Statistical model of inclusive distributions: Correlations in transverse momentum

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We investigate correlations in the transverse momenta of secondary particles emitted in inclusive reactions initiated by high-energy primary particles. The investigation is carried out within the framework of a statistical model of inclusive reactions originally proposed by Scalapino and Sugar and extended by us in a previous paper. We find substantial positive correlations between the transverse momenta of secondaries emitted within any given rapidity cluster. The typical correlation length in transverse momentum is a few times the mass of the secondaries.

I. INTRODUCTION AND SUMMARY

In a previous paper[1] we extended a statistical model originally proposed by Scalapino and Sugar.[2] Our principal aim has been an investigation of inclusive distributions in transverse momentum. The extension of the Scalapino-Sugar model is based upon the following observations.

(a) The basic *physical* assumption of the model is that all relevant inclusive distributions can be obtained as moments of a complex random field Φ (the "source function"). Hence, in principle, all inclusive quantities can be computed once a (physically acceptable) entropy functional[3] is conjectured.

(b) While the authors of Ref. 2 considered Φ to be defined on the longitudinal phase space only, we pointed out in I that the model can be extended by considering random fields on the entire three-dimensional single-particle phase space. Ignoring kinematical limitations due to the finiteness of the energy available for particle production, the latter is a symmetric space isomorphic to the hyperboloid SO(3, 1)/SO(3).

For the sake of brevity, and for obvious physical reasons, the random system characterized by Φ on SO(3, 1)/SO(3) is referred to as the "Feynman gas."

We argued in I that in the spirit of a statistical model, any anisotropy observed in inclusive distributions must be a "spontaneous" one; in other words, it must emerge as a consequence of the dynamics of the Feynman gas. By using semiclassical arguments and a crude approximation based on a technique of "rapidity averaging," we made it plausible that this is a real possibility: The Feynman gas may develop an anisotropic condensate.

The aim of this work is to explore some of the consequences of such a model. We are mainly interested in correlation functions in transverse momentum. The latter serves the purpose of exploring further details of the distributions; in particular, they indicate whether our statistical model is capable of predicting jets of emitted secondaries as observed experimentally.[4] To this end, the model has to be somewhat refined; in particular, we have to reconsider the techniques of rapidity averaging. This is done in Sec. II. We show that once some results[2] concerning rapidity clustering are accepted, the *effective* dimensionality of the problem can be reduced. Thus transverse-momentum distributions can be investigated individually in each rapidity cluster, while the effect of rapidity correlations is simulated by an "effective potential" within that cluster. While the arguments presented in Sec. II are not rigorous ones, they are physically plausible; more important, they are independent of a classical ("mean-field") approximation to the entropy. Thus, in contrast to the method used in I, we are now permitted to go beyond the mean-field approximation.

In Sec. III, we compute Gaussian fluctuations around the mean-field results derived in I. Not surprisingly, we find that the mean-field expression of the single-particle distribution is modified by the fluctuations. More interestingly, however, we also find nonvanishing irreducible correlation functions in transverse momentum.

In Sec. IV we argue that for our picture to be valid, the Feynman gas has to be a rather cold one: Otherwise, it would make no sense to decompose the inclusive distributions into contributions coming from a mean field and small fluctuations around it. (Such arguments are standard in statistical mechanics; however, it is worth reemphasizing them in the present contest.) Bearing this in mind, we present the results of a numerical evaluation of the single-particle distribution and of the correlation function in suitably chosen variables, for a *very cold* Feynman gas [$kT \approx 7\times10^{-11}\, m$, where m is the mass of a typical emitted secondary (i.e., of a pion)]. The single-particle distribution we find is in reasonable agreement with observed distributions. In addition, we predict strong positive correlations in transverse momentum with a "correlation length" $\Delta p_T^2 \approx 10 m^2$. In qualitative terms, this means that particles emitted from the same rapidity cluster tend to have the same transverse momentum too: *they form jets*. We comment on the possible significance of our results at the end of Sec. IV.

Some mathematical results used in deriving the expressions of inclusive distributions are collected in an appendix. Throughout this paper, we use a natural system of units: $\hbar = c = m = 1$; the metric on Minkowski space is given by the Cartesian form $g_{00} = -1$, $g_{ik} = \delta_{ik}$, with all other components vanishing.

