Quantum effects in the early universe. III. Dissipation of anisotropy by scalar particle production

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The dissipation of small amounts of anisotropy by particle production in homogeneous, spatially flat cosmologies is studied using the effective-action method. As a particular model we consider the production of conformally invariant scalar particles in a universe which also contains some classical radiation. We find that there is significant dissipation of anisotropy near the singularity. Calculations of the classical geometry, the particle production and vacuum persistence amplitudes, the spectrum of produced particles and their back reaction on the geometry are all discussed to lowest nonvanishing order in the deviation from isotropy.

I. INTRODUCTION

Anisotropy in the very early universe will be dissipated by the quantum-mechanical production of elementary particle pairs in the strongly time-dependent geometry of the big bang. Since the first suggestion of this mechanism by Zel'dovich,[1] the efficiency of this process has been demonstrated in homogeneous, but anisotropic model cosmologies by the calculations of Zel'dovich and Starobinsky[2,3] and Hu and Parker.[4] Since these calculations have been carried out in the test-field approximation, or for other reasons, they have been restricted to spacetime regions which do not include the initial singularity. In this paper we shall continue the study begun in two earlier papers[5,6] of a model calculation of anisotropy damping which can be carried out over the whole of spacetime because it includes the quantum effects on the dynamics of the universe in a consistent way.

The assumptions of the model and its strengths and weaknesses have been discussed in detail in paper II, but the major features are the following: We consider the production of scalar particles described by the conformally invariant wave equation

$$\Box^2\varphi - \tfrac{1}{6}R\varphi = 0 \tag{1.1}$$

in a homogeneous spatially flat classical geometry described by the line element

$$ds^2 = a^2[-d\eta^2 + (e^{2\beta})_{ij}dx^i dx^j]\,. \tag{1.2}$$

Here β_{ij} is a traceless 3×3 matrix, and both it and the scale factor a are functions of η alone. The universe is assumed to contain radiation described classically and whose pressure p_r and energy density ρ_r are related by the equation of state $p_r = \frac{1}{3}\rho_r$. The basic simplifying assumption of the model is that the anisotropy is small. Calculations of particle production probabilities, the vacuum persistance amplitude, and the back reaction of the produced particles on the geometry can therefore be developed in a perturbation series in β_{ij} and only the lowest nonvanishing order discussed. To compute these lowest-order terms we shall use the effective-action method.

The central quantity in the effective-action method is the effective-action functional $\Gamma[\bar{g}_{\alpha\beta}]$. This gives the vacuum persistence amplitude in the presence of classical external sources $\mathcal{T}^{\alpha\beta}(x)$ through the relation

$$\langle 0_+|0_-\rangle = \exp(iW)\,, \tag{1.3}$$

where

$$W = \Gamma[\bar{g}_{\alpha\beta}] + \int d^4x\, \bar{g}_{\alpha\beta}\mathcal{T}^{\alpha\beta}\,, \tag{1.4}$$

and $\bar{g}_{\alpha\beta}$ is the classical geometry which solves the equation

$$(\delta\Gamma/\delta\tilde{g}_{\alpha\beta})_{\bar{g}} = -\mathcal{T}^{\alpha\beta}\,. \tag{1.5}$$

Provided it is small, the total probability to produce a particle pair over the entire history of the universe is, from Eq. (1.3),

$$P = 2\,\mathrm{Im}W\,. \tag{1.6}$$

The summary of the effective-action method in the preceding paragraph is slightly more general than that of papers I and II, in that it explicitly includes the possibility of an external source throughout. The derivation of the generalization is easily accomplished. Equations (1.3) and (1.4) are direct consequences of Eqs. (I.2.5) and (I.2.4) generalized to include an external source. Equation (1.5) is Eq. (I.2.11). The generalization is not needed here to treat the classical radiation in the model. As before, the degrees of freedom representing that radiation will be explicitly included in the classical action and not treated as an external source. Rather, it is needed to enforce the constraint that the model contain some anisotropy. In the absence of such a constraint the extremum of the effective action would be an isotropic geometry. This constraint may be thought of as being maintained by sources located at the boundary of the spacetime region under consideration and in particular at the singularity. These constraint-maintaining external sources will thus not affect the local dynamical equations for the classical geometry which continue to be

$$(\delta\Gamma/\delta\tilde{g}_{\alpha\beta})_{\tilde{g}}=0\,, \tag{1.7}$$

but they will enter into the calculation of the total probability through Eqs. (1.6) and (1.4).

In paper I Eq. (1.7) was solved and the classical geometry was calculated in the limit of exact isotropy—the zeroth order of the perturbation problem described above. In this limit the quantum effect on the dynamics of the classical geometry arises from the anomalies in the trace of the vacuum matrix element of the stress-energy tensor of the scalar field. There is a class of physically reasonable classical geometries which extremize the effective action. In each of these the singularity is softer than that in the corresponding solution of Einstein's equations in the sense that the curvature diverges less strongly as a function of the scale factor a.

In paper II we calculated the effective action to second order in the anisotropy and displayed explicitly the dynamical equations [(II.3.5) and (II.3.8)] which determine the classical geometry in this order. In this paper we shall discuss the solutions of these equations. We shall find that there is only one member of the class of isotropic classical geometries derived in paper I which permits a solution of the dynamical equations which is consistent with the assumptions entering into the derivation of the effective action in paper II. This is the geometry referred to as the "marginal case" in paper I for which the scale factor varies near the singularity as $a(\eta)\sim\exp(\text{const}\times\eta)$, $\eta\to-\infty$. This geometry has a number of interesting properties. It is conformally related to a complete flat spacetime. (For this reason we shall also refer to it as the "conformally complete" case.) It has no cosmological particle horizons. Finally, the production of massive particles in a model cosmology whose scale factor behaves at all times as $\exp(\text{const}\times\eta)$ yields a thermal distribution of produced pairs at high energies.[7]

Our main result, obtained in Sec. II, is that for this conformally complete geometry the softening of the singularity which arises from the trace anomalies in zeroth order is already sufficient to make the particle production amplitudes finite to second order in the anisotropy without the inclusion of further back-reaction effects. In Sec. III we will display a crude calculation of the total particle production probability which omits the nonlocal parts of the dynamical equation. In Sec. IV the calculation of the spectrum of produced particles will be discussed. Section V contains a brief discussion of the back reaction.

II. PARTICLE PRODUCTION TO SECOND ORDER IN THE ANISOTROPY

The effective-action functional Γ for argument geometries of the form in Eq. (1.2) is a functional of a and β_{ij} alone. We write its expansion in powers of β_{ij} as

$$\Gamma[a,\beta]=\Gamma_0[a]+\Gamma_2[a,\beta]+\cdots\,, \tag{2.1}$$

where the first two terms are constant and quadratic in β_{ij}, respectively, and were calculated in papers I and II [Eqs. (II.2.13) and (II.2.29)]. The classical geometry is the solution of the dynamical equations

$$\frac{\delta\Gamma}{\delta a}=0 \tag{2.2a}$$

and

$$\frac{\delta\Gamma}{\delta\beta_{ij}}=-\tau^{ij}\,, \tag{2.2b}$$

where the external sources τ^{ij} [combinations of the $\mathcal{T}^{\alpha\beta}$ in Eq. (1.5)] that fix the amount of the anisotropy in the model vanish everywhere except on the boundaries of the spacetime region under consideration.

The classical geometry which solves Eqs. (2.2) can itself be formally expanded in powers of the anisotropy in the model. For the *solution* for the scale factor we write

$$a(\eta)=a_0(\eta)+a_2(\eta)+\cdots\,. \tag{2.3}$$

Here a_0 is one of the scale factors for an exactly isotropic universe, determined in paper I as an extremum of $\Gamma_0[a]$. The quantity a_2 represents the corrections to the scale factor which are quadratic in the anisotropy, including the back reaction of the produced particles. Since the action contains no terms linear in β_{ij}, this is the lowest order in the anisotropy in which corrections to the scale factor occur.

The solutions of Eq. (2.2b) have a similar expansion in powers of the anisotropy. The lowest order is linear in the anisotropy and is determined by the linear part of Eq. (2.2b). In the interior of the spacetime region under consideration, it reads

$$\delta\Gamma_2[a_0,\beta]/\delta\beta_{ij}=0\,. \tag{2.4}$$

As this is the only order we shall need to calculate in subsequent calculations, we shall henceforth understand by β_{ij} this *solution* which is linear in the anisotropy without additional notational devices.

