Spin-zero mass spectrum in the one-loop approximation in a linear SU(4) σ model

H. B. Geddes
Department of Physics, Carleton University, Ottawa, Canada
(Received 2 April 1979)

We investigate the spin-zero mass spectrum and the leptonic decay constants in a linear SU(4) meson σ model with $(4,4^*)\oplus(4^*,4)$ chiral-symmetry breaking. Calculations are carried out in the one-loop approximation. A number of solutions are presented.

I. INTRODUCTION

A large number of numerical studies have been undertaken with the SU(2)[1,2] and SU(3)[3-6] σ models. The SU(3) model in particular gives an accurate description of low-energy meson phenomenology. With the advent of charm[7], and perhaps additional flavors,[8] it is interesting to consider the model at the SU(4) level.

Several investigations have been carried out using the SU(4) model with mesons in the tree approximation.[9-13] Reasonable agreement with the experimental mass spectrum has been obtained, although experimental evidence for several members of the SU(4) meson 16-plets is scarce.

Numerical calculations in the one-loop approximation provide a much more stringent test of the model and the theoretical ideas it incorporates. In general, in the spirit of perturbation theory, we require that the values calculated in the one-loop approximation be within 10–20% of their physical values. In addition, the difference between the values in the tree and one-loop approximations should also be within this limit. These conditions impose highly nontrivial constraints on the model solutions.

In this paper we employ the SU(4) linear σ model with mesons incorporating both spontaneous symmetry breaking[14] and explicit symmetry-breaking terms linear in the fields. The symmetric Lagrangian contains the most general nonderivative chiral-invariant couplings. The currents obey the SU(4) current algebra. The axial-vector current divergences obey operator PCAC (partial conservation of axial-vector current). This model has been demonstrated to be renormalizable in the one-loop approximation.[15]

We will be primarily concerned with the mass spectrum and the leptonic decay constants. The masses are obtained for all particles using the two-point function. These quantities have been investigated in the SU(3) model in the one-loop approximation for many solutions, with good results being obtained.[4,6] We are interested in how easily the transition can be made to the SU(4) model.

The SU(4) model has essentially one more parameter than the SU(3) model, but it has six additional masses and a much larger mass splitting to accommodate. This problem is reflected in our SU(4)-model solutions. The calculated high and low masses approximate their experimental values less well than those midrange in the mass spectrum. However, overall the solutions adhere to the perturbation-theory criteria a stated above.

Our solutions also reflect the inherent SU(2) $\times$SU(2) Lagrangian symmetry. The small SU(2) $\times$SU(2)-symmetry breaking supports the conjecture that chiral SU(2) $\times$SU(2) symmetry is almost as good a symmetry as isospin, with corrections to it being of the order of 5–10%.[16] We also find that chiral SU(3) $\times$SU(3) is as good a Lagrangian symmetry as SU(3), although neither approach the success of SU(2) $\times$SU(2).

It has been suggested[11] that the leptonic decay constants of the charmed pseudoscalar mesons are larger than that of the pion by a factor of about 6. We find a much more moderate enhancement in our solutions, with F_D usually less than 200 MeV.

The paper is organized as follows: Sec. II presents a brief description of the linear SU(4) meson σ model. Sections III and IV outline the calculations in the tree and one-loop approximations, respectively. Our numerical inputs and results are discussed in Sec. V, where five solutions are presented. Finally, our conclusions are summarized in Sec. VI.

II. THE SU(4) LINEAR σ MODEL

The SU(4) linear σ model is a straightforward extension of the SU(3) model. The most general, renormalizable, chiral-SU(4) $\times$SU(4)-invariant Lagrangian density is

$$\mathcal{L}_0=\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M\partial^\mu M^\dagger)-\tfrac{1}{2}\mu^2\,\mathrm{Tr}(MM^\dagger)+f_1[\mathrm{Tr}(MM^\dagger)]^2+f_2\,\mathrm{Tr}(MM^\dagger MM^\dagger)+g(\det M+\det M^\dagger)\,. \qquad (2.1)$$

All terms but the last one are invariant under U(4) × U(4).

M and $M^\dagger$ are the 4 × 4 matrices of fields that transform as the (4, 4*) and (4*, 4) representations of chiral SU(4) × SU(4), respectively. M can be expressed as

$$M=(1/\sqrt{2})\lambda^i(\sigma_i+i\phi_i)\,, \tag{2.2}$$

where σ_i and ϕ_i represent 16-plets of scalar ($\epsilon, \kappa, D_s, F_s, \sigma, \sigma', \sigma_c$) and pseudoscalar ($\pi, K, D, F, \eta, \eta', \eta_c$) mesons, respectively, and the λ^i are the usual 4 × 4 SU(4) matrices with $\lambda^0=(1/\sqrt{2})\,I$ adjoined. Repeated Latin indices are summed from 0 to 15.

The symmetry-breaking Lagrangian density is chosen to transform as the $(4,4^*)\oplus(4^*,4)$ representation of SU(4) × SU(4). The simplest choice is then

$$\mathcal{L}_{SB}=-\epsilon_0\sigma_0-\epsilon_8\sigma_8-\epsilon_{15}\sigma_{15}\,. \tag{2.3}$$

The complete Lagrangian can be rewritten in the 16-component form as

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\sigma_i\partial^\mu\sigma_i+\tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\tfrac{1}{2}\mu^2(\sigma_i\sigma_i+\phi_i\phi_i)+\tfrac{1}{3}F_{ijkl}(\sigma_i\sigma_j\sigma_k\sigma_l+\phi_i\phi_j\phi_k\phi_l)+2\hat{F}_{ij,kl}\sigma_i\phi_j\phi_k\phi_l-\epsilon_i\sigma_i\,, \tag{2.4}$$

where

$$F_{ijkl}=\tfrac{1}{2}gA_{ijkl}+(f_1+\tfrac{1}{4}g)J^1_{ijkl}+\tfrac{1}{2}(f_2-\tfrac{1}{2}g)J^2_{ijkl}\,, \tag{2.5}$$

$$\hat{F}_{ij,kl}=-\tfrac{1}{2}gA_{ijkl}+f_1\delta_{ij}\delta_{kl}+\tfrac{1}{2}f_2J^3_{ijkl}-\tfrac{1}{4}g(J^1_{ijkl}-J^2_{ijkl})\,, \tag{2.6}$$

$$A_{ijkl}=8\delta_{i0}\delta_{j0}\delta_{k0}\delta_{l0}-2(\delta_{i0}\delta_{j0}\delta_{kl}+\text{five symmetric terms})+\sqrt{2}(\delta_{i0}d_{jkl}+\text{three symmetric terms})\,, \tag{2.7}$$

$$J^1_{ijkl}=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\,, \tag{2.8}$$

$$J^2_{ijkl}=d_{ijm}d_{mkl}+d_{ikm}d_{mjl}+d_{ilm}d_{mjk}\,, \tag{2.9}$$

$$J^3_{ijkl}=d_{ijm}d_{mkl}+f_{ikm}f_{mjl}+f_{ilm}f_{mjk}\,, \tag{2.10}$$

and

$$\epsilon_i=\delta_{i0}\epsilon_0+\delta_{i8}\epsilon_8+\delta_{i15}\epsilon_{15}\,. \tag{2.11}$$