II. APPROXIMATE REDUCTION OF THE DIMENSIONALITY: THE INDEPENDENT-CLUSTER APPROXIMATION

Let us work with Cartesian components of the complex random field $\Phi(p)$. In a well-known way, if $\Phi = \Phi_1 + i\Phi_2$, the pair of functions (Φ_1, Φ_2) may be regarded as components of a two-dimensional Euclidean vector $\vec{\Phi}$. In effect we are taking advantage of the local isomorphism,[5] $U(1) \simeq SO(2)$. With this, the generating functional of correlation functions becomes

$$\mathfrak{z} = e^{-F} = \int D\vec{\Phi}\, e^{-\beta\int_V S\, dV}\, e^{i\int_V dV \vec{J}\cdot\vec{\Phi}}, \qquad (2.1)$$

where, obviously, F generates the connected correlation functions whereas dV is the invariant volume element of the phase space. Using a Feynman parametrization as in I, the accessible volume V of the phase space is approximately given by the inequality

$$(1+t)(\cosh y)^2 \le \frac{s}{4} \quad (s \gg 1), \qquad (2.2)$$

where $s^{1/2}$ is the total energy in the center-of-mass system (c.m.s.). The entropy density is of the form

$$S = \tfrac{1}{2} g^{ab} \nabla_a \vec{\Phi} \cdot \nabla_b \vec{\Phi} + U(\vec{\Phi}\cdot\vec{\Phi}), \qquad (2.3)$$

where, as computed in I, the nonvanishing components of g^{ab} are

$$g^{yy} = \frac{1}{1+t}, \quad g^{tt} = 4t(1+t), \quad g^{\varphi\varphi} = \frac{1}{t}. \qquad (2.4)$$

We argued in I that the problem defined by Eqs. (2.1)–(2.4) can be considerably simplified by an effective reduction of dimensionality. We carry this out in two steps.

First, we restrict the function space to cylindrically symmetric functions; this is permissible if we are not interested in azimuthal correlations. With this, the entropy density is reduced to a two-dimensional one.

Second, a further reduction of the dimensionality was achieved in I by an approximate rapidity averaging of the distributions, based on the results of Scalapino and Sugar (see Ref. 2). These authors found evidence for clustering in rapidity by examining a one-dimensional model of the type (2.1). We argued that the physically interesting distributions are those which are averaged over one rapidity cluster. The argument presented in I is acceptable as long as one is interested in the "classical" (mean-field) results only; however, in order to be able to examine fluctuations as well, we have to improve on it.

Let us observe that, *in principle*, the problem can be *always* reduced to a one-dimensional one by using a transfer-matrix formalism. In practice, however, this is difficult: We cannot compute the transfer matrices exactly. Therefore, we resort to the following approximate procedure. In order to simulate the effect of rapidity clustering, we introduce a weak attractive interaction in rapidity. Instead of writing (2.3) (with $\partial\vec{\Phi}/\partial\varphi = 0$), we replace it by the following effective entropy density:

$$\tilde{S} = \frac{1}{2(1+t)}\left(\frac{\partial\vec{\Phi}}{\partial y}\right)^2 + 2t(1+t)\left(\frac{\partial\vec{\Phi}}{\partial t}\right)^2 + U - v\left(\vec{\Phi}\left(y + \frac{\Delta y}{2}, t\right)\cdot\vec{\Phi}\left(y - \frac{\Delta y}{2}, t\right)\right), \qquad (2.5)$$

where Δy is roughly the size of the rapidity clusters. The exact form of v should not matter very much; we argue, however, that the effective attraction in rapidity should vary with t at the same rate as the derivative term does (since y is a cyclic coordinate in phase space). Therefore, we take

$$v = \frac{2C}{1+t}\,\vec{\Phi}\left(y = \frac{\Delta y}{2}\right)\cdot\vec{\Phi}\left(y - \frac{\Delta y}{2}\right), \qquad (2.6)$$

where C is some constant. Now we compute a transfer matrix T in a rapidity interval of length Δy at a fixed value of t, with the effective Hamiltonian in y:

$$H = \tfrac{1}{2}(1+t)\vec{P}^2 - v, \qquad (2.7)$$

where $\vec{P}$ is the variable canonically conjugate to $\vec{\Phi}(y)$ [see Eq. (2.5)]. If the effective interaction is weak and Δy is not too large, we have approximately

$$\langle\vec{\Phi}(y+\Delta y/2)\,|\,T\,|\,\vec{\Phi}(y-\Delta y/2)\rangle \approx \left\{\exp\left[\frac{2C\beta\Delta y}{1+t}\,[\vec{\Phi}(y)]^2\right]\right\}\langle\vec{\Phi}(y+\Delta y/2)\,|\exp[-\tfrac{1}{2}\beta\Delta y(1+t)\vec{P}^2]\,|\,\vec{\Phi}(y-\Delta y/2)\rangle$$

$$=\frac{1}{2\pi\beta\Delta y(1+t)}\left\{\exp\left[\frac{2C\beta\Delta y}{1+t}\,[\vec{\phi}(y)]^2\right]\right\}\exp\left[-\frac{[\vec{\Phi}(y+\Delta y/2)-\vec{\Phi}(y-\Delta y/2)]^2}{2\beta\Delta y(1+t)}\right].\qquad(2.8)$$

