Birkhoff theorems for $R+R^2$ gravity theories with torsion

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A Birkhoff theorem recently proved by Ramaswamy and Yasskin for one specific $R+R^2$ gravity theory with torsion is extended to include most $R+R^2$ theories. Out of the five-parameter family of $R+R^2$ theories, two four-parameter families are shown to obey a Birkhoff theorem. In addition, the entire five-parameter family is shown to obey a weakened Birkhoff theorem: If the assumption of asymptotic flatness is added to the assumption of spherical symmetry, then the only solution is the Schwarzschild metric with zero torsion.

I. INTRODUCTION

In a recent paper Ramaswamy and Yasskin (RY henceforth) proved that a gravity theory based upon the Lagrangian

$$L=-(c^4/16\pi G)(R+\chi R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha}{}^{\mu\nu}) \tag{1.1}$$

possesses a Birkhoff theorem.[1] In Eq. (1.1) χ is a constant and the curvature is computed from a Cartan connection (which is metric compatible but may have a nonzero torsion).[2-4] By a "Birkhoff theorem" RY mean the following: When they insert the spherically symmetric ansatz

$$ds^2=e^{2\Phi}dt^2-e^{2\Lambda}dR^2-r^2d\Omega^2 \tag{1.2}$$

into the field equations they find that the only possible solution is the torsion tensor $Q^{\alpha}{}_{[\beta\gamma]}=0$, metric tensor = Schwarzschild metric. Since most empirical tests of general relativity utilize a spherically symmetric gravity source, any nonstandard gravity theory which obeys a Birkhoff theorem would be very difficult to distinguish experimentally from standard gravity theory. Thus the RY theorem raises the intriguing possibility that torsion may exist in nature, but may have been overlooked experimentally.

In this situation two questions come to mind. Are there $R+R^2$ Lagrangians other than Eq. (1.1) which possess a Birkhoff theorem? Also, if torsion does not manifest itself in the spherically symmetric sector, where does it show up? The answer to the second question is that the source must have unnatural parity (e.g., angular momentum); we defer discussion of such sources to a later paper. In the present paper, we focus on spherically symmetric sources and answer the first question in the affirmative: The most general $R+R^2$ Lagrangian one could construct contains five invariants of order R^2, hence five parameters rather than the single parameter χ of Eq. (1.1). For most choices of these constants, the theory possesses a Birkhoff theorem (see Sec. II). The remaining choices may allow non-Schwarzschild solutions, but even these solutions are ruled out if one imposes the physically reasonable boundary condition of asymptotic flatness (see Sec. III).

We wish to consider *all* Lagrangians of the form $R+R^2$, not just the special case (1.1), because the linearized version of Lagrangian (1.1) possesses a ghost pole in the propagator for those components of torsion which have spin-parity $J^P=2^-$ (Ref. 5). The ghost can be removed easily by adding an additional order-R^2 invariant to the Lagrangian (1.1).[6] Once this is done, however, there is no longer anything special about the choice (1.1), and one is obliged to consider all invariants of order R^2. Of course, one could go on to consider order-R^3, -R^4, etc., invariants; but the $R+R^2$ Lagrangians considered here represent the simplest ones which can give rise to a nontrivial torsion. (If the Lagrangian contains *no* order-R^2 terms, the field equations imply that torsion vanishes.[7])

In addition to the $R+R^2$ Lagrangians, there is another class of simple Lagrangians which give rise to nontrival torsion. Since the torsion $Q^{\alpha}{}_{\beta\gamma}$ is a tensor rather than a connection, it could be used by itself to construct invariants. All invariants linear in torsion turn out to be total derivatives, but there are quadratic invariants of the form Q^2 and $(\partial Q)^2$. These invariants are no more complex than the R^2 terms we consider here, and they would produce a nontrivial torsion if present in the Lagrangian. Would they also allow a Birkhoff theorem?

