2}-n$, for $n=0,1,2,\ldots$ where simple poles occur. We shall give the residues at these poles as well as the finite part. A number of other useful expressions are also given.

We have,[18] upon performing the summation in Eq. (A1) for $\mathrm{Re}(s) > \frac{1}{2}$,

$$F(s;\nu) = \tfrac{1}{2}\nu^{1-2s}\Gamma(\tfrac{1}{2})\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \tfrac{1}{2}(1+\nu^2)^{-s} - \int_0^1 (\nu^2+x^2)^{-s}dx + i\int_0^\infty \frac{[(1+ix)^2+\nu^2]^{-s} - [(1-ix)^2+\nu^2]^{-s}}{e^{2\pi x}-1}\,dx. \tag{A2}$$

It is seen from this expression that the only singularities of $F(s;\nu)$ are simple poles coming from the first term in Eq. (A2) given by $s=\frac{1}{2}-n$, for $n=0,1,2,\ldots$. [The last integral in Eq. (A2) may be shown to be analytic in s following a proof given for a similar integral on p. 270 of Ref. 34.]

We may expand $F(s;\nu)$ in a Laurent series about the pole at $s=\frac{1}{2}-n$ as

$$F(s;\nu) = \frac{F_{-1}(n;\nu)}{s+n-\frac{1}{2}} + F_0(n;\nu) + O(s+n-\tfrac{1}{2}). \tag{A3}$$

A straightforward calculation leads to

$$F_{-1}(n;\nu) = \frac{1}{2}\left(\frac{\nu^2}{4}\right)^n \frac{(2n)!}{(n!)^2}, \tag{A4}$$

$$F_0(n;\nu) = \frac{1}{2}\left(\frac{\nu^2}{4}\right)^n \frac{(2n)!}{(n!)^2}\left[\sum_{k=1}^{n}\frac{1}{k} - 2\sum_{k=1}^{n}\frac{1}{2k-1} - \ln\left(\frac{\nu^2}{4}\right)\right] + \tfrac{1}{2}(1+\nu^2)^{n-1/2} - \int_0^1 (\nu^2+x^2)^{n-1/2}dx$$
$$+ i\int_0^\infty \frac{[(1+ix)^2+\nu^2]^{n-1/2} - [(1-ix)^2+\nu^2]^{n-1/2}}{e^{2\pi x}-1}\,dx. \tag{A5}$$

Of particular interest are the cases $n=1,2$. In the case $n=1$, one obtains

$$F_{-1}(1;\nu) = \tfrac{1}{4}\nu^2, \tag{A6}$$

$$F_0(1;\nu) = -\tfrac{1}{4}\nu^2 + \tfrac{1}{4}\nu^2\ln 4 - \tfrac{1}{2}\nu^2\ln[1+(1+\nu^2)^{1/2}] + i\int_0^\infty \frac{[(1+ix)^2+\nu^2]^{1/2} - [(1-ix)^2+\nu^2]^{1/2}}{e^{2\pi x}-1}\,dx. \tag{A7}$$

Although we cannot obtain an analytic result for $F_0(1;\nu)$, we may obtain expansions for both large and small ν. For small ν we may show that

$$F_0(1;\nu) = -\tfrac{1}{12} + \tfrac{1}{2}(\gamma-1)\nu^2 - \tfrac{1}{8}\zeta_R(3)\nu^4 + \tfrac{1}{16}\zeta_R(5)\nu^6 + \cdots, \tag{A8}$$

where $\zeta_R(n)$ denotes the Riemann ζ function, and γ is the Euler constant. For large ν, we have

$$F_0(1;\nu) \simeq -\tfrac{1}{4}\nu^2\ln\left(\frac{\nu^2}{4}\right) - \tfrac{1}{4}\nu^2 - \tfrac{1}{2}\nu. \tag{A9}$$