The off-diagonal elements of this transfer matrix are very rapidly decreasing because of the presence of the last factor in Eq. (2.8). Therefore, we do not commit a substantial error if we replace T by its diagonal part. The second and third terms in the epxression (2.5) of the entropy density are local in y, so that their rapidity averages can be safely approximated with the help of the mean-value theorem. For instance,

$$\int_{y-\Delta y/2}^{y+\Delta y/2} dy'\left(\frac{\partial\vec{\Phi}(y',t)}{\partial t}\right)^2 \approx \Delta y\left(\frac{\partial\vec{\Phi}(y,t)}{\partial t}\right)^2.$$

Putting these results together, we find that $\mathfrak{z}$ approximately breaks up into a product of generating functionals over the rapidity clusters,

$$\mathfrak{z}=\prod_{(\text{clusters})} Z_i \equiv \exp\sum F_i\ ,$$

where each Z_i is of an identical structure, viz.,

$$Z_i=\int D\vec{\Phi}\exp\left\{-\Delta y\,\beta\int_0^{T(y_i)} dt\left[2t(t+1)\left(\frac{\partial\vec{\Phi}}{\partial t}\right)^2+U-\frac{2C}{1+t}\,\vec{\Phi}^2+i\vec{j}\cdot\vec{\Phi}\right]\right\}.\qquad(2.9)$$

The upper limit of the integration over t is determined by Eq. (2.2) taken at the center of the ith rapidity cluster,

$$T(y_i)=\frac{s}{4(\cosh y_i)^2}-1.\qquad(2.10)$$

It is obvious from the foregoing that the "independent cluster approximation" developed here breaks down near the end points of the available rapidity interval $t\approx 0$, $4(\cosh y)^2\approx s$. Our main interest lies, however, in studying distributions far away from these points; hence, we expect this approximation to reflect the main qualitative features of the distributions reasonably well.

In what follows, we study models in which U is of the Landau-Ginzburg form[1,2]

$$U=2A\vec{\Phi}^2+B(\vec{\Phi})^4.\qquad(2.11)$$

We notice that the four parameters (A,B,C,β) of such models are not all independent. The parameter B may be put equal to unity by means of a rescaling of the random variable $\vec{\Phi}$. The scale of $\vec{\Phi}$ in turn can be fixed by means of the normalization condition

$$\int_0^{T(y)} dt\left[\frac{1}{Z}\,\frac{\delta^2 Z}{\delta\vec{j}(t)\cdot\delta\vec{j}(t)}\right]_{\vec{j}=0}=-\Delta N,\qquad(2.12)$$

where ΔN is the multiplicity of secondaries produced in a rapidity interval of width Δy centered around y. However, from the calculational point of view, it will be convenient to retain a redundant set of parameters until the end of the calculation.

III. CORRELATIONS IN TRANSVERSE MOMENTUM

In the independent-cluster approximation the distributions in t are determined by one-dimensional path integrals. Using Eqs. (2.9) and (2.4) we standardized the model by absorbing a factor $(4\Delta y)$ into β. We thus have with a trival rescaling of $\vec{j}$:

$$Z=\int D\vec{\Phi}\exp\left\{-\beta\int_0^T dt\left[\tfrac{1}{2}t(t+1)\left(\frac{d\vec{\Phi}}{dt}\right)^2+\tfrac{1}{2}\left(A-\frac{C}{1+t}\right)\vec{\Phi}^2+\tfrac{1}{4}B(\vec{\Phi})^4+i\vec{j}\cdot\vec{\Phi}\right]\right\}$$

$$\equiv\int D\vec{\Phi}\exp\left[-\beta\int_0^T dt(W+i\vec{j}\cdot\vec{\Phi})\right].\qquad(3.1)$$

As discussed in I, this one-dimensional entropy has locally stable minima; these are found by looking at stable solutions of the Euler-Lagrange equation

$$-\frac{d}{dt}\left(t(t+1)\frac{d\vec{f}}{dt}\right)+\left(A-\frac{C}{1+t}\right)\vec{f}+B(\vec{f})^2\vec{f}=0.\qquad(3.2)$$

We adjust phases such that $\vec{f}=(f(t),0)$; the parameter A is chosen so as to give the classical (mean-field) distribution the desired asymptotic behavior at large t. In the mean-field approximation we have

$$\frac{dN}{dt}\equiv\frac{1}{\sigma}\frac{d\sigma}{dt}\propto\vec{f}^2.$$

We demand $dN/dt\sim t^{-4}$; this gives $A=2$. We notice in passing that a scaling law corresponding to standard quark models,[6] viz., $dN/dt\sim t^{-2}$, would demand $A=0$. We argue, however, that events in which free-quark properties reveal themselves are statistically rare ones, and thus they are beyond the domain of applicability of this model.

Thus, we fix A corresponding to the intermediate scaling region of the inclusive cross section. The classical solution is sought with the boundary conditions $f(0)=f_0$, $f(T)=0$. We found that letting $T\to\infty$ has very little effect on the general behavior of $f(t)$; thus, we use this approximation whenever it is convenient. The requirement of stability establishes a relationship between f_0 and the values of the remaining parameters B and C. We could establish such a relationship by means of a numerical experimentation. The calculations reported in Sec. IV are carried out with such values of the parameters.