The only sure way to answer this question is to add such invariants, one by one, and work out their consequences in detail. However, one can make a strong case that adding these invariants will destroy the Birkhoff theorem, and at the same time achieve considerable insight into what makes $R+R^2$ theories so unique, by studying the linearized limit of the $R+R^2$ field equations. This we have done in a recent paper (hereafter referred to as I), where we worked out the propagator (Green's function) for the linearized theory.[5] If the exact theory has a Birkhoff theorem, one would expect the linearized theory to have a propagator with $1/r$ behavior, as does the linearized limit of standard general relativity; if the exact theory has no Birkhoff theorem, one would expect a less standard propagator containing Yukawa terms e^{-mr}/r in addition to the Newtonian terms $1/r$. Naively, one would expect Yukawa terms to be present in $R+R^2$ theory, because the R term, when expanded in vierbein fields and Q's, contains a Q^2 term which will give rise to mass terms in the linearized field equations for the torsion. Such "mass" terms are indeed present, but in most of the field equations they have a very unorthodox form which forbids a Yukawa solution. Let us sketch a proof of this statement. (Full details of the straightforward but rather lengthy calculation are given in I.) Since there are 24 independent $Q^{\alpha}{}_{\beta\gamma}$ components and they are coupled by the field equations in a complex way, the first step is to choose a more natural basis, one not so strongly coupled by the linearized equations. Such a basis is constructed in I, and we denote its members by $Q_1, Q_2, \cdots, Q_{24}$. Use of this basis enormously simplifies the linearized field equations: Each Q_i is coupled to at most one other Q_j, or to at most one member of the corresponding vierbein basis V_i. Let Q_1 and Q_2 be a pair coupled by the linearized field equations. Then the Lagrangian for Q_1 and Q_2 is also simple:

$$L = \Lambda\partial^{\mu}(Q_1+Q_2)\partial_{\mu}(Q_1+Q_2) + (a/8\pi G)(Q_1+Q_2)(Q_1-Q_2)+\cdots, \qquad (1.3)$$

where Λ is a coupling parameter appearing in the order-R^2 terms, a is a constant of order unity, and the dots indicate terms which do not contribute to the linearized field equations for Q_1 and Q_2.

Already at this point we see that the field equations for Q_1 and Q_2 are going to be unorthodox: There is no "kinetic" term in L for the Q_1-Q_2 combination; and the mass term in L is indefinite. This unorthodoxy is our "punishment" for insisting that all Q and ∂Q terms in the Lagrangian occur only as part of a Cartan curvature tensor R, and not independently. If we allowed Q^2 and $(\partial Q)^2$ terms in the Lagrangian, not just $R+R^2$ terms, we would approximately triple the number of allowed invariants. Since such a wide choice of invariants is not available, it should not be surprising that the linearized Lagrangian (1.3) turns out to be a bit unusual.

The rest of the proof is elementary. From Eq. (1.3) we can derive the linearized field equations and write them in a matrix notation

$$\begin{bmatrix} \Lambda k^2 - a/8\pi G & \Lambda k^2 \\ \Lambda k^2 & \Lambda k^2 + a/8\pi G \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = 0, \qquad (1.4)$$

where we have shifted to k space. The propagator in k space is the inverse of the matrix (1.4):

$$P = \begin{bmatrix} \Lambda k^2 + a/8\pi G & -\Lambda k^2 \\ -\Lambda k^2 & \Lambda k^2 - a/8\pi G \end{bmatrix}\left[\frac{8\pi G}{a}\right]^2. \qquad (1.5)$$

Despite the presence of mass terms in Eq. (1.3), this propagator will have no Yukawa terms in configuration space because there are no $(k^2+m^2)^{-1}$ poles in k space. Q.E.D.

Indeed, this propagator has no poles of any type. The term "propagator" is a misnomer here, because Q_1 and Q_2 do not propagate; they are nonzero only inside matter. (The propagators for some of the Q_i do have $1/k^2$ poles and propagate in Newtonian fashion. There are also Q_i with Yukawa propagators; but these Q_i also have unnatural parity and therefore are irrelevant to Birkhoff's theorem.)