Using

$$S_i=\sigma_i-\xi_i\,, \tag{2.12}$$

we define new scalar fields S_i with vanishing vacuum expectation values, where

$$\langle 0|\sigma_i|0\rangle=\xi_i\,. \tag{2.13}$$

Introducing this translation into the Lagrangian gives

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu S_i\partial^\mu S_i+\tfrac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i\partial^\mu\phi_i-\tfrac{1}{2}m^{S^2}_{ij}S_iS_j-\tfrac{1}{2}m^{\phi^2}_{ij}\phi_i\phi_j+\tfrac{1}{3}F_{ijkl}(S_iS_jS_kS_l+\phi_i\phi_j\phi_k\phi_l)\\&+2\hat{F}_{ij,kl}S_iS_j\phi_k\phi_l+G^S_{ijk}S_iS_jS_k-3G^\phi_{ij,k}\phi_i\phi_jS_k-E_iS_i\,,\end{aligned} \tag{2.14}$$

where

$$m^{S^2}_{ij}=\mu^2\delta_{ij}-4F_{ijkl}\xi_k\xi_l\,, \tag{2.15}$$

$$m^{\phi^2}_{ij}=\mu^2\delta_{ij}-4\hat{F}_{ij,kl}\xi_k\xi_l\,, \tag{2.16}$$

$$G^S_{ijk}=\tfrac{4}{3}F_{ijkl}\xi_l\,, \tag{2.17}$$

$$G^\phi_{ij,k}=-\tfrac{4}{3}\hat{F}_{ij,kl}\xi_l\,, \tag{2.18}$$

and

$$E_i=\epsilon_i+\mu^2\xi_i-\tfrac{4}{3}F_{ijkl}\xi_j\xi_k\xi_l\,. \tag{2.19}$$

This Lagrangian is not normal-ordered, owing to difficulties inherent in the translation.[17]

Perturbation theory is defined as an expansion in powers of λ, which is introduced via

$$\mathcal{L}(M,\lambda)=(1/\lambda^2)\mathcal{L}(\lambda M)\,. \tag{2.20}$$

λ is employed solely for power counting and is set to unity at the end of the calculation. This is, in effect, an expansion in the number of closed loops for a given process. The symmetry properties of the Lagrangian are preserved order by order in this expansion.[18]

The Lagrangian to second order with λ factors and counterterms is

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu S_i\partial^\mu S_i+\tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\tfrac{1}{2}(m^2+\lambda^2\delta m^2)^S_{ij}S_iS_j-\tfrac{1}{2}(m^2+\lambda^2\delta m^2)^\phi_{ij}\phi_i\phi_j$$
$$+\tfrac{1}{3}\lambda^2(F+\lambda^2\delta F)_{ijkl}(S_iS_jS_kS_l+\phi_i\phi_j\phi_k\phi_l)+2\lambda^2(\hat{F}+\lambda^2\delta\hat{F})_{ij,kl}S_iS_j\phi_k\phi_l$$
$$+\lambda(G+\lambda^2\delta G)^S_{ijk}S_iS_jS_k-3\lambda(G+\lambda^2\delta G)^\phi_{ij,k}\phi_i\phi_jS_k-\frac{1}{\lambda}(E+\lambda^2\delta E)_iS_i\,, \tag{2.21}$$

where the second-order counterterms are denoted by δ. As demonstrated in I, these counterterms can be separated into divergent (D) and finite (Δ) components, i.e.,

$$\delta=D+\Delta\,, \tag{2.22}$$

in a well-defined manner. The divergent parts of the counterterms are used to cancel the divergent parts of the integrals, and the physical quantities are finite. The Feynman-diagram rules for this Lagrangian are given in Fig. 1.

When the second-order counterterms are introduced, one must keep terms to only second order in δ to ensure the correct symmetry properties. To enforce this the ξ_i must be considered separately. For example, for the scalar-field vaccum expectation value one has

$$\delta E_i=E_i(\delta\mu^2,\delta f_1,\delta f_2,\delta\epsilon)+m^{S2}_{ij}\delta\xi_j\,. \tag{2.23}$$

The vector and axial-vector currents after translation are

$$V^\mu_k=\tfrac{1}{2}f_{kij}(S_i\overleftrightarrow{\partial}^\mu S_j+\phi_i\overleftrightarrow{\partial}^\mu\phi_j)+f_{kij}\xi_j\partial^\mu S_j \tag{2.24}$$

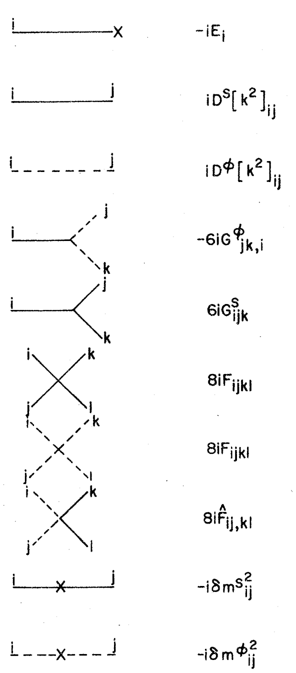

FIG. 1. Feynman-diagram rules for the Lagrangian of Eq. (2.21). Solid lines represent scalar fields and dashed lines pseudoscalar fields.

and

$$A^\mu_k=d_{kij}\phi_i\overleftrightarrow{\partial}^\mu S_j-d_{kij}\xi_i\partial^\mu\phi_j\,, \tag{2.25}$$

respectively. Their divergences are

$$\partial_\mu V^\mu_i=f_{ijk}\epsilon_jS_k\,, \tag{2.26}$$

$$\partial_\mu A^\mu_i=-d_{ijk}\epsilon_j\phi_k\,.\quad (i\neq 0) \tag{2.27}$$

Following Hu,[9] we define

$$a=\left(\frac{2}{3}\right)^{1/2}\frac{\epsilon_8}{\epsilon_0}\,, \tag{2.28}$$

$$b=\frac{\epsilon_{15}}{\sqrt{3}\epsilon_0}\,, \tag{2.29}$$

$$c=\left(\frac{2}{3}\right)^{1/2}\frac{\xi_8}{\xi_0}\,, \tag{2.30}$$

and

$$d=\frac{\xi_{15}}{\sqrt{3}\xi_0}\,. \tag{2.31}$$

The Lagrangian is invariant under SU(3), SU(2) $\times$ SU(2), and SU(3) $\times$ SU(3) when $a=0$, $1+a+b=0$, and $a=0$ with $b=-1$, respectively. Similar statements apply for c and d and the vacuum.

Finally we consider three-particle mixing. It is convenient to define a new basis such that the fields are orthogonal and the mass matrix diagonal in the tree approximation. Second-order calculations are simplified if we use this new basis for the internal lines in diagrams, allowing us to treat all internal lines on the same footing. Consequently, the orthogonal matrix $U^{S,\phi}_{\alpha i}$ is defined such that

$$U_{\alpha i}m^2{}_{ij}\hat{U}_{j\beta}=m_\alpha{}^2\delta_{\alpha\beta}\,, \tag{2.32}$$

where Latin indices are used to denote the original basis and Greek indices the new.