The total particle production probability P is given by Eq. (1.6). Making use of Eqs. (2.1) and (2.3), the contribution of the effective action evaluated at the classical geometry may be expanded in powers of the anisotropy in the model. To quadratic order one has

$$\Gamma[\bar{g}_{\alpha\beta}] = \Gamma_0[a_0] + \int_{-\infty}^{+\infty} d\eta \left(\frac{\delta \Gamma_0}{\delta a}\right)_{a_0} a_2(\eta) + \Gamma_2[a_0, \beta] + \cdots . \tag{2.5}$$

The second term vanishes because a_0 is an extremum of Γ_0. As far as its functional dependence on β_{ij} is concerned, Γ_2 has the general quadratic form

$$\Gamma_2 = \tfrac{1}{2} \int d^4x \int d^4x' \, \beta_{ij}(x) G^{ijkl}(x, x') \beta_{kl}(x') , \tag{2.6}$$

where the form of the operator G^{ijkl} is given in Eq. (II.2.19). Equation (2.2b) then reads

$$\int d^4x' \, G^{ijkl}(x, x') \beta_{kl}(x') = -\tau^{ij}(x) . \tag{2.7}$$

It is then an elementary calculation to verify from Eqs. (2.7), (2.6), (2.5), and (1.4) that the functional W at the constrained extremum of the effective action is

$$W = \Gamma_0[a_0] - \tfrac{1}{2} \int d^4x \int d^4x' \, \beta_{ij}(x) G^{ijkl}(x, x') \beta_{kl}(x') . \tag{2.8}$$

The total probability P to produce a pair over the history of the universe is thus, through Eqs. (1.6), (2.8), and (2.6),

$$P = -2 \, \mathrm{Im} \Gamma_2[a_0, \beta] , \tag{2.9}$$

where the effective action is understood to be evaluated at the solution of Eq. (2.4) for β_{ij}. Thus, only a_0 and this solution for β_{ij} are required to calculate P to quadratic order in the anisotropy.

The class of physically reasonable a_0 has already been determined in paper I. Each member represents a universe which starts from a singularity where $a = 0$ and evolves to a Friedmann solution at late times with

$$a_0(\eta) \sim (\tilde{\rho}_r/6)^{1/2} l \eta , \quad \eta \to \infty \tag{2.10a}$$

where $l = (16\pi G)^{1/2}$ and $\tilde{\rho}_r$ is the constant giving the density of classical radiation according to

$$\rho_r = \tilde{\rho}_r / a_0{}^4 . \tag{2.10b}$$

The expansions of a, β_{ij}, and Γ in powers of the model's anisotropy have a purely formal status until the finiteness of the coefficients in the expansion can be demonstrated. This is not merely a technical issue. The question is whether or not the back reaction of the produced particle pairs is a significant correction to the classical geometry for arbitrarily small anisotropy. If, for example, Eq. (2.9) yields an infinite answer when evaluated with a_0, then it is clear that a_0 is not a reasonable approximation to $a(\eta)$, and that back-reaction effects must give rise to significant corrections to a_0 near the singularity. In this case neither Eq. (2.3) nor (2.5) would be valid, and it would be necessary to solve the dynamical equations (2.2) without making an expansion in powers of the initial anisotropy. The central result of this paper is that for the conformally complete $a_0(\eta)$ calculated in paper I, the total particle production probability given by Eq. (2.9) is finite and the back reaction to the classical geometry can be calculated as a perturbation on the isotropic classical geometry.

To demonstrate the finiteness of the total particle production probability in Eq. (2.9), we must solve Eq. (2.4) for β_{ij}. Expressions for Γ_2 and for a first integral of Eq. (2.4) have been given in paper II in Eqs. (II.2.29) and (II.3.5), respectively. A more convenient set of working variables than a and η are the scale-invariant combinations b and χ defined as in paper I by

$$a_0(\eta) = l \tilde{\rho}_r{}^{1/4} b(\chi) , \tag{2.11a}$$

$$\eta = 6^{1/2} \tilde{\rho}_r{}^{-1/4} \chi \equiv \gamma \chi , \tag{2.11b}$$

where γ is here defined to be $6^{1/2}/\tilde{\rho}_r{}^{1/4}$. With $\kappa_{ij} = d\beta_{ij}/d\eta$ then one has

$$\Gamma_2 = \gamma^{-1} V \left(\int_{-\infty}^{+\infty} d\chi \left\{ \left[6b^2 - \lambda \left(\frac{b''}{b}\right) - \lambda \left(\frac{b'}{b}\right)^2 \right] \kappa_{ij} \kappa^{ij} + 3\lambda \left[\frac{i\pi}{2} + \ln(\mu_1 b) \right] \kappa'_{ij} \kappa'^{ij} - 3\lambda \kappa'_{ij} K \kappa'^{ij} \right\} \right) . \tag{2.12}$$

Here $\lambda = (2880\pi^2)^{-1}$, μ_1 is the regularization scale, V is the spatial volume under consideration, $\kappa'_{ij} = d\kappa_{ij}/d\chi$, and, as before, $b' = db/d\chi$. The nonlocal operator K acting on a function f is defined in terms of the Fourier transform of f

$$\hat{f}(\sigma) = \int_{-\infty}^{+\infty} d\chi \, e^{i\sigma\chi} f(\chi) , \tag{2.13a}$$

by

$$Kf(\chi)=(2\pi)^{-1}\int_{-\infty}^{+\infty} d\sigma\, e^{-i\sigma\chi}\ln|\sigma|\hat{f}(\sigma)\,. \qquad (2.13b)$$

An explicit expression for Kf in terms of $f(\chi)$ is[8]

$$Kf(\chi)=Cf(\chi)-\tfrac{1}{2}\int_{-\infty}^{+\infty} d\chi'\,\epsilon(\chi-\chi')\ln|\chi-\chi'|\frac{df}{d\chi'}\,, \qquad (2.14)$$

where C is the Euler's constant and $\epsilon(x)=1$, $x>0$ and $\epsilon(x)=-1$, $x<0$.

The first integral of the dynamical equation which results from varying Eq. (2.12) with respect to β_{ij} is

$$3\lambda\frac{d}{d\chi}\left\{-\left[\frac{i\pi}{2}+\ln(\mu_1 b)\right]\frac{d\kappa_{ij}}{d\chi}+K\frac{d\kappa_{ij}}{d\chi}\right\}+\left[6b^2-\lambda\left(\frac{b'}{b}\right)^2-\lambda\left(\frac{b''}{b}\right)\right]\kappa_{ij}=\gamma^2 c_{ij}\,. \qquad (2.15)$$

The integration constant c_{ij} sets the magnitude and orientation of the anisotropy. This equation is to be solved for β_{ij}.

Two boundary conditions are necessary to single out the physically relevant solution for κ_{ij} from Eq. (2.15). At large times when the universe becomes large we require that it behave classically. Then a_0 is given by Eq. (2.10), and in these regions the classical behavior for κ_{ij} is

$$\kappa_{ij}\sim\gamma^2 c_{ij}/(6b^2)\sim\gamma^2 c_{ij}/(6\chi^2)\,. \qquad (2.16)$$

It is not difficult to verify that this asymptotic behavior is consistent with Eq. (2.15).

At the singularity we must consider the possibilities. We shall show that, if it exists, there is a unique solution of Eq. (2.15) with the correct behavior at large χ [Eq. (2.16)] and which gives a finite total particle production probability through Eq. (2.9). To begin we write Eq. (2.15) in the general form

$$(\mathcal{R}+i\mathcal{J})\kappa_{ij}=\gamma^2 c_{ij}\,, \qquad (2.17)$$

where $\mathcal{R}$ and $\mathcal{J}$ are differential operators with real coefficients. In particular,

$$\mathcal{J}=-\frac{3\pi\lambda}{2}\frac{d^2}{d\chi^2}\,. \qquad (2.18)$$

Both $\mathcal{R}$ and $\mathcal{J}$ are Hermitian in the space of square-integrable 3×3 matrices with the natural scalar product

$$(f,h)=\int_{-\infty}^{+\infty} d\chi\,\bar{f}^{ij}(\chi)h_{ij}(\chi)\,. \qquad (2.19)$$

In this notation the second-order contribution to the effective action can be written as

$$\Gamma_2[a,\beta]=\gamma^{-1}V[(\bar{\kappa},\mathcal{R}\kappa)+i(\bar{\kappa},\mathcal{J}\kappa)]\,. \qquad (2.20)$$

From Eq. (2.17) it follows that

$$(\bar{\kappa},\mathcal{R}\kappa)+i(\bar{\kappa},\mathcal{J}\kappa)=\gamma^2(\bar{\kappa},c) \qquad (2.21)$$

and

$$(\kappa,\mathcal{R}\kappa)+i(\kappa,\mathcal{J}\kappa)=\gamma^2(\kappa,c)\,. \qquad (2.22)$$

The matrix c_{ij} must be real in order that the solution become classical at late times according to Eq. (2.6). It then follows that

$$(\gamma V)^{-1}\mathrm{Im}\Gamma_2=\mathrm{Im}(\bar{\kappa},c)=-\mathrm{Im}(\kappa,c)=-\gamma^{-2}(\kappa,\mathcal{J}\kappa)\,. \qquad (2.23)$$

The total particle production probability can be written using the first equality and Eq. (2.9) as

$$P=2\gamma V c_{ij}\int_{-\infty}^{+\infty} d\chi[-\mathrm{Im}(\kappa^{ij})]\,. \qquad (2.24)$$

More importantly, using the last equality and a single integration by parts we can write

$$P=\frac{\gamma^{-1}V}{960\pi}\int_{-\infty}^{+\infty} d\chi\,\frac{d\bar{\kappa}_{ij}}{d\chi}\frac{d\kappa^{ij}}{d\chi}\,, \qquad (2.25)$$

or equivalently, using Eqs. (II.B.10) and (2.11),

$$P=\frac{1}{1920\pi}\int d^4x(-\tilde{g})^{1/2}\bar{C}_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\,, \qquad (2.26)$$

where this expression is understood to hold to second order in β_{ij}. Equations (2.25) and (2.26) show that the probability is manifestly positive and covariant under real coordinate transformations. Equation (2.26) agrees with that independently and previously derived for this process by Zel'dovich and Starobinsky[3] in the test-field approximation when the geometry is real.