One motivation for adding Q^2 and $(\partial Q)^2$ terms is that a pure $R+R^2$ theory probably will not be renormalized when quantized. Lack of $(k^2+m^2)^{-1}$ poles in k space implies poor high-energy behavior of the theory. Once Q^2 and $(\partial Q)^2$ terms are added, the theory should be much easier to renormalize, since the high-energy behavior of propagators and vertices is no worse than that of standard (torsion-free) $R+R^2$ theory, which is renormalizable.[8] Of course, once $Q^2+(\partial Q)^2$ terms are added, we would also expect the Birkhoff theorem to disappear.

II. BIRKHOFF THEOREMS

The most general $R+R^2$ Lagrangian was written down in I.[5,9] The $F^{ab}_{\mu\nu}$ curvature of I differs from the $R^{\rho\sigma}{}_{\mu\nu}$ curvature used in RY only by some vierbein fields:

$$R^{\rho\sigma}{}_{\mu\nu} = V^{\rho}_{a}V^{\sigma}_{b}F^{ab}_{\mu\nu}. \qquad (2.1)$$

(We have set the Yang-Mills coupling g of I equal to unity.) Therefore, we can rewrite the Lagrangian in I as ($\kappa = 8\pi G$)

$$(\det V)\mathcal{L} = \tfrac{1}{8}R^{\alpha\beta}{}_{\sigma\rho}\bar{R}^{\sigma\rho}{}_{\alpha\beta} - R/2\kappa, \qquad (2.2)$$

where $\bar{R}$ depends on six parameters λ_i:

$$\begin{aligned}\bar{R}^{\sigma\rho}{}_{\alpha\beta} &= 2\lambda_1(\delta^{\sigma}_{\alpha}\delta^{\rho}_{\beta})R + 4\lambda_2(\delta^{\rho}_{\beta}R^{\sigma}{}_{\alpha} - \delta^{\sigma}_{\beta}R^{\rho}{}_{\alpha}) \\ &\quad + 2\lambda_6 R^{\sigma\rho}{}_{\alpha\beta} + 4\lambda_3(\delta^{\sigma}_{\alpha}R_{\beta}{}^{\rho} - \delta^{\rho}_{\alpha}R_{\beta}{}^{\sigma}) \\ &\quad + 2\lambda_4 R_{\alpha\beta}{}^{\sigma\rho} + 4\lambda_5(R^{\sigma}{}_{\beta\alpha}{}^{\rho} - R^{\rho}{}_{\beta\alpha}{}^{\sigma}) - (\alpha-\beta). \end{aligned} \qquad (2.3)$$

The RY case is $\lambda_4 \neq 0$, all other $\lambda_i = 0$. The λ_1 term will not contribute because the field equations (3′) below will predict $R=0$ in empty space.

Even in cases where the λ_1 invariant does not vanish, the field equations derived from Lagrangian (2.2) will depend on five parameters only, not on six. As pointed out in the concluding remarks of I, one linear combination of the λ_1, λ_2, and λ_6 terms is a total derivative because of a Bianchi identity and therefore may be dropped. We shall not eliminate this combination from Eq. (2.2) because its presence serves as a check on the algebra. All final answers should depend on the combinations $\lambda_1+\lambda_2$, $\lambda_2+\lambda_6$, never on the combination $\lambda_1-2\lambda_2+\lambda_6$.

Next, we obtain the field equations. We apply the usual Euler-Lagrange technique to Eq. (2.2). Following RY, we choose the connection $\Gamma^{\alpha}{}_{\beta\gamma}$ and vierbein V^{α}_{a} as our independent variables, and we introduce the abbreviation $B^{\alpha}{}_{\beta\gamma}$ for the torsion-dependent difference $\Gamma^{\alpha}{}_{\beta\gamma}-\{^{\alpha}{}_{\beta\gamma}\}$, where $\{^{\alpha}{}_{\beta\gamma}\}$ is the Christoffel connection.[10] We get

$$\tfrac{1}{2}R^{\alpha\beta}{}_{\nu\sigma}\overline{R}_{\mu}{}^{\sigma}{}_{\alpha\beta}-\tfrac{1}{8}g_{\mu\nu}R^{\alpha\beta}{}_{\sigma\rho}\overline{R}^{\sigma\rho}{}_{\alpha\beta}-G_{\mu\nu}/\kappa=0\,, \tag{3$'$}$$