The nontrivial component of U consists of

$$R(\theta)=R(\theta_8,\theta_{15},\theta_0)=R(\theta_8)R(\theta_{15})R(\theta_0)\,, \tag{2.33}$$

where θ_8, θ_{15}, and θ_0 are the 15-0, 0-8, and 8-15 mixing angles, respectively, and

$$R(\theta_8)=\begin{pmatrix}1&0&0\\0&\cos\theta_8&\sin\theta_8\\0&-\sin\theta_8&\cos\theta_8\end{pmatrix}, \tag{2.34}$$

$$R(\theta_{15})=\begin{pmatrix}\cos\theta_{15} & 0 & -\sin\theta_{15}\\ 0 & 1 & 0\\ \sin\theta_{15} & 0 & \cos\theta_{15}\end{pmatrix}, \tag{2.35}$$

and

$$R(\theta_0)=\begin{pmatrix}\cos\theta_0 & \sin\theta_0 & 0\\ -\sin\theta_0 & \cos\theta_0 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.36}$$

This gives, for example,

$$\begin{pmatrix}\eta\\ \eta_c\\ \eta'\end{pmatrix}=R_{\alpha i}\begin{pmatrix}\phi_8\\ \phi_{15}\\ \phi_0\end{pmatrix}. \tag{2.37}$$

III. TREE-APPROXIMATION CALCULATIONS

The tree-approximation Lagrangian contains ten parameters. We fix various masses and leptonic decay constants to determine these parameters. Naturally, it would be convenient to input the better-known quantities (e.g., $m_\pi, m_K, m_\eta, m_{\eta'}$) for this evaluation. However, the resulting equations are highly nonlinear, especially when mixed fields are involved.

To circumvent this, we first evaluate the nonzero ϵ_i and ξ_i using F_π, F_K, m_π, m_K, m_D, and m_F; then the remaining parameters can be easily evaluated. The values of m_D and m_F are adjusted somewhat to achieve acceptable values for m_η and $m_{\eta'}$, which cannot be input directly.

The leptonic decay constants for the pseudoscalar fields are defined via

$$[(2\pi)^3 2\omega_\alpha]^{1/2}\langle 0|A_i^\mu(0)|\phi_\alpha(p)\rangle = ip^\mu F_{i\alpha}^5 . \tag{3.1}$$

For the unmixed particles in the tree approximation this gives [Eq. (2.25)]

$$F_{ij}^5 = d_{ijk}\xi_k . \tag{3.2}$$

We need

$$F_\pi=\frac{1}{\sqrt{2}}\xi_0(1+c+d) \tag{3.3}$$

and

$$F_K=\frac{1}{\sqrt{2}}\xi_0(1-\tfrac{1}{2}c+d). \tag{3.4}$$

Multiplying Eq. (3.1) by p_μ, evaluating on mass shell in the tree approximation using Eq. (2.27), and substituting for $F_{i\alpha}^5$ one obtains

$$m_\pi^2\xi_0(1+c+d)=-\epsilon_0(1+a+b), \tag{3.5}$$

$$m_K^2\xi_0(1-c/2+d)=-\epsilon_0(1-a/2+b), \tag{3.6}$$

$$m_D^2\xi_0(1+c/2-d)=-\epsilon_0(1+a/2-b), \tag{3.7}$$

and

$$m_F^2\xi_0(1-c-d)=-\epsilon_0(1-a-b). \tag{3.8}$$

These linear equations can easily be solved for the ϵ's and ξ's.

Using the expressions given in Table I for three of the above masses, the values of $\mu^2-2f_1\xi_0^2(1+3c^2+6d^2)$, f_2, and g can be obtained from linear equations. Finally, one of the $I=0$ neutral-scalar-meson masses must be input. This will give a linear constraint to fix μ^2 and f_1. By construction the translated scalar fields have vanishing vacuum expectation values. This affords the condition

$$E_i=0 \tag{3.9}$$

which provides a useful check of the parameter evaluations.

With the Lagrangian parameters fixed, the remaining masses may be found using Table I. For the mixed fields this requires finding the eigenvalues of the 3×3 mass matrix. The triplets of mixing angles θ_p and θ_s for the pseudoscalar and scalar cases, respectively, can be found from the eigenvectors of the mass matrices.

The remaining leptonic decay constants can be computed. F_D and F_F are available directly from Eq. (3.2). For mixed fields in the tree approximation, Eq. (3.1) reduces to

$$F_{i\alpha}^5=U_{\alpha j}^\phi d_{ijk}\xi_k . \tag{3.10}$$

For scalar fields the decay constants are defined by

$$[(2\pi)^3 2\omega_j]^{1/2}\langle 0|V_i^\mu(0)|S_j(p)\rangle = ip^\mu F_{ij} . \tag{3.11}$$

In the tree approximation with Eq. (2.24) this gives

$$F_{ij}=f_{ijk}\xi_k . \tag{3.12}$$

IV. ONE-LOOP APPROXIMATION CALCULATIONS

This model has been demonstrated to be renormalizable in the one-loop approximation in I. It was shown that only the parameters of the symmetric Lagrangian (μ^2, f_1, f_2, and g) acquire divergent second-order parts. However, all parameters may acquire finite corrections. In this section we consider the calculation of the mass spectrum and the leptonic decay constants to second order using one- and two-point vertices.

First consider the evaluation of the finite second-order corrections to the Lagrangian parameters. Ten constraints are necessary to fix these corrections. As all equations must be linear in the second-order terms, this evaluation is numerically easier than the first-order case.

The vacuum expectation values of the scalar fields vanish by construction to second order.

TABLE I. The expressions for the nonvanishing pseudoscalar and scalar tree-approximation masses squared. From Eqs. (2.15) and (2.16)

$m^2{}_{ij}=\mu^2\delta_{ij}-2f_1\xi_0{}^2A^1_{ij}-f_2\xi_0{}^2A^2_{ij}-\frac{1}{2}g\,\xi_0{}^2A^3_{ij}$.