Suppose that there were two solutions $\kappa_{ij}^{(1)}$ and $\kappa_{ij}^{(2)}$ of Eqs. (2.15) or (2.17), both of which give a finite total particle production probability according to Eq. (2.25). The difference between the two solutions,

$$\kappa_{ij}^{(h)}=\kappa_{ij}^{(1)}-\kappa_{ij}^{(2)}\,, \qquad (2.27)$$

must satisfy the homogeneous equation corresponding to (2.17):

$$(\mathcal{R}+i\mathcal{J})\kappa_{ij}^{(h)}=0\,. \qquad (2.28)$$

From Eq. (2.27) and the triangle inequality we have, writing generically $p=d\kappa/d\chi$,

$$(p^{(h)},p^{(h)})^{1/2}\leq|(p^{(1)},p^{(1)})^{1/2}+(p^{(2)},p^{(2)})^{1/2}|\,. \qquad (2.29)$$

Both the integrals on the right-hand side of Eq. (2.29) must exist in order that the two solutions yield finite particle production probabilities. The integral $(p^{(h)},p^{(h)})$ must therefore also be finite. This can be computed, however, by constructing the scalar product of Eq. (2.28) with $\kappa^{(h)}$ and taking the imaginary part. One finds after an allowed integration by parts

$$(p^{(h)}, p^{(h)}) = 0\,. \tag{2.30}$$

Therefore, $d\kappa_{ij}^{(h)}/d\chi = 0$ and, since a constant is clearly not a solution of Eq. (2.15), $\kappa_{ij}^{(h)} = 0$. Thus, the solution to Eq. (2.15) is unique, if it exists, and yields a finite total pair production probability.

We as yet do not have a rigorous proof that solutions to Eq. (2.15) exist. We can, however, investigate whether there are asymptotic behaviors of β_{ij} near the singularity which are consistent both with the equation and with a finite pair production probability. Two cases can be distinguished. There is the conformally complete case in which the scale factor $b(\chi)$ has the behavior

$$b(\chi) \sim \text{const} \times \exp[(6/\lambda)^{1/4}\chi]\,, \quad \chi \to -\infty\,. \tag{2.31}$$

In the remaining cases the singularity occurs at a finite value χ_0 of the coordinate χ, and the behavior of $b(\chi)$ near this point is

$$b(\chi) \sim \text{const} \times (\chi - \chi_0)^{\sqrt{6}/2}\,, \quad \chi \to \chi_0\,. \tag{2.32}$$

[Equations (2.31) and (2.32) are (I.4.15) and (I.4.13), respectively, restricted to the trace-anomaly parameters of the scalar field $\alpha = \beta = \lambda = (2880\pi^2)^{-1}$, so that $\sigma = 6^{-1/2}$.] We shall now consider these two cases separately.

In the conformally complete case the structure of the integrodifferential equation (2.15) in the limit $\chi \to -\infty$ is of interest. The nonlocal term gives rise to the following asymptotic behavior for a general function f:

$$K\frac{df}{d\chi} \sim \text{const}\left(\frac{\ln(-\chi)}{\chi^2}\right), \quad \frac{df}{d\chi} = O\left(\frac{1}{\chi^2}\right), \tag{2.33}$$

$$K\frac{df}{d\chi} \sim \text{const}\left(\frac{1}{\chi^2}\right), \quad \frac{d^2f}{d\chi^2} = O\left(\frac{1}{\chi^{3+\epsilon}}\right). \tag{2.34}$$

From Eq. (2.31) the asymptotic form of the equation is

$$3\lambda\frac{d}{d\chi}\left[\left(\frac{6}{\lambda}\right)^{1/4}\chi\left(\frac{d\kappa_{ij}}{d\chi}\right) + K\frac{d\kappa_{ij}}{d\chi}\right] - 2(6\lambda)^{1/2}\kappa_{ij} = \gamma^2 c_{ij}\,. \tag{2.35}$$

From Eqs. (2.33) and (2.34) it is not difficult to verify that the following is a consistent asymptotic behavior for a solution of Eq. (2.35) as $\chi \to -\infty$:

$$\kappa_{ij} = c_{ij}\left[-\frac{\gamma^2}{2(6\lambda)^{1/2}} + \frac{\text{const}}{\chi^3} + \cdots\right]. \tag{2.36}$$

This behavior leads to a finite particle production probability through Eq. (2.25).

In order to interpret the behavior in Eq. (2.36) further, it is useful to introduce two quantities: The anisotropy energy density, defined by Misner[9] as the square of the shear $\sigma_{ij} = \kappa_{ij}/a$:

$$\rho_\beta = l^{-2}\sigma_{ij}\sigma^{ij} = \kappa_{ij}\kappa^{ij}/(la)^2\,, \tag{2.37}$$

and the magnitude of the square of the Weyl tensor given by

$$F = \bar{C}_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{2}{\gamma^2 a^4}\frac{d\bar{\kappa}_{ij}}{d\chi}\frac{d\kappa^{ij}}{d\chi}\,. \tag{2.38}$$

For the conformally complete case in Eq. (2.36) the anisotropy energy becomes infinite as the singularity is approached. It becomes infinite, however, less rapidly than predicted by the classical theory. In the classical limit $a^6\rho_\beta = \text{const}$. Here $a^6\rho_\beta \sim 0$ as the singularity is approached—specifically, $a^2\rho_\beta \sim \text{const}$. The square of the Weyl tensor becomes infinite but $(-g)^{1/2}F$ remains finite so that the integral in Eq. (2.25) giving the total particle production probability is finite.

In the test-field approximation to this problem, b and κ_{ij} would solve Einstein's equations and would be given by

$$b(\chi) = \chi\,, \quad \kappa_{ij} = \gamma^2 c_{ij}/(6\chi^2)\,. \tag{2.39}$$

At the singularity ($\chi = 0$), the integral in Eq. (2.25) diverges and the total particle production probability per unit volume is infinite in the test-field approximation. Equation (2.36) shows that in the conformally complete case the softening of the singularity following from the trace anomalies is sufficient to yield a finite particle production amplitude per unit volume without the inclusion of the back-reaction effects arising from the anisotropy itself. This is our main result. In particular, it justifies self-consistently the expansion of the classical geometry and effective action in powers of the anisotropy.

For the remaining cases in which isotropic scale factor has the behavior in Eq. (2.32), it is not possible to find solutions of Eq. (2.15) which are consistent with the assumptions invoked in evaluating the effective action in paper II. There the propagator of a conformally related field in a conformally flat spacetime was chosen to be conformally related to the flat-space Feynman propagator. This choice leads, for example, to a vanishing production rate of conformally invariant particles in conformally flat spacetimes and in other ways seems to be a natural one. The integrals over products of propagators necessary to evaluate the effective action were transformed into integrals of Feynman propagators over the whole of the conformally related flat spacetime. It seems clear, however, that in the cases represented by Eq. (2.32), where the conformally related flat spacetime is incomplete and restricted to $\chi \geq \chi_0$, the calculation should be restricted to the same domain. Such a restriction would be consistent with the assumptions employed in evaluating the effective action if there were a solution of the dynamical equation [Eq. (2.15)] with $\kappa'_{ij}=0$ for $\chi \leq \chi_0$. The nonretarded nature of the nonlocal term, however, shows that no such solution is possible. Whether there are other boundary conditions for calculating the effective action for these geometries which are physically reasonable and which do lead to consistent solutions for small anisotropies is an open question.