We now set $\vec{\Phi}=\vec{f}+\vec{x}$ and expand W up to second order in $\vec{x}$; this allows us to compute Gaussian fluctuations around the mean-field solution. We have

$$W=W(\vec{f})+w(\vec{f},\vec{x}),$$

where

$$w=\tfrac{1}{2}t(t+1)\left(\frac{d\vec{x}}{dt}\right)^2+\tfrac{1}{2}\left(A-\frac{C}{1+t}+B\vec{f}^{\,2}\right)\vec{x}^{\,2}+B(\vec{f}\cdot\vec{x})^2. \tag{3.3}$$

The variable $\vec{x}$ is subject to the boundary conditions $\vec{x}(0)=\vec{x}(T)=0$.

Formally, our problem is equivalent to the task of calculating the correlation functions for an anisotropic oscillator in two dimensions. The problem is complicated, however, by the fact that the "mass" and the "oscillator strength" of this fictitious oscillator are "time"-dependent quantities; moreover, the classical solution is known only numerically. Under such circumstances a reliable calculation of the functional integral would be very difficult.

Instead, we resort to an equivalent canonical formulation[7] in terms of a density operator $\Omega(t,t_0)$ which satisfies a Bloch equation with the standard initial condition

$$\frac{\partial\Omega}{\partial t}+H\Omega=0,\quad \Omega(t_0,t_0)=1, \tag{3.4}$$

where $H=H_1+H_2$, with

$$H_1=\frac{1}{2\beta t(t+1)}p_1^{\,2}+\frac{\beta}{2}\left(A-\frac{C}{1+t}+3Bf^2\right)x_1^{\,2},$$
$$H_2=\frac{1}{2\beta t(t+1)}p_2^{\,2}+\frac{\beta}{2}\left(A-\frac{C}{1+t}+Bf^2\right)x_2^{\,2}. \tag{3.5}$$

Here p_i, x_i are canonically conjugate operators $[x_k,p_l]=i\delta_{kl}$ $(k,l=1,2)$. For the sake of brevity we write

$$H_i=h(t)p_i^{\,2}+g_i(t)x_i^{\,2}\quad(i=1,2), \tag{3.6}$$

where the functions h_i and g_i can be read off from Eq. (3.5). Correspondingly, the density operator factorizes, $\Omega=\Omega_1\Omega_2=\Omega_2\Omega_1$. The one-dimensional Bloch equations with H_i given by (3.6) can be solved by the following Ansatz:

$$\Omega=\exp[\phi_1(t,t_0)p^2]\exp[\phi_2(t,t_0)x^2]\times\exp\{\phi_3(t,t_0)[p^2,x^2]\}, \tag{3.7}$$

where the functions ϕ satisfy the ordinary differential equations

$$\frac{d\phi_1}{dt}-4\phi_1^{\,2}g(t)+h(t)=0,$$
$$\frac{d\phi_2}{dt}+8\phi_1\phi_2g(t)+g(t)=0, \tag{3.8}$$
$$\frac{d\phi_3}{dt}=g(t)\phi_1.$$

These differential equations together with other useful relations are derived in the Appendix. (For the sake of simplicity, the subscripts labeling the degrees of freedom have been suppressed.) The advantage of the canonical formulation is now evident. Once Ω is given by (3.7), all the correlation functions can be calculated explicitly in terms of the functions ϕ, whereas the system of differential equations (3.8) is easily amenable to a numerical treatment.

We record here the expressions of the normalized inclusive cross section and of the irreducible correlation function in the Gaussian approximation:

$$\frac{dN}{dt}\equiv\frac{1}{\sigma}\frac{d\sigma}{dt}=\langle\vec{\Phi}(t)^2\rangle=\vec{f}(t)^2+\langle\vec{x}(t)^2\rangle,$$
$$C(t_1,t_2)\equiv\frac{1}{\sigma}\frac{d^2\sigma}{dt_1dt_2}-\frac{1}{\sigma^2}\frac{d\sigma}{dt_1}\frac{d\sigma}{dt_2}$$
$$=\langle\vec{\Phi}(t_1)^2\vec{\Phi}(t_2)^2\rangle-\langle\vec{\Phi}(t_1)^2\rangle\langle\vec{\Phi}(t_2)^2\rangle$$
$$=4\langle\vec{f}(t_1)\cdot\vec{x}(t_1)\vec{f}(t_2)\cdot\vec{x}(t_2)\rangle, \tag{3.9}$$

where the averages are to be taken with respect to the full density operator Ω; for instance,

$$\langle\vec{x}(t_1)\vec{x}(t_2)\rangle=\Theta(t_1-t_2)\frac{\langle 0|\Omega(T,t_1)\vec{x}\Omega(t_1t_2)\vec{x}\Omega(t_2,0)|0\rangle}{\langle 0|\Omega(T,0)|0\rangle}+(t_1\leftrightarrow t_2). \tag{3.10}$$