$$\nabla_{\sigma}\overline{R}^{\tilde{\sigma}\tilde{\gamma}}{}_{\tilde{\alpha}\hat{\beta}}-(2/\kappa)(\delta^{\gamma}_{\alpha}B^{\delta}{}_{[\beta\delta]}-\delta^{\gamma}_{\beta}B^{\delta}{}_{[\alpha\delta]}+B^{\gamma}{}_{[\alpha\beta]})=0\,, \tag{4$'$}$$

which are readily seen to be generalizations of RY's field equations (3) and (4). As in RY, we use tildes over an index to signify that covariant differentiation of that index is to be carried out using the Christoffel connection. Similarly, carets over an index signify use of the Cartan connection. As advertised earlier, Eq. (3′) implies $R=0$ in free space. We shall refer to Eqs. (3′) and (4′) as the Einstein and Cartan field equations, respectively.

Next we specialize Eqs. (3′) and (4′) to the case of spherical symmetry with metric (1.2). RY use symmetry plus metric compatibility to show that there are only five independent nonzero Cartan connection components. Four of these have nonzero torsion: $\Gamma^{T}{}_{RT}$, $\Gamma^{R}{}_{\theta\theta}$, $\Gamma^{T}{}_{RR}$, $\Gamma^{T}{}_{\theta\theta}$. Following RY, we denote these V, W, X, and Y, respectively; and we denote their torsion-dependent parts $\Gamma^{\alpha}{}_{\beta\gamma}-\{^{\alpha}{}_{\beta\gamma}\}$ by f, g, h, and k, respectively. Also, the independent nonvanishing components of the Cartan curvature are $R^{T}{}_{RTR}$, $R^{T}{}_{\theta T\theta}$, $R^{T}{}_{\theta R\theta}$, $R^{R}{}_{\theta T\theta}$, $R^{R}{}_{\theta R\theta}$, $R^{\theta}{}_{\phi\theta\phi}$, denoted by $-A$, $-C$, $+D$, $-G$, $+H$, and $+L$, respectively. Using this notation, we find that Eq. (3′) implies the following generalizations of RY's Einstein field equations (43)–(47):

$$(D-G)[1+(C-H)(2\alpha-4\overline{\alpha})]=0\,, \tag{43$'$}$$

$$D+G-[8\chi(CD-GH)-2\alpha(C-H)(D+G)$$
$$-4\overline{\alpha}(C+H)(D-G)]=0\,, \tag{44$'$}$$

$$C+H-[(4\chi-2\alpha)(C^2-H^2)$$
$$+4(\chi-\overline{\alpha})(D^2-G^2)]=0\,, \tag{45$'$}$$

$$(A-L)+2(C-H)=0\,, \tag{46$'$}$$

$$(A+L)[1+(C-H)(8\chi-4\alpha)]=0\,. \tag{47$'$}$$

The single parameter χ of RY is replaced by three parameters χ, α, $\overline{\alpha}$ which are related to the Lagrangian parameters λ_i of I as follows:

$$\chi=\kappa(2\lambda_2+2\lambda_3+\lambda_4+2\lambda_5+\lambda_6)\,, \tag{2.4}$$

$$\alpha=\kappa(\lambda_2+\lambda_3)\,, \tag{2.5}$$

$$\overline{\alpha}=\kappa(2\lambda_2+\lambda_5+\lambda_6)\,. \tag{2.6}$$

Similarly, the RY Cartan field equations (35)–(38) generalize to

$$-2(W+e^{-\Lambda}r'/r)+4(e^{-\Lambda}/r^2)\{r^2[\chi A+\alpha(C-H)]\}'$$
$$-8Y[(\chi-\overline{\alpha})G+\overline{\alpha}D]+8W[\chi C-\alpha(C-H)/2]=0\,, \tag{35$'$}$$

$$-2(Y-e^{-\Phi}\dot{r}/r)-4(e^{-\Phi}/r^2)\{r^2[\chi A+\alpha(C-H)]\}^{\cdot}$$
$$-8Y[\chi H+\alpha(C-H)/2]+8W[(\chi-\overline{\alpha})D+\overline{\alpha}G]=0\,, \tag{36$'$}$$