III. AN APPROXIMATION TO THE CLASSICAL GEOMETRY

To find the classical geometry for a given initial anisotropy the integrodifferential equation (2.15) must be solved for $\beta_{ij}(\chi)$. For a particular matrix c_{ij} the problem can always be written in terms of a single function $h(x)$ by the definition

$$\kappa_{ij}(\chi) = (\gamma^2/3\lambda)c_{ij}h(\chi)\,. \tag{3.1}$$

The equation for h is then

$$\frac{d}{d\chi}\left(A\frac{dh}{d\chi}\right) + Bh = 1 - \frac{d}{d\chi}\left(K\frac{dh}{d\chi}\right), \tag{3.2}$$

where the operator K is defined as before and the functions A and B are

$$A = -\frac{i\pi}{2} - \ln(\mu_1 b)\,, \tag{3.3}$$

$$B = \frac{2b^2}{\lambda} - \frac{1}{3}\left(\frac{b'}{b}\right)^2 - \frac{1}{3}\left(\frac{b''}{b}\right). \tag{3.4}$$

Several methods suggest themselves for solving Eq. (3.2). It could be differenced and solved by matrix inversion. It can be converted into a linear integral equation by inverting the differential operator on the right-hand side, applying it to both sides of the equation, and integrating by parts all of the remaining derivatives of h. A unique inverse of the differential operator can be shown to exist. The kernel of the resulting integral equation, however, is singular, so that the usual Fredholm methods are not directly applicable. An approximation to the solution could be obtained by iterating the integral equation, although there is no guarantee that the series thus generated would converge. The first term in the iteration would be the solution to the local differential equation

$$\frac{d}{d\chi}\left(A\frac{dh}{d\chi}\right) + Bh = 1\,. \tag{3.5}$$

In this section we will investigate this crude approximation to h. We shall call it the "local truncation" since it results from omitting the nonlocal term in the original equation.

The solution to Eq. (3.5) is determined by the boundary conditions implicit in Eqs. (2.16) and (2.36). The solutions to the homogeneous equation corresponding to Eq. (3.5) are a linear combination of solutions behaving as

$$\exp\{\pm\chi^2/[2\lambda\ln(\mu_1\chi)]^{1/2}\} \tag{3.6}$$

at large positive χ and linear combinations of solutions behaving as

$$(-\chi)^{-1/4}\exp[\pm\delta(-\chi)^{1/2}] \tag{3.7}$$

at large negative χ where $\delta^2 = (\frac{8}{3})(6/\lambda)^{1/2}$. There are no solutions of the homogeneous equation satisfying the boundary conditions at $\eta\to\pm\infty$. The demonstration is the same as that already given for Eq. (2.15). There is thus a unique solution to Eq. (3.5). This satisfies

$$h(\chi) = \lambda/(2\chi^2) + O(\chi^{-6})\,,\quad \chi\to\infty \tag{3.8a}$$

$$h(\chi) = -(3\lambda/8)^{1/2} + O((-\chi)^{-1/4}\exp[-\delta(-\chi)^{1/2}])\,,\quad \chi\to-\infty\,. \tag{3.8b}$$

The approach of h to a constant value is thus faster in the local truncation than in the case where the nonlocal term is included [cf. Eq. (2.36)].

The function h in the local truncation is shown in Fig. 1 for $\mu_1 = 1$. (A discussion of the numeri-

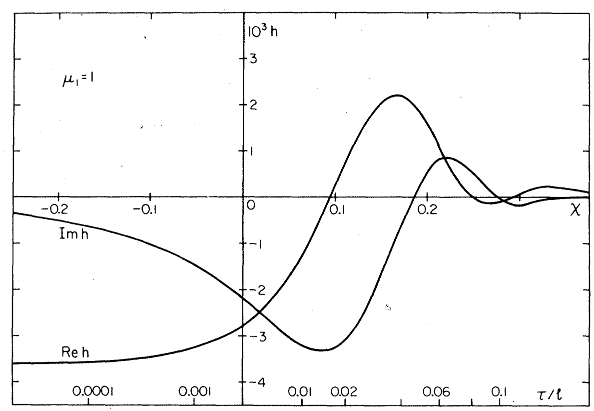

FIG. 1. The function h which measures the anisotropic part of the classical geometry calculated in the local truncation with the regularization scale $\mu_1 = 1$. The bottom scale measures the cosmic proper time from the singularity in units of the Planck time. The function h approaches a real constant as $\chi\to-\infty$ and vanishes as $\chi\to+\infty$. The imaginary part, which is a consequence of the particle production, is significant between $\chi=-3\times10^{-1}$ and $\chi=3\times10^{-1}$. The evolution is essentially classical after $\chi=0.35$ or $\tau=10^{-1}l$ with h decaying as (real const)/χ^2 and a negligible imaginary part.

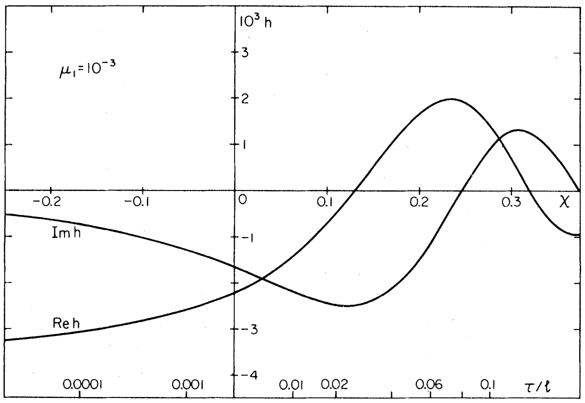

FIG. 2. The function h calculated in the local truncation for $\mu_1 = 10^{-3}$. The curve does not differ substantially from that computed with $\mu_1 = 1$ in Fig. 1 either in qualitative form or in magnitude of the real and imaginary parts.

cal solution is in the Appendix.) The imaginary part is significant in the range $-3\times 10^{-1} < \chi < 3\times 10^{-1}$. The upper end of this range corresponds to a proper time away from the singularity of about $10^{-1}l$. After this, the geometry is nearly real and the anisotropy decays according to the classical law $h = \text{const}/\chi^2$, appropriate for a radiation-dominated universe. Figures 2 and 3 show h computed in the local truncation for $\mu_1 = 10^{-3}$ and $\mu_1 = 10^{+3}$, respectively. The qualitative features of the function h are the same throughout the range bounded by these two cases. The quantitative values also do not differ greatly over this range, reflecting in part the fact that μ_1 enters the dynamical equation only through the logarithm.

In the local truncation the total probability for

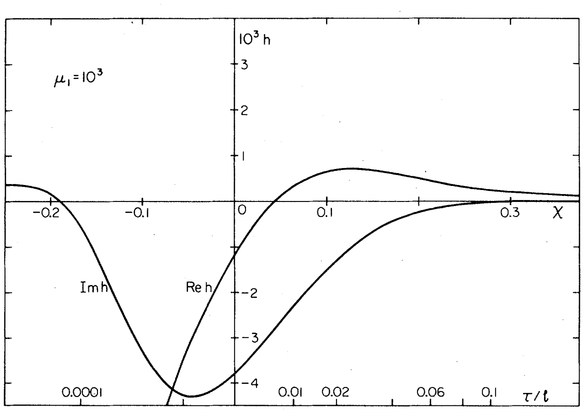

FIG. 3. The function h calculated for $\mu_1 = 10^{+3}$. The deviations from the $\mu_1 = 1$ case of Fig. 1 are more substantial than those of the $\mu_1 = 10^{-3}$ case in Fig. 2, but still commensurate with the large change in the renormalization scale.

producing a particle pair in a given comoving volume V may now be expressed as an integral of the dimensionless scale-invariant function h and the magnitude of the anisotropy. In epochs where the geometry evolves according to classical laws a convenient measure of the anisotropy is the parameter Δ defined in terms of the shear $\sigma_{ij} = \kappa_{ij}/a$ by

$$\Delta^2 = \frac{\sigma_{ij}\sigma^{ij}}{l^4 \rho_r^{\,3/2}} . \tag{3.9}$$

The quantity Δ is dimensionless, scale invariant, and constant in time in classical regimes. It may be written in the alternative forms

$$\Delta^2 = \frac{1}{6}\left(\frac{\Delta H}{H}\right)^2 \frac{1}{l^2 \rho_r^{\,1/2}} = \frac{\rho_B}{l^2 \rho_r^{\,3/2}} . \tag{3.10}$$

Here H is the average and ΔH the rms difference of the three principal Hubble constants defined as the eigenvalues of $(a'/a^2)\delta_{ij} + \sigma_{ij}$. Thus if we denote these eigenvalues by H_1, H_2, H_3 we can put, following Misner,[9]

$$H = H_1 + H_2 + H_3 , \tag{3.11}$$

$$3(\Delta H)^2 = (H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2 . \tag{3.12}$$