The matrix elements of Ω are easily calculable. We find

$$\langle x_2|\Omega(t',t)|x_1\rangle=\frac{1}{2[-\pi\phi_1(t',t)]^{1/2}}e^K,$$

where

$$K = 2\phi_3(t', t) + \phi_2(t', t)\, e^{8\phi_3(t', t)} x_1^2 + \frac{(x_2 - e^{4\phi_3(t', t)} x_1)^2}{4\phi_1(t', t)}$$

(see the Appendix). With the help of these formulas the calculation of the averages can be reduced to quadratures. Throughout these equations, $|0\rangle$ stands for the coordinate eigenvector with vanishing eigenvalue.

We now proceed to define certain combinations of the functions $\phi_1, \ldots, \phi_3$ for each degree of freedom in a way which allows us to express the inclusive cross section and the correlation function in a convenient form.

Let us define

$$\begin{aligned} R(t_3, t_2, t_1) &= e^{8\phi_3(t_3, t_2)}\left(\phi_2(t_3, t_2) + \frac{1}{4\phi_1(t_3, t_2)}\right) + \frac{1}{4\phi_1(t_2, t_1)}, \\ G(t_2, t_1) &= \frac{-e^{4\phi_3(t_2, t_1)}}{4\phi_1(t_2, t_1)}. \end{aligned} \tag{3.11}$$

After a somewhat lengthy but straightforward calculation we obtain

$$\frac{dN}{dt} = f(t)^2 + \frac{1}{2}\sum_{k=1}^{2}\left(\frac{G_k(T, t)G_k(t, 0)}{G_k(T, 0)}\right)^{1/2}[-R_k(T, t, 0)]^{-3/2},$$

$$C(t_2, t_1) = 2\Theta(t_2 - t_1) f(t_2) f(t_1)\left(\frac{G_1(T, t_2)[G_1(t_2, t_1)]^3 G_1(t_1, 0)}{G_1(T, 0)}\right)^{1/2}[R_1(T, t_2, t_1)R_1(t_2, t_1, 0) - G_1^2(t_2, t_1)]^{-3/2} + (t_2 \leftrightarrow t_1).$$

By repeated use of the addition theorem derived in the Appendix, these expressions can be further simplified. We obtain finally

$$\begin{aligned} \frac{dN}{dt} &= f^2(t) + \frac{1}{2}\sum_{k=1}^{2}\frac{G_k(T, 0)}{G_k(T, t)G_k(t, 0)}, \\ C(t_2, t_1) &= 2\Theta(t_2 - t_1) f(t_2) f(t_1)\frac{G_1(T, 0)}{G_1(T, t_2)G_1(t_1, 0)} + (t_2 \leftrightarrow t_1). \end{aligned} \tag{3.12}$$

In all these equations it is understood that G_k and R_k are computed via Eqs. (3.8) and (3.11) with the input functions $h(t)$ and $g_k(t)$, cf. Eqs. (3.5) and (3.6).

Let us observe that in the neighborhood of $t=0$, $h(t) \propto t^{-1}$ and hence, $t=0$ is a singular point of the system (3.8). For the purpose of a numerical treatment, we shift this singularity to some unphysical value of t, by writing $h^{-1}(t) = (t+\lambda)(t+1)$ with some small positive λ. [Other regularization methods are also tried. The exact manner in which $h(t)$ is regularized has little effect on the end result.] More importantly, it can be verified that the averages (3.9) exist if and only if the following convergence conditions are satisfied:

$$\begin{aligned} &R_k(T, t_2, t_1) + R_k(t_2, t_1, 0) < 0, \\ &R_k(T, t_2, t_1)R_k(t_2, t_1, 0) - G_k^2(t_2, t_1) > 0, \\ &\phi_1(t_2, t_1) < 0. \end{aligned} \tag{3.13}$$

These conditions are closely related to the local stability of the classical solution $f(t)$ (i.e., to the positivity of the functional w); details of these considerations are reported elsewhere.[8]

IV. RESULTS OF A TYPICAL NUMERICAL CALCULATION AND CONCLUSIONS

Below we present the results[9] of a typical numerical estimate of dN/dt and $C(t_1, t_2)$. No serious attempt has been made to fit actual measured distributions. [In particular, as far as we know, no detailed measurements of $C(t_1, t_2)$ have been made so far.] Before presenting the results, however, two qualitative questions have to be settled.

First, we observe that the parameter β^{-1} represents an effective temperature of the Feynman gas as characterized by Eqs. (2.1)–(2.3). A semiclassical expansion around the mean-field results has a meaning only if the effective temperature is sufficiently low; otherwise, the fluctuations increase rapidly. In the Gaussian approximation both $\langle \vec{x}^2 \rangle$ and the correlation function scale as β^{-1}, provided the parameters A, B, C, are kept fixed. (This is just the classical scaling law.)