(i,j)	A^1_{ij}	A^2_{ij}	A^3_{ij}
π	$2+3c^2+6d^2$	$(1+c+d)^2$	$1-2(c+d)+6cd-3d^2$
K	$2+3c^2+6d^2$	$1-c+2d-cd+7c^2+d^2$	$1+c-2d-3cd-3d^2$
D	$2+3c^2+6d^2$	$1+c-2d+5cd+c^2+13d^2$	$1-c+2d-cd-2c^2+d^2$
F	$2+3c^2+6d^2$	$1-2(c+d)-10cd+4c^2+13d^2$	$(1+c+d)^2$
η_{88}	$2+3c^2+6d^2$	$1-2(c-d+cd)+3c^2+d^2$	$1+2(c-d)-6cd-3d^2$
η_{1515}	$2+3c^2+6d^2$	$1-4d+\frac{1}{2}c^2+7d^2$	$1+4d-\frac{3}{2}c^2+3d^2$
η_{00}	$2+3c^2+6d^2$	$1+\frac{3}{2}c^2+3d^2$	$-3(1-\frac{1}{2}c^2-d^2)$
η_{08}	0	$\sqrt{6}c(1+d-\frac{1}{2}c)$	$\sqrt{6}c(1-d+\frac{1}{2}c)$
η_{015}	0	$2\sqrt{3}(d+\frac{1}{4}c^2-d^2)$	$2\sqrt{3}(d-\frac{1}{4}c^2+d^2)$
η_{815}	0	$\sqrt{2}c(1+d-\frac{1}{2}c)$	$-\sqrt{2}c(1+3d+\frac{3}{2}c)$
ϵ	$2+3c^2+6d^2$	$3(1+c+d)^2$	$-(1-2c-2d+6cd-3d^2)$
κ	$2+3c^2+6d^2$	$3(1-c+2d-cd+c^2+d^2)$	$-(1+c-2d-3cd-3d^2)$
D_s	$2+3c^2+6d^2$	$3+3c-6d-cd+c^2+7d^2$	$-(1-c+2d-cd-2c^2+d^2)$
F_s	$2+3c^2+6d^2$	$3-6c-6d+2cd+4c^2+7d^2$	$-(1+c+d)^2$
σ_{88}	$2+9c^2+6d^2$	$3(1-2c+2d-2cd+3c^2+d^2)$	$-(1+2c-2d-6cd-3d^2)$
σ_{1515}	$2+3c^2+18d^2$	$3-12d+\frac{3}{2}c^2+21d^2$	$-(1+4d-\frac{3}{2}c^2+3d^2)$
σ_{00}	$6+3c^2+6d^2$	$3+\frac{9}{2}c^2+9d^2$	$3(1-\frac{1}{2}c^2-d^2)$
σ_{08}	$2\sqrt{6}c$	$3\sqrt{6}c(1+d-\frac{1}{2}c)$	$-\sqrt{6}c(1-d+\frac{1}{2}c)$
σ_{015}	$4\sqrt{3}d$	$6\sqrt{3}(d+\frac{1}{4}c^2+d^2)$	$-2\sqrt{3}(d-\frac{1}{4}c^2+d^2)$
σ_{815}	$6\sqrt{2}cd$	$3\sqrt{2}c(1+d-\frac{1}{2}c)$	$\sqrt{2}c(1+3d+\frac{3}{2}c)$

This provides three constraints. From Fig. 2

$$E_i+\lambda^2\delta E_i+\lambda^2\,(\text{loop contribution})=0\,. \tag{4.1}$$

As E_i vanishes in the tree approximation, only the second-order contribution need be considered.

Evaluating the diagrams of Fig. 2 one finds

$$\Delta E_i+DE_i-3\sum_\alpha G^S_{i\alpha\alpha}i\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-m^{S\,2}_\alpha)}+3\sum_\alpha G^\phi_{\alpha\alpha,i}i\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-m^{\phi\,2}_\alpha)}=0\,. \tag{4.2}$$

The divergent part of this expression may be isolated using the prescription of I. The remaining equation for the finite part is

$$\Delta E_i-3\sum_\alpha G^S_{i\alpha\alpha}(m^{S^2}_\alpha-\nu^2)\overline{B}(0,m^{S^2}_\alpha,\nu^2)+3\sum_\alpha G^\phi_{\alpha\alpha,i}(m^{\phi^2}_\alpha-\nu^2)\overline{B}(0,m^{\phi^2}_\alpha,\nu^2)=0\,, \tag{4.3}$$

where

$$\overline{B}(p^2,x^2,y^2)=B(p^2,x^2,y^2)-B(0,\nu^2,\nu^2)\quad(\text{Ref. 19})\,, \tag{4.4}$$

$$B(p^2,x^2,y^2)=i\int\frac{d^4l}{(2\pi)^4}\frac{1}{[(l-p)^2-x^2][l^2-y^2]}\,, \tag{4.5}$$

and ν^2 is arbitrary.

Employing Eq. (2.23), ΔE_i can be rewritten as

$$\Delta E_i=E_i(\Delta\mu^2,\Delta f_1,\Delta f_2,\Delta g,\Delta\epsilon_i)+m_{ij}{}^{S^2}\Delta\xi_j\,. \tag{4.6}$$

The $\Delta\xi_i$ term is written separately as it can only appear linearly in the expressions to second order. Three useful constraints are then provided by combining Eqs. (4.3) and (4.6) for $i=0$, 8, and 15.

We choose to input several masses. From Figs. 3 and 4, respectively, the finite second-order mass corrections to the pseudoscalar and scalar masses are

$$\Sigma^{\phi}_{ij}(s)=m^{\phi 2}_{ij}(\Delta\mu^2,\Delta f_1,\Delta f_2,\Delta g)+6G^{\phi}_{ij,k}\Delta\xi_k$$
$$-4\sum_{\alpha}F_{ij\alpha\alpha}(m^{\phi^2}_{\alpha}-\nu^2)\bar{B}(0,m^{\phi^2}_{\alpha},\nu^2)-4\sum_{\alpha}\hat{F}_{ij,\alpha\alpha}(m^{S^2}_{\alpha}-\nu^2)\bar{B}(0,m^{S^2}_{\alpha},\nu^2)$$
$$+36\sum_{\alpha,\beta}G^{\phi}_{i\alpha,\beta}G^{\phi}_{j\alpha,\beta}\bar{B}(s,m^{\phi^2}_{\alpha},m^{S2}_{\beta}) \tag{4.7}$$

and

$$\Sigma^{S}_{ij}(s)=m^{S2}_{ij}(\Delta\mu^2,\Delta f_1,\Delta f_2,\Delta g)-6G^{S}_{ijk}\Delta\xi_k$$
$$-4\sum_{\alpha}\hat{F}_{\alpha\alpha,ij}(m^{\phi^2}_{\alpha}-\nu^2)\bar{B}(0,m^{\phi 2}_{\alpha},\nu^2)-4\sum_{\alpha}F_{\alpha\alpha ij}(m^{S^2}_{\alpha}-\nu^2)\bar{B}(0,m^{S^2}_{\alpha},\nu^2)$$
$$+18\sum_{\alpha,\beta}G^{S}_{\alpha\beta i}G^{S}_{\alpha\beta j}\bar{B}(s,m^{S2}_{\alpha},m^{S2}_{\beta})+18\sum_{\alpha,\beta}G^{\phi}_{\alpha\beta,i}G^{\phi}_{\alpha\beta,j}\bar{B}(s,m^{S^2}_{\alpha},m^{\phi^2}_{\beta}). \tag{4.8}$$

Each unmixed mass provides a constraint via

$$\mathrm{Re}[D^{-1}{}_{ij}(M^2)]=0\,, \tag{4.9}$$

where the unrenormalized propagator is given by

$$D^{-1}{}_{ij}(s)=s\delta_{ij}-m^2{}_{ij}-\Sigma_{ij}(s) \tag{4.10}$$

and M denotes the mass to second order. For stable particles the wave-function renormalization constant is given by

$$Z_i=1+\Sigma_i'(M_i^2)\,, \tag{4.11}$$

where the prime denotes differentiation with respect to s.