In the case $n=2$, Eqs. (A4) and (A5) give

$$F_{-1}(2;\nu) = \tfrac{3}{16}\nu^4, \tag{A10}$$

$$F_0(2;\nu) = -\tfrac{7}{32}\nu^4 - \tfrac{1}{8}(1+\nu^2)^{3/2} + \tfrac{3}{8}(1+\nu^2)^{1/2} + \tfrac{3}{16}\nu^4\ln 4$$
$$- \tfrac{3}{8}\nu^4\ln[1+(1+\nu^2)^{1/2}] + i\int_0^\infty \frac{[(1+ix)^2+\nu^2]^{3/2} - [(1-ix)^2+\nu^2]^{3/2}}{e^{2\pi x}-1}\,dx. \tag{A11}$$

Again we may obtain expansions of $F_0(2;\nu)$ in the cases of large and small ν. For small ν we have

$$F_0(2;\nu)=\tfrac{1}{120}-\tfrac{1}{8}\nu^2+(\tfrac{3}{8}\gamma-\tfrac{1}{2})\nu^4 -\tfrac{1}{16}\zeta_R(3)\nu^6+\cdots. \quad (A12)$$

For large ν we have

$$F_0(2;\nu)\simeq-\tfrac{3}{16}\nu^4\ln\left(\frac{\nu^2}{4}\right)-\tfrac{7}{32}\nu^4-\tfrac{1}{2}\nu^3. \quad (A13)$$

Another useful result is that

$$F(s=-(n+\tfrac{1}{2});\nu=0)=\frac{n}{2(n+1)} -\frac{1}{2}\sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k\frac{B_{k+1}}{(k+1)}, \quad (A14)$$

where the B_{k+1} are the Bernoulli numbers. The cases $n=0,1$ of Eq. (A14) agree with results contained in Ref. 18.

A related series is

$$D(s;\nu)=\sum_{n=-\infty}^{+\infty}(n^2+\nu^2)^{-s}, \quad (A15)$$

which converges for $\mathrm{Re}(s)>\frac{1}{2}$. In this region of the complex s plane, from Eq. (A1) we have

$$D(s;\nu)=2F(s;\nu)+\nu^{-2s}. \quad (A16)$$

We may then analytically continue $D(s;\nu)$ throughout the complex s plane, where it will have the same analytic structure as $F(s;\nu)$. Writing

$$D(s;\nu)=\frac{D_{-1}(n;\nu)}{s+n-\frac{1}{2}}+D_0(n;\nu)+O(s+n-\tfrac{1}{2}) \quad (A17)$$

for s in a neighborhood of $s=\frac{1}{2}-n$, it is easy to see that

$$D_{-1}(n;\nu)=2F_{-1}(n;\nu), \quad (A18)$$

$$D_0(n;\nu)=2F_0(n;\nu)+\nu^{2n-1}. \quad (A19)$$

Finally, the series arising in the case of twisted fields in Sec. III B may also be defined in terms of those given above. Define, for $\mathrm{Re}(s)>\frac{1}{2}$,

$$G(s;\nu)=\sum_{n=-\infty}^{+\infty}[(2n+1)^2+\nu^2]^{-s}. \quad (A20)$$

It is easy to see that

$$G(s;\nu)=D(s;\nu)-2^{-2s}D(s;\tfrac{1}{2}\nu). \quad (A21)$$

Expanding about the simple pole at $s=\frac{1}{2}-n$, we have

$$G(s;\nu)=\frac{G_{-1}(n;\nu)}{s+n-\frac{1}{2}}+G_0(n;\nu)+O(s+n-\tfrac{1}{2}), \quad (A22)$$

where

$$G_{-1}(n;\nu)=D_{-1}(n;\nu)-4^{n-1/2}D_{-1}(n;\tfrac{1}{2}\nu), \quad (A23)$$

$$G_0(n;\nu)=D_0(n;\nu)-4^{n-1/2}D_0(n;\tfrac{1}{2}\nu) +4^{n-1/2}D_{-1}(n;\tfrac{1}{2}\nu)\ln 4. \quad (A24)$$

Using Eqs. (A18) and (A4), it may be seen that Eq. (A23) may be simplified to

$$G_{-1}(n;\nu)=F_{-1}(n;\nu). \quad (A25)$$

---

[1]For a review of this subject see, for example, B. S. DeWitt, Phys. Rep. 19C, 295 (1975) or L. Parker, in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum, New York, 1977).