Second, we have to settle the question of the variables used in the correlation function $C(t_1, t_2)$. Intuitively, this function measures the correlation present between two points on the phase space which are separated by a "distance" d. In our case, the two points are within the same rapidity cluster, so they have the same rapidity coordinates (within an uncertainty of the order of Δy, which we neglect). In the same way, we take the two points to have the same azimuthal coordinate. In I we derived the infinitesimal line element on the phase space; we found

$$ds^2 = (1+t)dy^2 + \frac{1}{4}\frac{dt^2}{t(1+t)} + t\,d\phi^2.$$

Therefore, on setting $dy = d\phi = 0$, we integrate ds between two points in order to obtain their distance:

$$d(t_1,t_2) = \frac{1}{2}\int_{t_1}^{t_2} \frac{dt}{[t(t+1)]^{1/2}} = \frac{1}{2}\ln\frac{[t_2(t_2+1)]^{1/2}+t_2+\frac{1}{2}}{[t_1(t_1+1)]^{1/2}+t_1+\frac{1}{2}} \,. \tag{4.1}$$

We introduce therefore the coordinates

$$\tau = \tfrac{1}{2}\ln\{2[t(t+1)]^{1/2}+2t+1\}$$

and give the values of the correlation function as a function of $d = |\tau_1 - \tau_2|$ while keeping $\tau_1+\tau_2$ fixed. The coordinate τ is readily recognized as the "transverse rapidity", viz., $t=\sinh^2\tau$.

The numerical results presented here have been obtained with the following values of the parameters: $A=2$, $B=7.4\times10^{-4}$, and $C=5.2$. In order to suppress the fluctuations, we have taken a sufficiently low value of the temperature $\beta=1.5\times10^{10}$. (With this value of β, the single-particle cross section is dominated by the mean-field term up to $t\lesssim5000$; this is equivalent to $p_T\lesssim10$ GeV for pions.)

In Fig. 1 we plotted the resulting one-particle inclusive distribution; for comparison, sample data (appropriately normalized) are taken from Ref. 10 and they are plotted together with the theoretical curve. In Fig. 2, we plotted the normalized correlation function in transverse momentum, $C(d)/C(0)$, as a function of the distance on phase space as defined by Eq. (4.1). All calculations have been performed with $T\to\infty$; we have verified that the results are practically unaffected if we take finite, realistic values of T. (The statistical model cannot be expected to work well near the boundaries of the phase space where the finiteness of T has a substantial effect on the distributions.) At any rate, due to this approximation, our results can be expected to describe inclusive data near the central region in rapidity ($y\approx0$). Owing to the kinematical constraint (2.2), the available transverse momentum is small in the fragmentation regions and our approximation necessarily

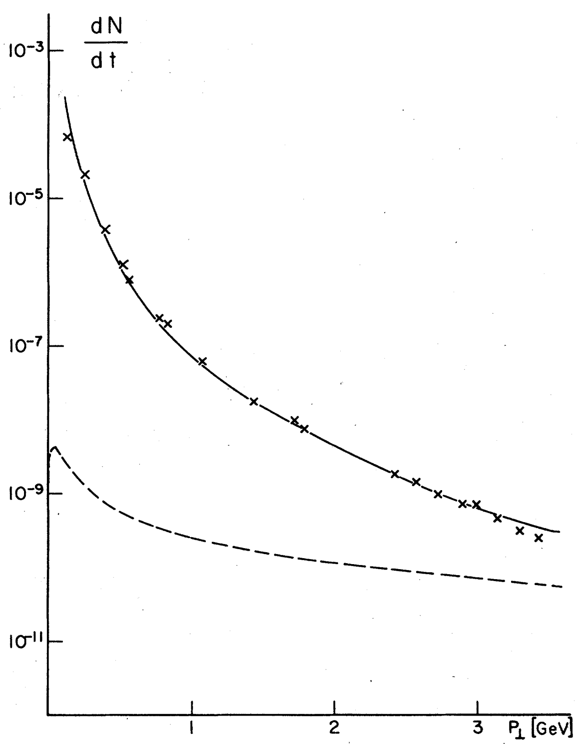

FIG. 1. Typical single-particle inclusive distribution in transverse momentum. All produced particles are assumed to be pions: $m=0.14$ GeV. The values of the parameters are $A=2$, $B=7\times10^{-4}$, and $C=5.2$. (The curve is normalized to a total multiplicity $\langle N\rangle=9$.) Full line: Mean field with Gaussian fluctuations. Dashed line: Gaussian fluctuation (scaled up by a factor 10^2.) Crosses: experimental data taken from Fig. 19 of Ref. 10. The data have been normalized to the theoretical curve at $t^{1/2}=2.8$ GeV.