We will characterize the magnitude of the anisotropy in the classical geometry by the parameter Δ evaluated in the late-time classical regime. From Eqs. (3.1) and (3.9), Δ is related to the integration constant c_{ij} by

$$\Delta^2 = (216)^{-1}\gamma^6 c_{ij}c^{ij} . \tag{3.13}$$

Reexpressing Eq. (2.26) in terms of h and Δ, one finds for the total probability to produce a pair of scalar particles in a comoving volume V

$$P = \frac{72\pi}{\lambda}\left(\frac{V}{\gamma^3}\right)\Delta^2 \int_{-\infty}^{+\infty} d\chi \left|\frac{dh}{d\chi}\right|^2 . \tag{3.14}$$

A natural comoving volume in which to evaluate this probability is the volume V_r occupied by one of the classical radiation quanta. This is

$$V_r = \frac{\pi^2}{2\zeta(3)}\left(\frac{\pi^2}{15\tilde{\rho}_r}\right)^{3/4} = 0.2041\gamma^3 , \tag{3.15}$$

where $\zeta(3) = 1.202$ is the Riemann ζ function at argument 3. The total probability P_r to produce a pair in the volume occupied by one classical radiation quantum is then

$$P_r = 1.312\times 10^6 \Delta^2 \int_{-\infty}^{+\infty} d\chi \left|\frac{dh}{d\chi}\right|^2 . \tag{3.16}$$

For the function h calculated in the local truncation with $\mu_1 = 1$ we find

$$P_r = 6.2\times 10^2 \Delta^2 . \tag{3.17}$$

In considering this final result for the particle production probability it is important to recall three facts. First, the calculation has been carried out in the local truncation so that the numerical coefficient in Eq. (3.17) can only be an order-of-magnitude approximation to the exact value. Second, the value of the constant depends on the regularization scale μ_1. Third, the calculation has been carried out assuming small anisotropy. In particular, it is restricted to those values of Δ for which P_r is a small number. We display the result in Eq. (3.17) chiefly in the spirit of showing that a calculation of the particle production probability can be consistently carried out, beginning at the cosmological singularity, and yield a finite result.

We shall conclude this section with some brief remarks on the local properties of the classical geometry. Two convenient measures of the local anisotropy were introduced in Sec. II: The magnitude of the squared Weyl tensor and the anisotropy energy—[Eqs. (2.37) and (2.38)]. They are related to h and Δ as follows:

$$F=\bar{C}_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}=4\Delta^2W^2/(\lambda^2l^4)\,, \tag{3.18}$$

where

$$W^2=\frac{2}{b^4}\left|\frac{dh}{d\chi}\right|^2 \tag{3.19}$$

and

$$\rho_\beta=4\Delta^2R_\beta/(\lambda^2l^4)\,, \tag{3.20}$$

where

$$R_\beta=h^2/b^2\,. \tag{3.21}$$

Figures 4 and 5 show curves of the magnitudes of the dimensionless, scale-invariant functions W^2 and R_β as a function of τ, the proper time away from the singularity. Both measures of the anisotropy are infinite at the singularity but decay very rapidly away from it.

The classical geometry by itself is not simply related to the local rate of particle production. However, since the total particle production probability [Eq. (3.14)] is expressed as an integral over time, the integrand can be viewed as a crude measure of the rate. If we write

$$P=\int_0^\infty d\tau\, p(\tau)\,, \tag{3.22}$$

where $d\tau=\sqrt{6}\,b\,d\chi$, then Eqs. (3.14) and (3.19) show that p is proportional to W^2b^3. A graph of this quantity is shown in Fig. 6. W^2b^3 approaches infinity at the singularity but slowly enough so that the total probability integral [Eq. (3.22)] converges. The major contribution to the total probability comes in the proper time interval between the singularity and several hundredths of a Planck time later.

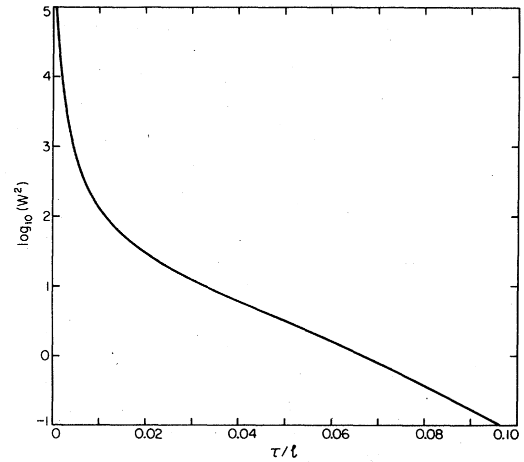

FIG. 4. The function W^2 which measures the magnitude of the square of the Weyl tensor plotted against the cosmic proper time from the singularity τ. The curve shown here was calculated in the local truncation with $\mu_1=1$. By this measure the universe is arbitrarily anisotropic at the singularity, but the anisotropy decreases rapidly away from the singularity.

IV. THE SPECTRUM OF PRODUCED PARTICLES

In this section we shall derive an expression for the spectrum of produced particles to lowest order in the anisotropy. To do this we calculate the amplitude $A_{\vec{k}\vec{k}'}$ that a pair of conformally invariant scalar particles with wave vectors $\vec{k}$ and $\vec{k}'$ is produced from the initial vacuum. A defini-

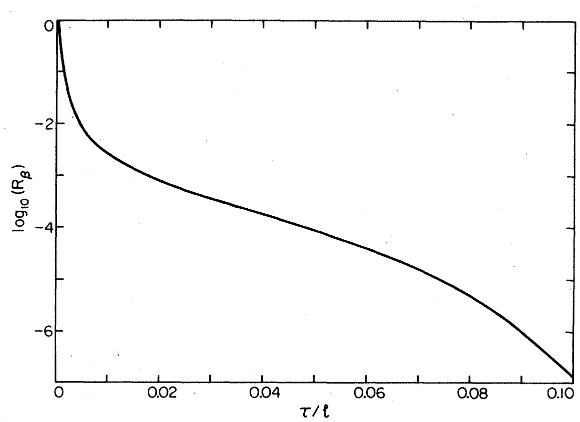

FIG. 5. The function R_β which measures the magnitude of the anisotropy energy plotted against cosmic proper time from the singularity τ. The curve shown here was calculated in the local truncation with $\mu_1=1$. By this measure the universe is arbitrarily anisotropic at the singularity, but the anisotropy decreases rapidly away from the singularity.

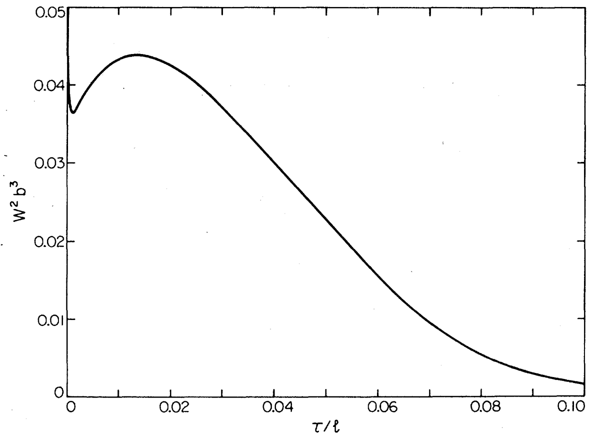

FIG. 6. The function W^2b^3 whose integral is proportional to the total particle production probability per unit comoving volume plotted as a function of the cosmic proper time from the singularity τ. The curve shown here was calculated in the local truncation with $\mu_1=1$. Since the integral of W^2b^3 is proportional to the total probability of producing a particle pair in a given comoving volume over the whole history of the universe, W^2b^3 itself may be thought of as a rough measure of the particle production rate. The rate becomes infinite at the singularity, but slowly enough so that the integral starting from the singularity is finite. There is structure in the curve near $\tau=0.001$, which would look less severe if plotted against χ. The rate then decays to zero with increasing time.

tion of initial and final particle states was implicit in our evaluation of the functional integral for the vacuum persistence amplitude. The definition can be recovered through the relation

$$G(x,x')=i\langle 0_+|T(\varphi(x)\varphi(x'))|0_-\rangle/\langle 0_+|0_-\rangle \qquad (4.1)$$

and the decomposition of the field into annihilation and creation operators of the appropriate vacuums. For example, write the field as

$$\varphi(x)=\sum_{\vec{k}}[f_{\vec{k}}(x)a_{\vec{k}}+\bar{f}_{\vec{k}}(x)a_{\vec{k}}^{\dagger}], \qquad (4.2)$$

where the $a_{\vec{k}}$ annihilate the final vacuum and $f_{\vec{k}}(x)$ are the solutions of the conformally invariant wave equation corresponding to the final particle states. The functions $f_{\vec{k}}(x)$ can then be found from Eq. (4.1) and a knowledge of $G(x,x')$ at late times. For the purpose of evaluating the particle production amplitudes to lowest order in β_{ij}, the functions $f_{\vec{k}}(x)$ need only be known in the limit of exact isotropy. There, since the propagator was assumed to be conformally related to the flat-space Feynman propagator [Eq. (II.2.14)], the functions $f_{\vec{k}}(x)$ are conformally related to the positive-frequency solutions of the flat-space wave equation. One has

$$f_{\vec{k}}(x)=\frac{1}{a(\eta)}\frac{e^{ik\cdot x}}{(2\omega_k V)^{1/2}}, \qquad (4.3)$$

where $\omega_k=|\vec{k}|$ and V is the spatial volume under consideration.