In second order we can also input mixed masses using

$$\mathrm{Re}[\mathrm{Det}D^{-1}(M_{\alpha}^2)]=0\,. \tag{4.12}$$

The propagator is given by

$$D(s)=\frac{D^{-1}_{\mathrm{adj}}(s)}{\mathrm{Det}D^{-1}(s)}\,. \tag{4.13}$$

Owing to the s dependence of $D^{-1}_{\mathrm{adj}}(s)$, $D(s)$ cannot be diagonalized, in general, for all masses with a single set of angles θ. That is possible only in the tree approximation. In this case we require three sets of angles θ_{α} such that

$$R(\theta_{\alpha})D^{-1}_{\mathrm{adj}}(M_{\alpha}^2)R^{-1}(\theta_{\alpha}) \tag{4.14}$$

is diagonal, where $R(\theta_{\alpha})$ is given in Eq. (2.33). Since its determinant vanishes, $D^{-1}_{\mathrm{adj}}(M_{\alpha}^2)$ has only one nonvanishing eigenvalue, which will correspond to the desired propagator; for example, $D_{\eta}(s)$ is then

$$D_{\eta}(s)=R_{1i}(\theta_{\eta})D_{ij}(s)R^{-1}{}_{j1}(\theta_{\eta})\,. \tag{4.15}$$

For stable particles the wave-function renormalization constant is

$$Z_{\alpha}=\frac{\mathrm{Tr}D^{-1}_{\mathrm{adj}}(M_{\alpha}^2)}{(d/ds)\,\mathrm{Det}D^{-1}(M_{\alpha}^2)}\,. \tag{4.16}$$

Then, for example, the renormalized η field is

$$\eta^R=Z_{\eta}^{-1/2}R_{1i}(\theta_{\eta})\phi_i\,. \tag{4.17}$$

The mixed states are no longer orthogonal owing to the s dependence of $\Sigma(s)$.

We input M_{π}, M_K, M_{η}, $M_{\eta'}$, M_D, and M_{σ}. The latter mass is input to determine $\Delta\mu^2$ and Δf_1.

Finally we input F_{π}. Setting

$$\Gamma^{5\,\mu}_{ij}(p)=p^{\mu}\Gamma^5_{ij}(p^2)$$
$$=-D^{\phi-1}{}_{jk}(p^2)\int d^4x\langle 0|T\{A^{\mu}_i(0)\phi_k(x)\}|0\rangle e^{-ipx}\,, \tag{4.18}$$

one has

$$F^5_{ij}=\sqrt{Z}_j\Gamma^5_{ij}(M^2)\,. \tag{4.19}$$

Thus

$$F_{\pi}=\sqrt{Z}_{\pi}\Gamma^5_{33}(M_{\pi}^2)\,. \tag{4.20}$$

The free-field axial-vector-current–field vertex relations are given in Fig. 5, where

$$\rho^5_{k,i}=-d_{ikm}\xi_m \tag{4.21}$$

and

$$\rho^5_{k,ij}=d_{kij}\,. \tag{4.22}$$

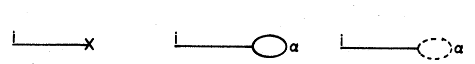

FIG. 2. Diagrams contributing to the vacuum expectation values of the $I=0$ scalar fields to second order.

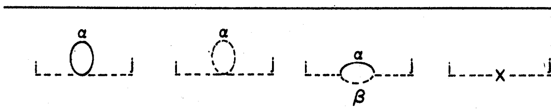

FIG. 3. Diagrams contributing to the second-order pseudoscalar mass.

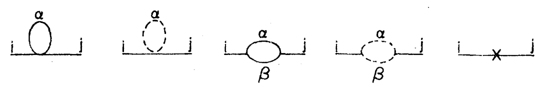

FIG. 4. Diagrams contributing to the scalar mass to second order.

FIG. 6. Diagrams for the pseudoscalar-field leptonic decay constant.

From the Feynman diagrams of Fig. 6 one has

$$\Gamma^{5\,\mu}_{ij}(p) = -p^{\mu}\rho^{5}_{i,j} - 6\sum_{\alpha,\beta} \rho^{5}_{i,\alpha\beta} G^{\phi}_{j\alpha,\beta} R^{\mu}(p, m^{S\,2}_{\beta}, m^{\phi\,2}_{\alpha}) , \tag{4.23}$$

where[20]

$$R^{\mu}(p, x^2, y^2) = i \int \frac{d^4 l}{(2\pi)^4} \frac{(2l-p)^{\mu}}{[l^2 - x^2][(l-p)^2 - y^2]} . \tag{4.24}$$

For mixed fields one has

$$F^{5}_{i\alpha} = \sqrt{Z_{\alpha}}\, U^{\phi}_{\alpha j}(M^2) \Gamma^{5}_{ij}(M^2) , \tag{4.25}$$

where $U^{\phi}_{\alpha j}(M^2)$ is constructed from the appropriate eigenvector of $D^{-1}_{\text{adj}}(M^2)$. Similar relations apply for the scalar fields.

The Ward-Takahashi identity involving the vertex of Eq. (4.18) is

$$s\Gamma^{5}_{ij}(s) = -d_{ijk}\epsilon_k + d_{ikl}\xi_l D^{\phi\,-1}_{kj}(s) . \tag{4.26}$$

Thus, for example, to have

$$[(2\pi)^3 2\omega]^{1/2} \langle 0 | \partial_{\mu} A^{\mu}_{3} | \pi^0(p) \rangle = F_{\pi} M_{\pi}^{\,2} , \tag{4.27}$$

one must have

$$d_{33l}\xi_l Z_{\pi}^{\,1/2} D_{\pi}^{\,-1}(M_{\pi}^{\,2}) = 0 . \tag{4.28}$$

In general this will be true only if $\Delta M_{\pi}^{\,2}$ vanishes. Consequently, the usual expression for operator PCAC (e.g., $\partial_{\mu} A^{\mu}_{3} = F_{\pi} M_{\pi}^{\,2} \phi^{R}_{3}$) is valid in the one-loop approximation only if ΔM^2 vanishes. We do not adhere to this constraint in our model solutions and, as a result, use the current-field vertex functions to evaluate the leptonic decay constants.

Once the Lagrangian parameters have been fixed, the remaining masses and leptonic decay constants can be evaluated.

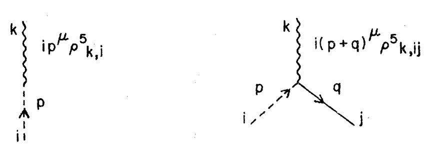

FIG. 5. Feynman-diagram rules for the free-field axial-vector-current–field vertex. The wavy line represents the axial-vector current. The factors $\rho^{5}_{k,i}$ and $\rho^{5}_{k,ij}$ are given in Eqs. (4.21) and (4.22), respectively.

V. NUMERICAL ANALYSIS

As indicated above, our goal was to approximate the scalar and the pseudoscalar mass spectrum and the known leptonic decay constants. The mass spectrum has not yet been completely determined; consequently, we shall briefly consider it before discussing our numerical results.

The SU(3) pseudoscalar octet mass spectrum is well known.[21] There is some uncertainty with the SU(4) singlet; however, the $X^0(958)$ is generally preferred to the $E(1420)$ as the η' meson. The $D(1863, 1868)$ meson[21] is now reasonably well established as the $I=\frac{1}{2}$, $S=0$, $|C|=1$ component of the pseudoscalar 15-plet. The F meson ($|C|=1$, $|S|=1$) mass is expected to be near that of the D meson. There is some evidence for the candidate $F(2030)$,[21] but this has not yet been confirmed. Finally, the $X(2830)$ meson[21] is a likely candidate for the η_c meson.