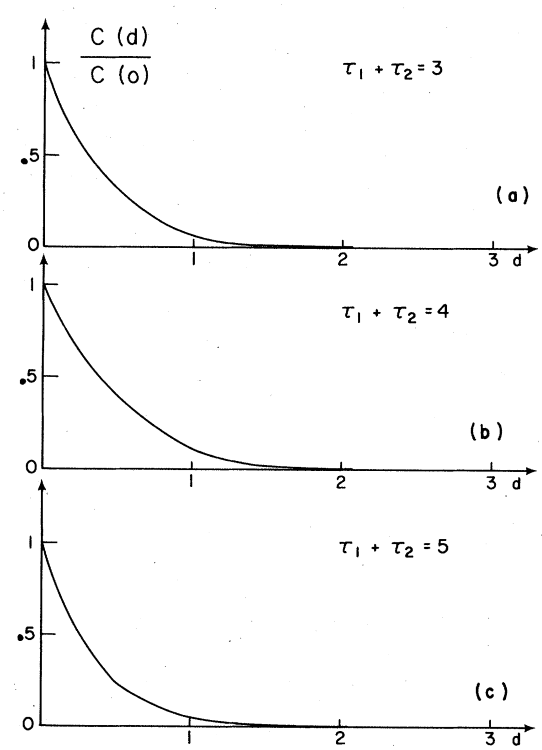

FIG. 2. Normalized irreducible correlations in transverse momentum plotted as a function of $d=|\tau_1-\tau_2|$. Figures (a)–(c) give the values of the correlation functions at $\tau_1+\tau_2=3$, 4, and 5, respectively. The values of the parameters A, B, and C are the same as in Fig. 1.

breaks down there. We have no reliable results to report for finite and small values of T at this time.

It is obvious from the figures that this statistical model reproduces the main features of the single-particle distributions reasonably well. In addition, we predict substantial positive correlations between the magnitudes of the transverse momenta of the emitted secondaries. Remarkably, the correlation functions depend rather weakly on the mean transverse rapidity, whereas they decrease rapidly with the distance d. [The range of correlations becomes somewhat smaller as the mean transverse rapidity increases (see Fig. 2).] Keeping in mind that our results have been worked out for *fixed* longitudinal rapidity (with an uncertainty Δy), this result can be converted into a prediction of angular correlations between secondaries emitted from the same cluster.

To this end we recall that the standard formula relating t to the emission angle Θ in the c.m.s. frame reads

$$t=\frac{\sinh^2 y \sin^2\Theta}{1-\cosh^2 y \sin^2\Theta}. \tag{4.2}$$

On assuming that the secondaries are emitted from a cluster of width Δy centered around $y=0$, we have approximately

$$t\approx(\Delta y)^2\tan^2\Theta[1+O((\Delta y)^2)]. \tag{4.3}$$

This expression can be substituted into (4.1) in order to express d in terms of the emission angles. The expression of d becomes particularly simple if we assume $t\gg 1$ (in conventional units, this is well satisfied for $t\gtrsim 1$ GeV). In that case we have

$$d(t_1,t_2)\approx|\ln\tan\Theta_2-\ln\tan\Theta_1| \tag{4.4}$$

so that, in essence, we predict positive angular correlations in Θ.

We claim that this result gives qualitative evidence for the presence of *jets* of secondaries emitted from the central rapidity region. In order to substantiate this claim, we would have to consider correlations both in the angle Θ and in the azimuthal angles. Whereas we have suppressed any dependence on azimuthal angles throughout this paper, it is not difficult to see that, *in the Gaussian approximation*, one necessarily obtains positive correlations both in transverse rapidity and in the azimuthal angles. (Phenomenologically, "*jets*" are characterized precisely by such correlations, cf. ref. 4.)

Indeed, on considering fluctuations which are not all cylindircally symmetrical, viz. $\vec{x}=\vec{x}(t,\phi)$, we can expand them in a Fourier series,

$$\vec{x}(t,\phi)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\vec{x}_m(t)e^{im\phi}\quad(\vec{x}_{-m}=\vec{x}_m^*).$$

By repeating the calculation described in Sec. III, we find that the density operator Ω factorizes in m, each Ω_m being determined by a Hamiltonian of the form (3.6). (However, for $m\neq 0$, the functions g_i pick up an additional term $m^2/4t$.) The correlation functions (3.9) are now generalized to

$$C(t_1,\phi_1;\ t_2,\phi_2)=4\sum_{m=0}^{\infty}\langle\vec{f}(t_1)\cdot\vec{x}_m(t_1)\vec{f}(t_2)\cdot\vec{x}_m^*(t_2)\rangle \times\cos m(\phi_1-\phi_2), \tag{4.5}$$

where each coefficient of this Fourier series can be shown to be *positive*. Consequently, the correlation is largest at $\phi_1=\phi_2$. [Unfortunately, the series (4.5) is converging rather slowly near $\phi_1=\phi_2$; therefore, a reliable computation of the joint correlation function requires a large amount of numerical work, which is beyond the scope of this paper.]