$$-[X+Y-(e^{-\Phi-\Lambda}/r)(re^{\Lambda})^{\cdot}]-4(e^{-\Phi-\Lambda}/r)\{re^{\Phi}[(\chi-\overline{\alpha})D+\overline{\alpha}G]\}'-4(e^{-\Phi-\Lambda}/r)\{re^{\Lambda}[\chi C-\alpha(C-H)/2]\}^{\cdot}$$
$$-4V[(\chi-\overline{\alpha})G+\overline{\alpha}D]-4\chi[\chi H+\alpha(C-H)/2]-4Y[\chi L-\alpha(C-H)]=0\,, \tag{37$'$}$$

$$-[W-V+(e^{-\Phi-\Lambda}/r)(re^{\Phi})']-4(e^{-\Phi-\Lambda}/r)\{re^{\Phi}[\chi H+\alpha(C-H)/2]\}'-4(e^{-\Phi-\Lambda}/r)\{re^{\Lambda}[(\chi-\overline{\alpha})G+\overline{\alpha}D]\}^{\cdot}$$
$$-4V[\chi C-\alpha(C-H)/2]-4X[(\chi-\overline{\alpha})D+\overline{\alpha}G]-4W[\chi L-\alpha(C-H)]=0\,. \tag{38$'$}$$

The primes and dots denote derivatives with respect to R and t, respectively.[11]

The following are sufficient conditions for the theory defined by the above field equations to possess a Birkhoff theorem:

either

$$\overline{\alpha}=\chi,\quad \alpha=\text{anything} \tag{2.7}$$

or

$$2\overline{\alpha}=\alpha,\quad \chi=\text{anything}\,. \tag{2.8}$$

The proof in each case is identical to that given in RY, except for the substitution of primed Eqs. (35′)–(38′) and (43′)–(47′) for the corresponding unprimed equations of RY.

III. NON-SCHWARZSCHILD SOLUTIONS

Since we have been able to prove a Birkhoff theorem only for some choices of the parameters χ, α, $\overline{\alpha}$, we must consider the possibility that for other choices of the parameters a non-Schwarzschild solution may exist. Indeed we shall construct such a solution explicitly below. The following theorem, however, implies that all such solutions may be ruled out by imposing the boundary condition of asymptotic flatness at $R\to\infty$: If the combination of curvature components $-C+H \equiv R^{T}{}_{\theta T\theta}+R^{R}{}_{\theta R\theta}$ equals zero (or any nonconstant function of R and T), then the solution is the Schwarzschild metric with zero torsion. Note, asymptotic flatness guarantees that the assumptions of the theorem will be satisfied (because curvature must fall off to zero as $R\to\infty$).

We outline the proof of this theorem. From the assumptions of the theorem and Eqs. (43′) and (47′),

$$A+L=D-G=0. \tag{3.1}$$

Equations (44′) and (45′) then predict

$$G=D=C+H=0. \tag{3.2}$$

From this result plus Eq. (46′), all the assumptions of RY's case III are satisfied. The rest of the proof is identical to that used by RY to prove a Birkhoff theorem for case III.

We now construct a specific solution which violates Birkhoff's theorem. From the theorem just proven, we must pick $-C+H=$ nonzero constant. We choose

$$-C+H=(8\chi-4\alpha)^{-1}\neq 0 \tag{3.3}$$

so that Eq. (47′) is satisfied. We substitute Eq. (3.3) into Eqs. (35′) and (36′) and use the following two Bianchi identities to simplify the resulting equations:

$$-(e^{-\Lambda}/r^2)(r^2L)'+2YD-2WH=0, \tag{39}$$

$$(e^{-\Phi}/r^2)(r^2L)^{\cdot}+2YC-2WG=0. \tag{40}$$

Equations (35′) and (36′) then collapse to the following two equations:

$$-3W-8Y(\overline{\alpha}-\chi)(D-G)=0, \tag{3.4}$$

$$-3Y-8W(\overline{\alpha}-\chi)(D-G)=0. \tag{3.5}$$

We must avoid the solution $W=Y=0$, which would lead to $C-H=0$, contradicting Eq. (3.3). [See RY Eqs. (19)–(24), which express the curvature components C, H, D, G, $\cdots$ in terms of the connection components V, W, X, Y.] Hence, we choose

$$W=\pm Y, \tag{3.6}$$

$$\overline{\alpha}-\chi\neq 0. \tag{3.7}$$

We do not need to impose $D-G\neq 0$ because, if we use RY Eqs. (19)–(24) expressing curvature in terms of connection, we find from $W=\pm Y$ that

$$D=\pm H,\quad G=\pm C. \tag{3.8}$$

Hence, Eq. (3.3) guarantees $D-G\neq 0$. If we eliminate $D-G$ from Eqs. (3.4) and (3.5), using Eqs. (3.6) and (3.8) we find

$$4\chi+2\overline{\alpha}=3\alpha. \tag{3.9}$$

From this constraint plus Eq. (3.8), we find that all the Einstein equations except (46′) are satisfied. The remaining two Cartan equations (37′) and (38′) collapse to one equation

$$V-(e^{-\Phi-\Lambda}/r)(re^{\Phi})'\pm[(e^{-\Phi-\Lambda}/r)(re^{\Lambda})^{\cdot}-\chi]=0. \tag{3.10}$$

Let us summarize. The Cartan field equations have collapsed to the two equations (3.6) and (3.10); provided the parameters obey Eq. (3.9), the Einstein equations collapse to the two equations (46′) and (3.3). We therefore have four field equations to be satisfied by seven unknowns V, W, X, Y, Φ, Λ, r. (The $V, W, \cdots$ are linear functions of torsion $f, g, \cdots$ and are therefore independent of the metric parameters.) Obviously there will be an infinity of solutions. We write down one such solution. Let the solution be static with $r=R$ and

$$e^{\Phi}=e^{-\Lambda}=c_0[r/(4\chi-2\alpha)+1/r]+1. \tag{3.11}$$

c_0 is a constant of integration. As expected from the theorem, the solution is not asymptotically flat. We record also the expressions for f, g, h, and k, the torsion-dependent parts of V, W, X, and Y, respectively:

$$\pm h\equiv\pm(\Gamma^{T}{}_{RR}-\{^{T}{}_{RR}\})\equiv\pm(X-\{^{T}{}_{RR}\}) = e^{-\Phi}[r/(4\chi-2\alpha)-1/r+1/c_0]+e^{\Phi}(\phi'+1/r), \tag{3.12}$$

$$g\equiv\Gamma^{R}{}_{\phi\phi}-\{^{R}{}_{\phi\phi}\}\equiv W-\{^{R}{}_{\phi\phi}\} = e^{\Phi}/r+c_0/(4\chi-2\alpha-c_0)4, \tag{3.13}$$

$$f\equiv\Gamma^{T}{}_{RT}-\{^{T}{}_{RT}\}\equiv V-\{^{T}{}_{RT}\} = e^{\Phi}/r\pm h, \tag{3.14}$$

$$\pm k\equiv\pm(\Gamma^{T}{}_{\theta\theta}-\{^{T}{}_{\theta\theta}\})\equiv\pm(Y-\{^{T}{}_{\theta\theta}\}) = g-e^{\Phi}/r. \tag{3.15}$$

As expected from the discussion in the Introduction, there is not a Yukawa potential anywhere.

APPENDIX: THE NEWTONIAN LIMIT

We should check to see that the theories studied in this paper have a Newtonian limit; otherwise, they may easily be distinguished from Einstein's theory, even though they possess a Birkhoff theorem. Einstein's theory possesses a Newtonian limit because, when all velocities are low, we recover the Newtonian equation of motion for test particles $\ddot{X}=-\nabla\phi$, where ϕ obeys the usual field equation $\nabla^2\phi = -4\pi G T_{00}$. By "low velocity" we mean

$$GM/r \cong V^2/c^2 \equiv \beta^2 \ll 1\,, \tag{A1}$$

where V, M, and r are a typical velocity, mass, and length characterizing the matter distribution.