The spectrum of the scalar mesons is less well known; however, the SU(3) octet component is gradually taking shape. We associate the $I=1$ $\delta(980)$[21] and the $I=\frac{1}{2}$ $\kappa(1400)$[21] with the ϵ and κ, respectively. The $I=0$ $S^*(980)$[21] is identified with the σ. The SU(4) singlet σ' is associated with the $\epsilon(1300)$,[21] whose mass may be as large as 1700 MeV.[22] Little is known about the D_s and F_s mesons, but their masses are expected to be large (2–3 GeV). There are several candidates for the σ_c meson in the 3400–3500-MeV region including the $X(3415)$, $X(3510)$, and $X(3555)$; however, the $X(3415)$ is favored.[21]

We also considered the leptonic decay constants. These constraints provided additional information for the determination of the Lagrangian parameters, but imposed severe restrictions on the number of acceptable solutions. This will be discussed below. The known decay constants are F_{π} and F_K. F_{π} has been fixed at about 95 MeV. The ratio F_K/F_{π} is 1.25 ± 0.03.[21]

In both the tree and the one-loop approximation, ten parameters must be determined. Three are fixed in each case by requiring that the vacuum expectation values of the $I=0$ scalar fields vanish. The remainder in each case are determined by inputting various masses and decay constants.

As outlined in Sec. II, in the tree approximation we input F_π, F_K, m_π, and m_K. The mixed masses are difficult to input; consequently, we input m_D and m_F and adjust these somewhat (particularly m_F) to obtain reasonable values for m_η, m_{η_c}, and $m_{\eta'}$. Finally, the σ mass is input to determine μ^2 and f_1. We set $\nu^2 = |\mu^2|$ for the second-order calculations.

In the one-loop approximation we again input F_π, M_π, and M_D. Since the Δ quantities must appear linearly, we can easily input M_η, M_{η_c}, and $M_{\eta'}$. Finally, we again input M_σ. The quantities employed as input in the one-loop approximation were not necessarily set equal to their tree-approximation values.

A large number of solutions were investigated. In general, a particular tree-approximation solution will not give an acceptable second-order solution, since the second-order corrections will be too large. Consequently, considerable care was required in choosing solutions.

We present five solutions which are representative of the basic properties of the solutions found. Tables II and III contain the masses and leptonic decay constants, respectively, for the tree approximation solutions. Table IV contains the Lagrangian parameters for these solutions. These solutions were not the best available in the tree approximation, but were chosen on the basis of the resulting solutions to second order. The masses, decay constants and Lagrangian parameters for the second-order solutions are given in Tables V, VI, and VII, respectively. The complete calculations are given to enable a comparison between tree- and one-loop-approximation solutions.

Naturally, we were unable to fit our proposed mass spectrum exactly; however, the basic features could, in general, be reproduced quite well. Perturbation theory does not require that a given quantity acquire its physical value to any finite order. Nevertheless, in the spirit of the perturbation approach, the percentage difference between the tree and the one-loop values of physical quantities should on average not be too large (10–20%). Similarly, the difference between the second-order and physical values should, on average, be no larger than this. Finding solutions that obeyed these criteria required considerable effort; however, we feel that the solutions presented in the tables are acceptable in the above context.

The first major obstacles in finding a satisfactory solution were the leptonic decay constants of the charmed particles, in particular those of F_D and F_F. In the tree approximation these are generally of the order of 200–300 MeV. In second order, however, where one inputs masses near their physical values, these decay constants change sign. This problem was quite difficult to avoid. The price we had to pay was to raise the

TABLE II. Some tree-approximation solutions for the SU(4) σ model for the masses (in MeV) and mixing angles (in rad).

	1	2	3	4	5
m_π	355	324	390	353	582
m_K	533	514	532	533	638
m_η	550	537	551	547	651
$m_{\eta'}$	828	862	845	805	894
m_D	1537	1689	1602	1463	1501
m_F	1512	1664	1583	1439	1487
m_{η_c}	1747	1923	1831	1663	1682
m_ϵ	1012	1088	1089	974	1043
m_κ	1126	1199	1185	1093	1090
m_σ	902	963	989	868	980
$m_{\sigma'}$	1257	1318	1288	1231	1147
m_{D_s}	2067	2281	2192	1969	1960
m_{F_s}	2179	2388	2284	2086	2013
m_{σ_c}	3163	3478	3304	3006	3031
θ_8^ϕ	−0.57	−0.56	−0.56	−0.57	−0.56
θ_{15}^ϕ	−0.17	−0.15	−0.14	−0.19	−0.06
θ_0^ϕ	0.11	0.09	0.09	0.12	0.04
θ_8^s	2.24	−0.92	2.23	2.25	2.20
θ_{15}^s	3.95	−0.83	3.97	3.94	3.95
θ_0^s	3.86	0.76	3.90	3.85	3.91

TABLE III. The leptonic decay constants (in MeV) for the tree-approximation solutions presented in Table II.

	1	2	3	4	5
F_π	110	121	121	105	96.2
F_K	128	138	135	124	105
F_K/F_π	1.16	1.14	1.12	1.17	1.09
F_D	240	263	252	227	212
F_F	257	279	266	246	221
$F_{8\eta}$	8.26	8.17	7.09	8.11	5.02
$F_{8\eta_c}$	0.56	0.49	0.45	0.60	0.28
$F_{8\eta'}$	−18.7	−17.2	−14.7	−19.6	−8.40
$F_{15\eta}$	7.10	6.55	6.81	7.99	1.30
$F_{15\eta_c}$	317	348	329	300	282
$F_{15\eta'}$	72.5	77.9	76.6	69.8	61.4
F_κ	−17.7	−16.5	−14.1	−18.4	−8.48
F_{D_s}	−129	−142	−131	−122	−116
F_{F_s}	−112	−125	−116	−104	−107

tree-approximation value of the pion mass. We could then move the second-order pion mass down near its physical value, if desired, depending on the magnitude of the second-order shift one is willing to accept. A shift of 150 MeV is a large percentage shift for the pion, but this size of shift is common for the larger masses where it represents a much smaller percentage.

The other general problems were to move the η_c and σ_c masses near their physical values. M_{η_c} tends to be too small and M_{σ_c} too large. The only masses that are affected individually by $\Delta\mu^2$ and Δf_1 are M_σ, M_{σ_c}, and $M_{\sigma'}$ [all others employ $\Delta\mu^2 - 2\Delta f_1 \xi_0^2(2+3c^2+6d^2)$]. Thus they can be adjusted independently after the remainder of the solution has been chosen. The values in the tables represent a compromise for the three masses.

Solution 1 is characterized by relatively small second-order percentage corrections. Thus, for example, $m_\pi = 355$ MeV and $M_\pi = 323$ MeV. Similarly, M_K and M_η are larger than their physical values. $M_{\eta'}$, M_D, M_ϵ, and $M_{\sigma'}$ are quite good. M_{η_c}, M_κ, and M_σ are (15–20)% too small, whereas M_{σ_c} is about 25% too large. F_π is about 50% too large, but F_K/F_π is within 10%. F_D and F_F are both acceptable.