[2]An early exception to this is I. T. Drummond, Nucl. Phys. B94, 115 (1975). More recent papers include T. S. Bunch, P. Panangaden, and L. Parker, University of Wisconsin-Milwaukee report (unpublished); N. D. Birrell, King's College, London, report (unpublished).

[3]L. H. Ford, Proc. R. Soc. London A368, 305 (1979).

[4]L. H. Ford and T. Yoshimura, Phys. Lett. 70A, 89 (1979).

[5]B. S. Kay, Phys. Rev. D 20, 3052 (1979).

[6]D. J. Toms, Phys. Rev. D 21, 928 (1980).

[7]H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).

[8]C. J. Isham, Proc. R. Soc. London A362, 383 (1978).

[9]L. H. Ford, Phys. Rev. D 21, 933 (1980).

[10]B. S. DeWitt, C. F. Hart, and C. J. Isham, Physica (Utrecht) 96A, 197 (1979).

[11]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973); S. Coleman, in *Laws of Hadronic Matter*, edited by A. Zichichi (Academic, New York, 1975).

[12]S. Weinberg, Phys. Rev. D 7, 2887 (1973).

[13]R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[14]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

[15]K. Symanzik, Commun. Math. Phys. 16, 48 (1970). See, also, S. Coleman in Ref. 11.

[16]S. W. Hawking, Commun. Math. Phys. 55, 133 (1977).

[17]A different approach also using generalized ζ functions was proposed by J. S. Dowker and R. Critchley, Phys. Rev. D 13, 3224 (1976) and is used in J. S. Dowker and G. Kennedy, J. Phys. A 11, 895 (1978).

[18]G. Ghika and M. Visinescu, Nuovo Cimento 46A, 25 (1978).

[19]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); C. G. Bollini and J. J. Giambiagi, Nuovo Cimento 12B, 20 (1972).

[20]For those readers who are totally unfamiliar with the effective potential, Refs. 11–14 are recommended. There is also a discussion in E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973). Our notation is closest to that of Jackiw in Ref. 13.

[21]In the examples discussed below this does not cause any problems because the models are all static. It is done only for convenience.

[22]C. Bernard, Phys. Rev. D 9, 3312 (1974).

[23] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).

[24] S. Weinberg, Phys. Rev. D 9, 3357 (1974).

[25] This is true of course for all space-times of constant positive curvature. For models with a constant negative curvature, the conformal term is like having a negative $(\text{mass})^2$ term in the Lagrangian. If the manifold is not one of constant curvature then the calculation described in this paper is not so simple.

[26] Y. Nambu, Phys. Lett. 26B, 626 (1968).

[27] In the examples which are dealt with in this paper, we can show this by explicitly summing the series which occurs in Eq. (19). See Ref. 16 for a discussion of and references to cases in which this cannot be done. One still has $\zeta(S)$ analytic at $S=0$.

[28] M. B. Kislinger and P. D. Morley, Phys. Rev. D 13, 2771 (1976).

[29] These remarks are similar to those made in Ref. 11 in discussing the validity of their symmetry-breaking solutions.

[30] This differs by a factor of $\frac{1}{2}$ from the result presented in Ref. 7 because we have a scalar field rather than the electromagnetic field, and so there is one degree of freedom rather than two.

[31] See L. H. Ford, Phys. Rev. D 14, 3304 (1976), where references to the original sources may be found.

[32] L. H. Ford, Phys. Rev. D 11, 3370 (1975). See also Ref. 31.

[33] E. Lindelöf, *Le calcul des rêsidus* (Gautier-Villars, Paris, 1905).

[34] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1927), p. 145, Exercise 7.