Fairchild has checked the Newtonian limit for a specific $R+R^2$ theory,[3] but his procedure has been criticized. Fairchild shows that the Einstein field equations for this theory collapse to the usual field equations for general relativity, except for some torsion-dependent terms added to the stress tensor $T_{\alpha\beta}$. However, he does not estimate the size of these added terms. We shall show that these terms are zero in the Newtonian limit.

Since the present theory is more complicated than that of Einstein, we will need two additional assumptions to get a Newtonian limit:

$$S^{\alpha}{}_{\beta\gamma} = 0\,, \tag{A2}$$

$$\delta \equiv \lambda_i \hbar\kappa/r^2c^3 \ll 1\,. \tag{A3}$$

$S^{\alpha}{}_{\beta\gamma}$ is to the Cartan field equations what $T_{\alpha\beta}$ is to the Einstein field equations: $S^{\alpha}{}_{\beta\gamma}$ is the matter source. $S^{\alpha}{}_{\beta\gamma}$ therefore depends on our choice of matter Lagrangian. Fortunately, we do not have to be too specific about this choice: In most theories Eq. (A2) will be satisfied if the source has no net intrinsic spin.

Equation (A3) implies that typical lengths are greater than the Planck length times $\lambda_i^{1/2}$; this assumption is needed to prevent the order-R^2 terms (which are linear in the coupling constants λ_i) from dominating the R/κ term in the Lagrangian. Note, assumptions (A1) and (A3) are logically independent: r could be of the order of the Planck length, yet the phenomena being studied could be nonrelativistic if M were small enough. Also, one cannot drop the R^2 terms by arguing that they are at least quadratic in the V's, therefore (from our experience with standard general relativity) they contribute terms at most of order β^4 to the field equations. This β^4 is multiplied by λ_i, so that some assumption must be made which limits λ_i. Furthermore, the leading linear in V term contributed by R/κ can also be of order β^4.

If we assume $\delta \ll 1$, we can neglect the order-R^2 terms in the field equations. Once this is done, the Newtonian limit is immediate. In fact, without order-R^2 terms the theory collapses to Kibble's theory, which is identical to that of Einstein for $S^{\alpha}{}_{\beta\gamma}=0$ (Ref. 7). The exact Cartan equations in Kibble's theory predict that the quantity

$$\Gamma^{\alpha}{}_{\beta\gamma} - \left\{{}^{\alpha}{}_{\beta\gamma}\right\} \equiv B^{\alpha}{}_{\beta\gamma} \tag{A4}$$

is identically zero.

The foregoing result suggests that the Newtonian limit is the leading term in a *double* expansion

$$B^{\alpha}{}_{\beta\gamma} = \sum_{m,n} B^{(mn)\alpha}{}_{\beta\gamma}\,, \tag{A5a}$$

$$V^{\alpha}_{a} = \sum_{m,n} V^{(mn)\alpha}{}_{a}\,, \tag{A5b}$$

where

$$B^{(mn)} = O(\delta^m\beta^n/r), \quad V^{(mn)} = O(\delta^m\beta^n)\,, \tag{A6}$$

and $B^{(0n)}=0$. Furthermore, it is a plausible conjecture that "post-Newtonian" corrections may be calculated by a double successive approximation scheme, where results from lower-order calculations are used to linearize the equations for the higher-order corrections, exactly as in standard general relativity. The only sure way to verify this conjectured scheme is to carry it out explicitly. We shall not do this here, but we shall go through a simple dimensional analysis which indicates that the scheme is plausible, and that the lowest-order post-Newtonian corrections are of order δ or smaller. We write the Cartan and Einstein field equations in a schematic form designed to facilitate dimensional analysis (all tensor indices and powers of β and vierbein fields are suppressed; all derivatives are assumed to be of order $1/r$):

$$S = O(B/\kappa) + O(\lambda_i/r^3) + O(\lambda_i B/r^2, \lambda_i B^2/r, \lambda_i B^3)\,, \tag{A7a}$$

$$T = O(1/\kappa r^2) + O(B/\kappa r, \lambda_i/r^4) + O(B^2/\kappa, \lambda_i B/r^3, \lambda_i B^2/r^2, \lambda_i B^3/r, \lambda_i B^4)\,. \tag{A7b}$$