To first order in β_{ij} the action for the conformally invariant scalar field is the sum of Eqs. (II.2.6) and (II.2.7). Making a conformal transformation of the scalar field

$$\varphi=\tilde{\varphi}/a(\eta), \qquad (4.4)$$

this action can be written as

$$S_f[a,\beta]=-\tfrac{1}{2}\int d^4x(\eta^{\alpha\beta}\partial_\alpha\tilde{\varphi}\partial_\beta\tilde{\varphi}-2\beta^{ij}\partial_i\tilde{\varphi}\partial_j\tilde{\varphi}). \qquad (4.5)$$

This is the action for a free scalar field $\tilde{\varphi}$ in Minkowski space with an additional derivative interaction proportional to β^{ij}. Both the Green's functions and the asymptotic states of this flat-space field theory are identical to those of the curved-space theory to first order in β_{ij}. It follows that to first order in β_{ij} those amplitudes which do not involve the counteraction are identical to those of this conformally related flat-space field theory. This is true, in particular, for the amplitude to produce a pair of particles from the vacuum calculated to first order in β_{ij}. Using standard Feynman rules derived from Eq. (4.5), we may thus immediately write down the amplitude $A_{\vec{k}\vec{k}'}$ to first order in β_{ij} as

$$A_{\vec{k}\vec{k}'}=-\int d^4x\,\frac{e^{-ik\cdot x}}{(2\omega_k V)^{1/2}}\,\frac{e^{-ik'\cdot x}}{(2\omega_{k'}V)^{1/2}}(2k_ik_j'\beta^{ij}). \qquad (4.6)$$

Carrying out the spatial integration, this can be written as

$$A_{\vec{k}\vec{k}'}=-\delta_{\vec{k},-\vec{k}'}\omega_k^{-1}k_ik_j'\beta^{ij}(2\omega_k), \qquad (4.7)$$

where $\beta^{ij}(\omega)$ is

$$\beta^{ij}(\omega)=\int_{-\infty}^{+\infty}d\eta\,e^{i\omega\eta}\beta^{ij}(\eta). \qquad (4.8)$$

The δ function in the wave vectors in Eq. (4.7) is a reflection of the homogeneity of the spatial geometry and the consequent conservation of spatial momentum.

From Eq. (4.7) the frequency spectrum of produced pairs can be determined. The probability $p(\omega)d\omega$ of producing a pair, each member of which has frequency ω in the range $d\omega$, is found by multiplying $|A_{\vec{k}\vec{k}'}|^2$ by the number of states with wave vector $\vec{k}$ in this range, $dn=\omega^2d\omega\,d\Omega_{\vec{k}}[V/(2\pi)^3]$, summing over all directions of $\vec{k}$ and dividing by 2 since the particles in the final state are identical. The result is

$$p(\omega)=\frac{V}{30\pi^2}\omega^4\bar{\beta}_{ij}(2\omega)\beta^{ij}(2\omega). \qquad (4.9)$$

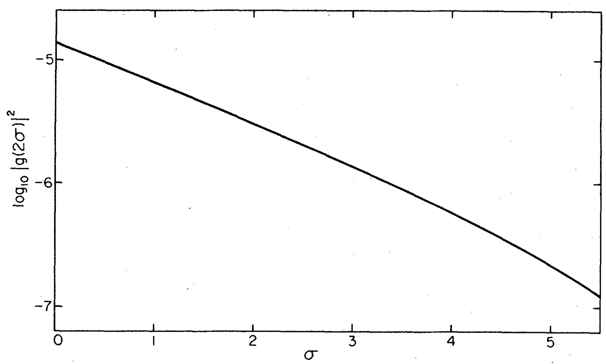

FIG. 7. The spectrum. If a real pair is produced from the vacuum in a spatially homogeneous cosmology, each member must have the same frequency because their spatial momenta are equal and opposite. The function $|g(2\sigma)|^2 d\sigma$ is proportional to the probability of producing a pair in a given comoving volume over the whole history of the universe, each member of which has a frequency in the range σ to $\sigma+d\sigma$. The energy of the produced pair differs from 2σ by a factor of $a^{-1}(\eta)$ so that the energy red shifts with the expansion in the usual way. Owing to the limitations of the numerical integration, the range of σ has been limited to values which are somewhat smaller than those characterizing the typical oscillations in Figs. 1–3. In this range the spectrum varies nearly exponentially with σ.

This gives the spectrum. As a check one can calculate the total probability P to produce a pair by integrating over all frequencies. The result is just Eq. (2.25).

The probability $p(\omega)d\omega$ can be expressed in terms of the scale-invariant function h introduced in Eq. (3.1). Defining $\sigma=\gamma\omega$ and writing

$$g(\sigma)=\int_{-\infty}^{+\infty} d\chi\, e^{i\sigma\chi}\frac{dh}{d\chi}\,, \tag{4.10}$$

one has

$$p(\omega)d\omega=\frac{d\sigma}{20\pi^2\lambda^2}\left(\frac{V}{\gamma^3}\right)\Delta^2|g(2\sigma)|^2\,. \tag{4.11}$$

In the local truncation $f(\chi)$ was calculated in Sec. III. Figure 7 shows the resulting factor $|g(2\sigma)|^2$. A thermal spectrum of produced particles would be characterized by a $p(\omega)$ which decreased exponentially with ω. The behavior displayed in Fig. 7 is close to an exponential decrease, but in view of the approximate nature of the calculation no firm significance can be attached to this fact.

V. CORRECTIONS TO THE SCALE FACTOR

The lowest-order corrections to the scale factor $a(\eta)$ are quadratic in the initial anisotropy. While these were not needed to compute the vacuum persistence amplitude to quadratic order, it is instructive to consider them both to complete the discussion of the classical geometry to quadratic order in the anisotropy and to demonstrate in this order that the back reaction is finite. We shall confine our attention to the case discussed extensively in the preceding sections for which the geometry is conformally complete in the limit of exact isotropy. Since we have numerically calculated the first-order corrections only in the local approximation, we shall not actually calculate the quadratic corrections, but we shall at least display the equation which determines them and remark on its properties.

The equation

$$\delta\Gamma[a,\beta]/\delta a=0 \tag{5.1}$$

is the simplest relation to employ to determine the quadratic corrections $a_2(\eta)$ to the scale factor [cf. Eq. (3.1)]. This equation yields the relation

$$R=-(l^2/2)T\,, \tag{5.2}$$

where T is the trace anomaly. This equation was written out in Eq. (II.3.8). Linearized in a_2 about the isotropic scale factor a_0, this relation will give a linear, fourth-order, *local* differential equation for a_2. In the expansion of Γ in powers of β_{ij}, only $\delta\Gamma_0/\delta a$ will involve a_2 when linearized; the term $\delta\Gamma_2/\delta a$ is already quadratic in β_{ij} and will give the driving term in the differential equation. A little computation gives the following result:

$$a_2'''' + p_3 a_2''' + p_2 a_2'' + p_1 a_2' + p_0 a_2 = d\,, \tag{5.3}$$

where

$$p_3=-4\frac{a_0'}{a_0}\,, \tag{5.4}$$

$$p_2=-\frac{2}{\lambda}\left(\frac{a_0}{l}\right)^2-6\frac{a_0''}{a_0}+8\left(\frac{a_0'}{a_0}\right)^2\,, \tag{5.5}$$

$$p_1=-4\frac{a_0'''}{a_0}+16\left(\frac{a_0''}{a_0}\right)\left(\frac{a_0'}{a_0}\right)-8\left(\frac{a_0'}{a_0}\right)^3\,, \tag{5.6}$$

$$p_0=-\frac{4}{\lambda}\frac{a_0a_0''}{l^2}+4\left(\frac{a_0'}{a_0}\right)\left(\frac{a_0'''}{a_0}\right)+3\left(\frac{a_0''}{a_0}\right)^2 -16\left(\frac{a_0'}{a_0}\right)^2\left(\frac{a_0''}{a_0}\right)+6\left(\frac{a_0'}{a_0}\right)^4\,, \tag{5.7}$$

$$d=\frac{a_0^{\,3}}{3\lambda l^2}\kappa_{ij}\kappa^{ij} +\frac{a_0}{3}\left[\tfrac{3}{2}(\kappa_{ij}'\kappa'^{ij})+2\left(\frac{a_0'}{a_0}\right)(\kappa_{ij}\kappa^{ij})' -\tfrac{1}{2}(\kappa_{ij}\kappa^{ij})''+2\left(\frac{a_0'}{a_0}\right)'\kappa_{ij}\kappa^{ij}\right]. \tag{5.8}$$

In these relations a_0 is the scale factor in the limit of exact isotropy and κ_{ij} is the solution for $d\beta_{ij}/d\eta$ discussed in Sec. II.