In solution 2 we keep basically the same tree-approximation solution as case 1, but move the second-order values of M_π, M_K, and M_η near their physical values; for example, $m_\pi = 324$ MeV and $M_\pi = 161$ MeV. The second-order corrections are thus somewhat larger than in solution 1, but we feel that the average correction is still reasonable. The values of M_π, M_K, M_η, $M_{\eta'}$, M_D, M_ϵ, M_κ, and $M_{\sigma'}$ are quite good. M_{η_c} is a little too small (at 2530 MeV). M_σ is too small and M_{σ_c} is

TABLE IV. The values of the Lagrangian parameters and the ratios a, b, c, and d for the tree-approximation solutions of Tables II and III.

	1	2	3	4	5
ξ_8 (MeV)	−20.4	−19.1	−16.3	−21.2	−9.79
ξ_{15} (MeV)	−151	−167	−154	−142	−138
ξ_0 (MeV)	260	283	273	248	224
μ^2 (GeV^2)	−0.571	−0.656	−0.576	−0.508	−0.506
f_1	−1.67	−1.46	−1.45	−1.68	−2.81
f_2	−11.1	−11.4	−11.5	−11.1	−11.9
g	3.34	3.29	3.16	3.35	4.16
ϵ_8 (GeV^3)	0.026	0.027	0.023	0.025	0.012
ϵ_{15} (GeV^3)	0.667	0.893	0.760	0.571	0.541
ϵ_0 (GeV^3)	−0.426	−0.556	−0.484	−0.369	−0.368
a	−0.050	−0.040	−0.039	−0.056	−0.026
b	−0.904	−0.927	−0.907	−0.894	−0.849
c	−0.064	−0.055	−0.049	−0.070	−0.036
d	−0.336	−0.340	−0.325	−0.331	−0.357

TABLE V. The masses (in MeV) in the one-loop approximation for the tree-approximation solutions of Tables II–IV.

	1	2	3	4	5
M_π	323	161	145	142	145
M_K	581	506	495	504	504
M_η	573	543	551	558	628
$M_{\eta'}$	975	953	951	944	994
M_D	1889	1917	1903	1788	1805
M_F	1916	1938	1920	1814	1792
M_{η_c}	2457	2530	2444	2288	2183
M_ϵ	1015	991	984	948	911
M_κ	1075	1131	1073	946	991
M_σ	779	822	866	738	915
$M_{\sigma'}$	1235	1300	1271	1203	1141
M_{D_s}	2290	2437	2350	2114	2060
M_{F_s}	2469	2625	2518	2286	2010
M_{σ_c}	4466	5019	4776	4308	4098

about 45% too large. The second-order shifts of F_D and F_F are probably too large to be acceptable. F_π is a bit large but F_K/F_π is again within about 10% of its target value.

Solution 3 is probably our best solution, although the second-order shift of F_D and F_F may again be a bit large. It represents a general improvement over solution 2. The tree-approximation solution is near that of cases 1 and 2. The second-order pseudoscalar mass spectrum is quite good, however, M_{η_c} is again a bit too small. The scalar mass spectrum again suffers from a small M_σ and a large M_{σ_c}, but is otherwise acceptable.

Solution 4 is similar to solution 2, but the value of M_{σ_c} is much better. However, the price to be paid for this is an M_σ which is about 250 MeV too small. Overall, it is probably not as good as solution 2, except that the second-order corrections to F_D and F_F are now in the region of 10%.

In solution 5 we show the effects of allowing M_π to increase to the region of 500–600 MeV. However, M_π is still kept near its physical value. The resulting mass spectrum is quite good with the exception of M_η being too large and M_{η_c} and M_κ being too small. The second-order corrections to F_D and F_F are less than 20%. The $I=0$ scalar masses have improved dramatically. This solution is probably not acceptable, as the second-order correction to the pion mass is too large.

If one allows a large pion mass in both the tree (500–600 MeV) and the one-loop (300–400 MeV) approximations, the whole SU(3) pseudoscalar octet masses shift upward somewhat. The remaining mass spectrum improves, however. The decay constants are also quite good. This solution is not included in the tables as we feel the large pion mass renders it unacceptable.

The symmetry of the Lagrangian and vacuum are indicated by a, b, c, and d of Eqs. (2.28)–(2.31). The Lagrangian is symmetric under SU(3), SU(2) × SU(2), and SU(3) × SU(3) if we have $a=0$, $1+a+b=0$, and ($a=0$, $b=-1$), respectively. In the tree-approximation solutions, a is small, in the order of −0.04 to −0.05, and b is near −1 at about −0.9. Consequently, the Lagrangian has approximate SU(3) and SU(3) × SU(3) symmetry,

TABLE VI. The leptonic decay constants (in MeV) in the one-loop approximation for the solutions of Table V.

	1	2	3	4	5
F_π	165	136	147	166	140
F_K	215	182	193	217	190
F_K/F_π	1.37	1.36	1.36	1.39	1.49
F_D	170	22.3	98.9	204	187
F_F	253	103	180	284	273
$F_{8\eta}$	122	164	168	154	160
$F_{8\eta_c}$	−1.17 +i.91	−2.17 +i.91	−2.16 +i.72	−1.86 +i.96	−2.03 +i.44
$F_{8\eta'}$	5.20	−26.4	−16.0	16.9	−9.16
$F_{15\eta}$	−162	16.6	33.0	−62.1	249
$F_{15\eta_c}$	256 +i3.11	282 +i3.76	268 +i3.29	243 +i3.06	232 +i2.95
$F_{15\eta'}$	82.1	114	107	86.3	82.1
F_κ	−34.2 −i10.1	−31.2 −i12.1	−29.2 −i7.96	−42.5 −i7.92	−10.8
F_{D_s}	−72.0 −i83.5	−93.0 −i99.1	−83.5 −i87.8	−84.2 −i72.4	155
F_{F_s}	−49.7 −i83.0	−72.7 −i98.8	−64.3 −i89.1	−58.5 −i71.8	−55.0

TABLE VII. The second-order finite corrections to the Lagrangian parameters of Table IV calculated using $\nu^2 = |\mu^2|$. The second-order values of the ratios a, b, c, and d are also presented.

	1	2	3	4	5
$\Delta\xi_8$ (MeV)	−8.12	−1.30	−7.22	−11.1	−29.2
$\Delta\xi_{15}$ (MeV)	181	345	249	132	122
$\Delta\xi_0$ (MeV)	−48.6	−211	−133	−7.49	−6.03
$\Delta\mu^2$ (GeV^2)	1.09	1.29	0.922	0.856	−0.495
Δf_1	24.9	24.9	25.5	25.5	33.7
Δf_2	60.4	49.6	62.0	68.1	89.7
Δg	17.5	18.2	18.7	18.3	21.7
$\Delta\epsilon_8$ (GeV^3)	0.032	0.022	0.027	0.031	0.039
$\Delta\epsilon_{15}$ (GeV^3)	0.537	−0.145	0.254	0.563	0.492
$\Delta\epsilon_0$ (GeV^3)	−0.345	0.078	−0.148	−0.343	−0.287
a	−0.070	−0.079	−0.073	−0.072	−0.093
b	−0.900	−0.908	−0.932	−0.943	−0.957
c	−0.102	−0.100	−0.094	−0.109	−0.143
d	0.003	0.111	0.041	−0.035	−0.051

but is very close to being SU(2) × SU(2) symmetric. In the one-loop approximation a decreases to the region −0.07 to −0.08, while b remains at about −0.9. Thus the SU(3) and SU(3) × SU(3)-symmetry breaking increases, whereas the SU(2) × SU(2) symmetry improves.