Equations (A7a) and (A7b) are nothing more than a catalog of every term allowed by dimensional analysis [B, λ_i, and κ have dimension 1/length, 1, and (length)^2, respectively, in natural units] except that the order-$(1/\kappa r)$ term is omitted from the Cartan equations (A7a). Detailed examination of those equations shows that no term of this type is present. That is, the R/κ term in the Lagrangian contributes no term involving only V's to the Cartan equations. Furthermore, when we wrote out the linearized Cartan equations in detail in I, we found that an order-B/κ term is always present. We now rediscover our earlier result that $B^{(0n)}=0$, together with a new result that $B^{(in)}\neq 0$ if the order-(λ_i/r^3) terms are nonzero. We see no reason for λ_i/r^3 terms to be absent, except perhaps for special values of the λ_i. Even if the λ_i/r^3 terms were absent, the ansatz (A5a) would probably continue to be satisfied, but for an uninteresting reason:

$B^{\alpha}{}_{\beta\gamma} \equiv 0$.

Of course to verify the Newtonian limit one must study the equations of motion of the sources as well as the field equations. Yasskin has worked out the equations of motion in his thesis.[10] It should come as no surprise that in the Newtonian limit $B^{(0n)} = 0$, these equations collapse to the usual ones $T^{\mu\nu}{}_{;\nu} = 0$.

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[1]Sriram Ramaswamy and Philip B. Yasskin, Phys. Rev. D 19, 2264 (1979). This paper will be referred to as RY in the text.

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[6]Equation (1.1) is the Lagrangian of Ref. 5, specialized to the case $\lambda_4 \neq 0$ and all other $\lambda_i = 0$. From the results of Ref. 5, to guarantee no 2^- ghost as well as no 0^- tachyon, one must set $\lambda_4 + \lambda_5 = 0$ and $\lambda_4 - 2\lambda_5 > 0$. Hence an additional R^2 invariant proportional to λ_5 must also be present in the Lagrangian.

[7]T. W. B. Kibble, J. Math. Phys. 2, 212 (1961).

[8]K. S. Stelle, Phys. Rev. D 16, 953 (1977).

[9]In I and at Eq. (1.2) we use a metric which has the opposite sign from the RY metric. However, we use the same conventions as RY for curvature and torsion tensors, so none of the field equations in Secs. II and III are affected.

[10]In the Introduction, and in Ref. 5, we used vierbein fields and torsion as independent variables; here we follow RY and use vierbein fields and Cartan connection. Yasskin has shown that these alternative choices of variable lead to equivalent field equations. [Philip B. Yasskin, Ph.D. thesis, University of Maryland, 1979 (unpublished).] Since our theory is metric compatible,

$$B^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma}).$$

We use the $B^{\alpha}{}_{\beta\gamma}$ notation of I rather than the $\lambda^{\alpha}{}_{\beta\gamma}$ notation of RY, because in the Appendix and in I λ is used to denote a coupling parameter in the order-R^2 terms.

[11]We must verify that the new field equations (43′)–(47′) and (35′)–(38′) depend only on $\lambda_1 + \lambda_2$ and $\lambda_2 + \lambda_6$, as required by the Bianchi identity. This will certainly be the case if we can add a quantity $+\lambda_1$ to each of the quantities in parentheses in Eqs. (2.4)–(2.6). Will these added λ_1 terms change any of the field equations? It is easy to verify that the new λ_1 terms drop out of the Einstein equations (43′)–(47′). In each of the Cartan equations (35′)–(38′) λ_1 multiplies a function of curvature which vanishes because of the Bianchi constraints on the curvature. These constraints have been worked out by RY in their Eqs. (39)–(42). [The algebra required is straightforward, provided one uses the condition $R = 0$ repeatedly, i.e., (46′).] Therefore, the field equations depend only on $\lambda_1 + \lambda_2$ and $\lambda_2 + \lambda_6$, a reassuring check on the algebra.