If the expansion of the classical geometry in powers of the anisotropy is to make sense, then the quadratic correction to the scale factor a_2 should be a small correction to a_0 for sufficiently small anisotropy. We shall show that there is a unique solution to Eq. (5.3) for which this is indeed the case.

If a solution to Eq. (5.3) exists which is a small correction to a_0, it will be unique provided there is no solution to the homogeneous equation which meets these criteria. To decide whether or not this is the case, it is sufficient to look at the possible asymptotic behaviors of the solutions to the homogeneous equation as $\chi \to \pm\infty$. In each limit there should be four linearly independent behaviors since the equation is of fourth order.

For large χ there is a two-parameter family of solutions which correspond to perturbations in $\tilde{\rho}_r$ and the origin of the coordinate η. These solutions have the large-η behavior

$$a_2(\eta) \sim A\eta + B + C/\eta + D/\eta^2 + \cdots , \qquad (5.9)$$

where A is arbitrary and proportional to the perturbation in $\tilde{\rho}_r$, B is arbitrary and proportional to the perturbation in the origin of η, and the coefficients C, D, ... are determined in terms of A and B. The remaining two linearly independent solutions have the large-η behavior

$$a_2(\eta) \sim K_\pm \eta^{\alpha_\pm} \exp[\pm \eta^2/(2\lambda\gamma^2)^{1/2}] \qquad (5.10)$$

for arbitrary constants $K_\pm$ and exponents $\alpha_\pm$ determined by the equation. The solutions corresponding to changes in $\tilde{\rho}_r$ and the origin of η are trivial and may be excluded without loss of generality. The growing solution in Eq. (5.10) must be excluded if a_2 is to be a small correction to a_0. There is thus only one linearly independent solution at large η which is a small correction to a_0.

The behavior of the homogeneous solutions to Eq. (5.3) as $\eta \to -\infty$ may be analyzed with the help of Eq. (2.31) giving the small-η behavior of a_0 and therefore of the coefficients p_i. As $\eta \to -\infty$ these coefficients become constants and the four linearly independent solutions are each of the form

$$a_2 \sim \text{const} \times \eta^q \exp[r(6/\lambda)^{1/4}(\eta/\gamma)] , \qquad (5.11)$$

where r is a root of

$$r^4 - 4r^3 + 2r^2 + 4r - 3 = 0 . \qquad (5.12)$$

These roots are 3, 1, 1, and -1. Since two of the roots are degenerate, the four linearly independent solutions correspond to values of (r, q) given by $(3, 0)$, $(1, 0)$, $(1, 1)$, and $(-1, 0)$. The solution $(1, 1)$ corresponds to a perturbation in $\tilde{\rho}_r$ and will not occur in any solution for which this perturbation vanishes. A perturbation in the origin of the coordinate η gives rise to a solution of the form $(1, 0)$. Of the remaining solutions, $(3, 0)$ is consistent with the requirement that a_2 be a small correction to a_0 as $\eta \to -\infty$, while $(-1, 0)$ is not.

Up to an overall normalization there is only one nontrivial solution of the homogeneous equation at $\eta = +\infty$ which is a small correction to a_0. Integrated backwards this solution will in general be a linear combination of all four possible behaviors at $\eta = -\infty$. There is thus very probably no solution of the homogeneous equation for which a_2 is a small correction to a_0 for all η. If it exists, a solution to the inhomogeneous equation which satisfies this criteria would then be unique.

It is worth remarking that this demonstration is contained in a less rigorous form in our analysis in paper I of the nonlinear equation $R = (-l^2/2)T$ in the isotropic limit. The homogeneous part of Eq. (5.3) is the linearization of that equation. For the nonlinear case we identified a one-parameter family of physically acceptable solutions at infinity [Eq. (I.A1)], a one-parameter family of conformally complete solutions at the singularity [Eqs. (I.A7)] up to choices in the origin of η, and a unique conformally complete geometry which joined the two.

It is not difficult to demonstrate the existence of a solution to Eq. (5.3) for which a_2 is a small correction to a_0. We have found one nontrivial solution of the homogeneous equation consistent with this criterion at $\eta = +\infty$ and one at $\eta = -\infty$. Both solutions decrease exponentially—from them the Green's function for Eq. (5.3) can be constructed. Equation (2.16), together with Eqs. (2.31) and (2.36), shows that the driving term d in Eq. (5.3) decreases as $O(\exp[(6/\lambda)^{1/4}(\eta/\gamma)])$ as $\eta \to -\infty$ and $O(\eta^{-1})$ as $\eta \to +\infty$. These behaviors, and those of the homogeneous solutions making up the Green's function, are sufficient to show that the inhomogeneous solution constructed by integrating the Green's function over the driving term exists and leads to an a_2 with the asymptotic behaviors $a_2 = O(\eta^{-1})$ as $\eta \to +\infty$ and $a_2 = O(\exp[(6/\lambda)^{1/4}(\eta/\gamma)])$ as $\eta \to -\infty$. Both behaviors are small corrections to a_0 for suitably small anisotropy. The finiteness of the back reaction to quadratic order in the anisotropy is thus demonstrated.

ACKNOWLEDGMENTS

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APPENDIX: ASYMPTOTIC SOLUTIONS TO THE EQUATION FOR THE ANISOTROPY

We present in this appendix asymptotic expansions of the solutions to the anisotropy equation in the local truncation [Eq. (3.5)] both at the singularity and at late times. These asymptotic expansions are useful to start numerical integrations of Eq. (3.5) forward and backward in time. For $b(\chi)$ we shall consider only the conformally complete case behaving near the singularity as in Eq. (2.31).

1. Singularity

To obtain an expansion of the solution of Eq. (3.5) near the singularity we expand the coefficients A and B, making use of the expansion of $b(\chi)$ obtained in paper I, Eq. (I.A6). For this purpose it is convenient to follow paper I and introduce some new variables defined by

$$(b')^{3/2}=6^{3/8}y^{1/2}[1+g(z)]\,, \tag{A1a}$$

$$y=z^{3/2}=\lambda^{-3/4}b^3\,, \tag{A1b}$$

$$\zeta=-\ln z\,. \tag{A1c}$$

The singularity occurs at $y=z=0$ or at $\zeta=\infty$. With ζ as an independent variable, Eq. (3.5) becomes

$$-\frac{d}{d\zeta}\left[(\rho-\tfrac{1}{2}\zeta)(1+g)^{3/2}\frac{dh}{d\zeta}\right]+F(\zeta)h=\frac{\lambda^{1/4}}{4\sqrt{6}\,(1+g)^{2/3}}\,, \tag{A2}$$

where

$$F(z)=\frac{1}{4}\left[\frac{(2/\sqrt{6})z}{(1+g)^{2/3}}-\tfrac{2}{3}(1+g)^{2/3}-\frac{4}{9}\,\frac{z}{(1+g)^{1/3}}\,\frac{dg}{dz}\right], \tag{A3}$$

and

$$\rho=\frac{i\pi}{2}+\ln(\mu_1\lambda^{1/4}) \tag{A4}$$

is a constant.