The values of c and d reflect the magnitudes of the SU(3) and the SU(4) symmetry breaking of the vacuum. In the tree approximation c is relatively small at approximately −0.05 and shifts to about −0.1 in the one-loop approximation. The value of d is much larger in the tree approximation, in the region of −0.3. However, this decreases to the region 0.1 to −0.05 in the one-loop approximation. This latter shift probably reflects our problem with F_D and F_F. In general, in the solutions where

TABLE VIII. The eigenvectors for the pseudoscalar and scalar mixed fields in the tree and one-loop approximations for solution 3 of Table V. In each case the upper row is the tree-approximation value and the second row is the second-order correction to this value.

	8	15	0
η	0.986	0.089	0.139
	0.047	−0.123	−0.251
η_c	−0.002	0.848	−0.530
	−0.008	0.113	0.181
η'	−0.165	0.522	0.837
	0.158	0.044	0.004
σ	0.493	0.463	0.736
	0.028	0.733	−0.480
σ_c	0.008	0.844	−0.536
	−0.003	0.025	0.040
σ'	−0.870	0.270	0.413
	−0.416	−0.121	−0.798

the second-order correction to these decay constants is small, the second-order value of d is more negative.

The wave-function renormalization constants follow the same general pattern in all five solutions. For the pseudoscalar mesons for solution 3 we have $Z_\pi^{1/2} = 0.814$, $Z_K^{1/2} = 0.817$, $Z_\eta^{1/2} = 0.815$, $Z_{\eta'}^{1/2} = 0.774$, $Z_D^{1/2} = 0.693$, and $Z_F^{1/2} = 0.754$.

Finally, the eigenvectors for the mixed pseudoscalar and scalar fields in the tree and one-loop approximations are presented for solution 3 in Table VIII.

VI. CONCLUSION

The pattern of the solutions seems relatively clear. We can approximate the overall trends in the mass spectra and the leptonic decay constants but not the details. Although the magnitude of the second-order shifts are often large, they are acceptable in general.

The solutions in the SU(4) model were not nearly as successful as those in the SU(3) version. However, as mentioned above, there is only one extra parameter to accommodate both the much larger mass splitting and the six additional masses. Consequently, the calculated mass spectrum deviates most from the experimental values at the high- and the low-mass extremes.

There are three immediate remedies for this problem. First, with such a large-mass splitting additional bilinear symmetry-breaking terms may be required. Secondly, the incorporation of other fields, in particular the baryons, may give a more realistic mass spectrum. Finally, higher-order calculations, in particular the two-loop approximation, may give better results. Although we would expect that the third-order corrections would be at most only 20% of the second-order ones, a calculation at this order would allow an adjustment in the lower-order computations, which may permit a more dramatic overall improvement.

ACKNOWLEDGMENT

This work was supported in part by the National Research Council of Canada under Grant No. A1574.

---

[1]The linear SU(2) σ model with mesons in the one-loop approximation was considered by J. L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970).

[2]For examples of calculations with the nonlinear SU(2) σ model in the one-loop approximation, see K. S. Jhung and R. S. Willey, Phys. Rev. D 9, 3132 (1974); and W.-L. Lin and R. S. Willey, *ibid.* 14, 196 (1976).

[3]For examples of calculations in the SU(3) σ model in the tree approximation, see M. Levy, Nuovo Cimento 52, 23 (1967); W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969); P. Carruthers, Phys. Rev. D 2, 2265 (1970); P. Carruthers and R. W. Haymaker, *ibid.* 4, 1808 (1971); 4, 1815 (1971); 6, 1528 (1972); R. Olshansky, *ibid.* 4, 2440 (1971); D. W. McKay, W. F. Palmer, and R. F. Sarraga, *ibid.* 8, 2532 (1973); W. Hudnall and J. Schechter, *ibid.* 9, 2111 (1974); H. B. Geddes and R. H. Graham, *ibid.* 12, 855 (1975); 13, 56 (1976).

[4]For a calculation of the one- and two-point functions in the one-loop approximation in the SU(3) meson σ model, see L.-H. Chan and R. W. Haymaker, Phys. Rev. D 7, 402 (1973).

[5]Meson-meson scattering in the SU(3) model in the one-loop approximation has been considered by L.-H. Chan and R. W. Haymaker, Phys. Rev. D 10, 4143 (1974).

[6]For a detailed study in the SU(3) meson model of the one- and two-point functions, the K_{l3} decay form factors and the theoretical predictions for these form factors derived using specific symmetry assumptions, see H. B. Geddes and R. H. Graham, Phys. Rev. D (to be published).

[7]J.-E. Augustin *et al.*, Phys. Rev. Lett. 33, 1406 (1974); J. J. Aubert *et al.*, *ibid.* 33, 1404 (1974); C. Bacci *et al.*, *ibid.* 33, 1408 (1974); G. S. Abrams *et al.*, *ibid.* 33, 1453 (1974).

[8]S. W. Herb *et al.*, Phys. Rev. Lett. 39, 252 (1977).

[9]B. Hu, Phys. Rev. D 9, 1825 (1974).

[10]M. Vaughn, Phys. Rev. D 13, 2621 (1976).

[11]J. Kandaswamy, J. Schechter, and M. Singer, Phys. Rev. D 13, 3151 (1976).

[12]For an example of calculations using a general SU(4) model in the tree approximation, see Y. Ueda, Phys. Rev. D 16, 841 (1977).

[13]Tree-approximation calculations using the nonlinear SU(4) model are demonstrated by M. Singer, Phys. Rev. D 16, 2304 (1977).

[14]Y. Nambu, Phys. Rev. Lett. 4, 380 (1960); Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961); J. Goldstone, Nuovo Cimento 19, 155 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[15]The SU(n) meson σ model has been demonstrated to be renormalizable in the one-loop approximation for $n \geq 4$ by H. B. Geddes, Phys. Rev. D 20, 531 (1979). Hereafter this will be referred to as I.

[16]H. Pagels, Phys. Rep. 16C, 219 (1975).

[17]B. W. Lee, Nucl. Phys. B9, 649 (1969).

[18]S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969).

[19]The $\bar{B}$ integral is evaluated in Ref. 4.

[20]The R^{μ} integral is discussed and evaluated in Ref. 17.

[21]Particle Data Group, Phys. Lett. 75B, 1 (1978).

[22]In a recent analysis the ϵ (1300) has been rejected in favor of the ϵ' (1540) by P. Estabrooks, Phys. Rev. D 19, 2678 (1979).

[23]The best experimental datum is $F_K/F_\pi f_+(0)=1.25 \pm 0.03$ given by G. Donaldson *et al.*, Phys. Rev. D 9, 2960 (1974). We expect that $f_+(0) \approx 1$. See Ref. 6 for further discussion.