An expansion of g in powers of z and $\ln z$ was obtained in Eq. (I.A7) from which follows an expansion of $F(\zeta)$ in powers of ζ and $\exp(-\zeta)$. A particular solution to Eq. (A2) which decays as the singularity is approached can be found by expressing f in the form

$$f_{\text{part}}=\sum_{n=0}^{\infty}\psi_n(\zeta)\,, \tag{A5}$$

where $\psi_n(\zeta)$ increases with ζ at most as a polynomial multipled by $\exp(-n\zeta)$. The relations determining ψ_n are found from Eq. (A2) by equating terms decaying as $\text{ex}(-n\zeta)$ on both sides. For ψ_0 we have immediately

$$\psi_0=-(\sqrt{6}/4)\lambda^{1/2}\,. \tag{A6}$$

For ψ_1 we have to solve

$$-\frac{d}{d\zeta}\left[\left(\rho-\frac{\zeta}{2}\right)\frac{d\psi_1}{d\zeta}\right]-\tfrac{1}{6}\psi_1=\frac{\lambda^{1/2}}{4\sqrt{6}}(f_1-\tfrac{2}{3}g_1)\,, \tag{A7}$$

where in the notation of Eq. (I.A6) and (I.A7)

$$f_1=\frac{1}{4}\left(\frac{2}{\sqrt{6}z}-\tfrac{4}{9}g_1-\tfrac{4}{9}z\frac{\partial g_1}{\partial z}\right), \tag{A8}$$

$$g_1=z(\varphi_{01}+\tfrac{3}{16}\sqrt{6}\ln z)\,. \tag{A9}$$

The constant φ_{01} for the conformally complete solution is equal to -0.37216. (See Fig. 3 of paper I.) Writing the solution to ψ_1 as an asymptotic series

$$\psi_1=e^{-\zeta}\left(a+\frac{b}{\zeta}+\frac{c}{\zeta^2}+\frac{d}{\zeta^3}+\frac{e}{\zeta^4}+\cdots\right) \tag{A10}$$

and equating the coefficients of successively lower orders of ζ, one finds

$$a=\frac{\lambda^{1/2}}{12}\,, \tag{A11a}$$

$$b=2\left[(\rho+\tfrac{2}{3})a+\frac{\lambda^{1/2}}{4\sqrt{6}}\left(\frac{3}{8\sqrt{6}}-\tfrac{8}{9}\varphi_{01}\right)\right], \tag{A11b}$$

$$c=2b(\rho-\tfrac{1}{3})\,, \tag{A11c}$$

$$d=2[(\rho-\tfrac{4}{3})c+(2\rho-\tfrac{1}{2})b]\,, \tag{A11d}$$

$$e=2[b(2\rho)+c(4\rho-2)+d(\rho-\tfrac{7}{3})]\,. \tag{A11e}$$

The general solution of Eq. (A2), which vanishes at the singularity, is the sum of Eq. (A5) plus any solution of the homogeneous equation which vanishes there. An asymptotic expansion for this decaying homogeneous solution is

$$h_{\text{homo}}=AK_0((\tfrac{8}{3})^{1/2}\xi)\,[1+O(\sqrt{\zeta}\,e^{-\zeta})]\,, \tag{A12}$$

where $\xi=(\zeta/2-\rho)^{1/2}$ and A is any complex constant. Here, $K_0(u)$ is the Bessel function having the asymptotic expansion

$$K_0(u)=\left(\frac{\pi}{2u}\right)^{1/2}e^{-u}\left(1-\frac{1}{8u}+\frac{9}{2!}\,\frac{1}{(8u)^2}+\cdots\right). \tag{A13}$$

The most general solution of Eq. (A2) which vanishes at the singularity is thus a one-parameter family with the asymptotic behavior

$$h=-\frac{\sqrt{6}}{4}\lambda^{1/2}+AK_0((\tfrac{8}{3})^{1/2}\xi)+O(e^{-\zeta})\,. \tag{A14}$$

2. Late times

At large χ, $b(\chi)$ approaches the Friedmann solution $b=\chi$ and Eq. (3.5) becomes

$$-\frac{d}{d\chi}\left\{\left[\frac{i\pi}{2}+\ln(\mu_1\chi)\right]\frac{dh}{d\chi}\right\}+\left[\frac{2\chi^2}{\lambda}-\frac{1}{3\chi^2}\right]h=1\,. \tag{A15}$$

Transforming χ to the variable t defined by

$$t=\frac{i\pi}{2}+\ln(\mu_1\chi)\,, \tag{A16}$$

and defining

$$r_0=e^{-i\pi/2}\lambda^{-1/4}\mu_1^{-1}\,, \tag{A17}$$

Eq. (A15) becomes

$$\frac{d^2h}{dt^2}+\left(\frac{1}{t}-1\right)\frac{dh}{dt}+\left(-2\frac{r_0^4e^{4t}}{t}\right)h=\lambda^{1/2}r_0^2\frac{e^{2t}}{t}\,. \tag{A18}$$

A particular solution of Eq. (A18) which decays at large t can be found by expanding f in the form

$$h_{\text{part}}=\sum_m \exp(-mt)s_m(t)\,, \tag{A19}$$

where the sum is over even m and $s_m(t)$ increase with t, at most like a power. Comparing coefficients of successively lower orders of $\exp(-mt)$ ($m=2,\ 0,\ -2,\ldots$), we obtain the following solutions for s_m:

$$s_2=\frac{\lambda^{1/2}}{2r_0^2}\,,\quad s_4=0 \tag{A20a}$$

$$s_6=\frac{1}{r_0^4}\,(3t-\tfrac{5}{6})s_2,\quad s_8=0 \tag{A20b}$$

$$s_{10}=\frac{3s_2t}{2r_0^8}\left(42t-\tfrac{91}{3}+\frac{139}{54t}\right). \tag{A20c}$$

The general solution of Eq. (A18) at large t is then this particular solution plus any solution of the corresponding homogeneous equations. There are solutions of this homogeneous equation which grow at large t and also ones which decay. While the $h(\chi)$ of interest must vanish at large χ, the growing solution of the homogeneous equation is also useful numerically, as will be described below.

To find an asymptotic expansion of the growing solution h^+, we write

$$h^+(t)=e^{k(t)}\,, \tag{A21}$$

wherein k obeys the equation

$$\frac{d^2k}{dt^2}+\left(\frac{dk}{dt}\right)^2+\left(\frac{1}{t}-1\right)\frac{dk}{dt}=\frac{2r_0^4e^{4t}}{t}\,. \tag{A22}$$

We then expand $k(t)$ in the form

$$k(t)=\sum_n k_n(t)\,, \tag{A23}$$

where the sum ranges over $n=2,\ 0,\ -2,\ -4,\ldots,$ and $k_n(t)$ has the form

$$k_n(t)=\frac{e^{nt}}{t^{1/2}}\left(C_0^{(n)}+\frac{C_1^{(n)}}{t}+\cdots\right). \tag{A24}$$

Substituting this expansion in Eq. (A22) and equating coefficients of $\exp(nt)$ one finds first

$$\left(\frac{dk_2}{dt}\right)^2=\frac{2r_0^4e^{4t}}{t}\,. \tag{A25}$$

The positive square root gives an h which grows with t and for this case

$$k_2=\frac{r_0^2e^{2t}}{\sqrt{2}\,t^{1/2}}\left(1+\frac{1}{4t}+\frac{3}{4\times4t^2}+\frac{1\times3\times5}{4\times4\times4t^3}+\cdots\right). \tag{A26}$$

Similarly, one obtains the following equations for k_0 and k_{-2} and their solutions:

$$\frac{dk_0}{dt}=\tfrac{1}{2}\left(1+\frac{1}{2t}\right)\,, \tag{A27}$$

$$k_0=-\tfrac{1}{2}(t+\tfrac{1}{2}\ln t)\,, \tag{A28}$$

$$\frac{dk_{-2}}{dt}=-\frac{\sqrt{2}\,t^{1/2}}{r_0^2e^{2t}}\left(\frac{1}{64t^2}+\frac{3}{16}\right), \tag{A29}$$

$$k_{-2}=\frac{3\sqrt{2}\,t^{1/2}}{32r_0^2e^{2t}}\left(1+\frac{1}{4t}+\frac{1}{48t^2}-\frac{1}{64t^3}+\cdots\right). \tag{A30}$$

The asymptotic expansion of the decaying solution is found by choosing the negative root in Eq. (A25). Writing the decaying solution h^- as

$$h^-(t)=e^{j(t)}\,, \tag{A31}$$

and setting

$$j(t)=\sum_n j_n(t) \tag{A32}$$

as in Eqs. (A23) and (A24), one easily finds that

$$j_2=-k_2,\ j_0=k_0,\ j_{-2}=-k_{-2},\ \text{etc.} \tag{A33}$$

The general solution of Eq. (A18) is thus

$$h=h_{\text{part}}+B^+h^++B^-h^-\,, \tag{A34}$$

where B^+ and B^- are arbitrary constants.

3. Numerical solution

The two-point boundary-value problem presented by Eq. (3.5) and the condition that h vanish at $\chi=\pm\infty$ was solved in the following way: The equation was integrated forward from $\chi=-\infty$ starting with the asymptotic expansion in Eq. (A14) for a particular value of A. At large positive χ the coefficient B^+ of the growing homogeneous solution was isolated using the expansion in Eq. (A23). The value of A was adjusted until B^+ vanished. This is simple to do since B^+ is linear A. This gives an accurate solution for small and moderate values of χ. To find the solution for large values, the equation was integrated backward from $\chi=+\infty$ starting with Eq. (A34) with $B^+=0$ and the asymptotic expansion in Eq. (A32). The value of B^- was adjusted until the solution thereby obtained matched that found from the forward integration. In this way the curves of $h(x)$ shown in Figs. 1–3 were obtained.

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