To summarize, we found that at least some important properties of the inclusive distributions can be qualitatively understood in terms of a conceptually simple statistical model. Obviously, further calculations and, in particular, improved approximations are necessary in order to test such models in detail. Owing to the fact that particle formation takes place in a region where the coupling strength of the fundamental interaction between quarks and gluons appears to be large (and, hence, details of the fundamental dynamics are hard to calculate), such an approach may be worth persuing.

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APPENDIX

A. The density matrix of a temperature-dependent oscillator

Let $h(t)$, $g(t)$ be real smooth functions of t in some interval $(T>t>t_0)$. Consider now a Bloch equation of the form

$$\frac{\partial\Omega}{\partial t}+[h(t)p^2+g(t)x^2]\Omega=0, \tag{A1}$$

where t may be formally regarded as an inverse temperature. This equation is the Lie differentail equation of a one-parameter subgroup on the dynamical group[11] of the oscillator, which in our case is isomorphic to SL(2, R). The dynamical group may be obtained by analytic continuation from SU(1,1); therefore, the treatment of Eq. (A1) parallels that of an oscillator with *time-dependent* coefficients, see e.g., Malkin *et al.*[12] A standard Cartesian basis of SL(2, R) could be obtained by taking, e.g.,

$$J_1 = \frac{-i}{8}[p^2, x^2], \quad J_2 = \frac{i}{4}(p^2 + x^2), \quad J_3 = \frac{i}{4}(p^2 - x^2). \tag{A2}$$

However, such a basis is not particularly useful here. The commutation relations are immediately verified with the help of the canonical commutation relations.

The group element Ω may be parametrized in various ways, see e.g., Wybourne, Ref. 11. We choose the following parametrization:

$$\Omega = \exp(\phi_1 p^2)\exp(\phi_2 x^2)\exp\{\phi_3[p^2, x^2]\}. \tag{A3}$$

Notice that J_1 is acting as a generator of "dilations,"

$$i[J_1, p^2] = -p^2, \quad -i[J_1, x^2] = x^2. \tag{A4}$$

The functions ϕ_k $(1 \leq k \leq 3)$ obey the standard initial conditions $\phi_k(t_0, t_0) = 0$. On inserting (A3) into (A1) written in the form

$$\frac{\partial\Omega}{\partial t}\Omega^{-1} + hp^2 + gx^2 = 0$$

and using the Baker-Campbell-Hausdorff formula, we obtain the differential equations (3.8).

The calculation of the matrix elements of Ω is straightforward. One inserts eigenvectors of the operators p and x between the product of operators (A3) and uses the commutation relations (A4). This reduces the problem to the computation of some Gaussian integrals. The end result is

$$\langle x_2|\Omega(t', t)|x_1\rangle = \frac{1}{2\sqrt{-\pi}\,\phi_1(t', t)}e^K,$$

with

$$K = 2\phi_3(t', t) + x_1^2 e^{8\phi_3(t', t)}\phi_2(t', t) + \frac{1}{4\phi_1(t', t)} + \frac{x_2^2}{4\phi_1(t', t)} - \frac{x_1 x_2 e^{4\phi_3(t', t)}}{2\phi_1(t', t)}. \tag{A5}$$

This has been already quoted in Sec. III.

B. Addition theorem

Addition theorems result from the multiplication of group elements. In particular, we have for $t_3 > t_2 > t_1$

$$\int dx \langle x_2|\Omega(t_3, t_2)|\Omega x\rangle\langle x|\Omega(t_2, t_1)|x_1\rangle = \langle x_2|\Omega(t_3, t_1)|x_1\rangle. \tag{A6}$$

On inserting the expressions (A5) into this equation and performing the resulting Gaussian integral, we can compare the coefficients of x_1^2, x_2^2, and x_1x_2 on both sides of the equation. Using the definition of the functions R and G [Eq. (3.11)], the resulting relationships can be written as

$$e^{8\phi_3(t_2, t_1)}\left(\phi_2(t_2, t_1) + \frac{1}{4\phi_1(t_2, t_1)}\right) - \frac{G^2(t_2, t_1)}{R(t_3, t_2, t_1)} = e^{8\phi_3(t_3, t_1)}\left(\phi_2(t_3, t_1) + \frac{1}{4\phi_1(t_3, t_1)}\right), \tag{A7}$$

$$\frac{1}{4\phi_1(t_3, t_2)} - \frac{G^2(t_3, t_2)}{R(t_3, t_2, t_1)} = \frac{1}{4\phi_1(t_3, t_1)}, \tag{A8}$$

$$\frac{G(t_3, t_2)G(t_2, t_1)}{R(t_3, t_2, t_1)} = -G(t_3, t_1). \tag{A9}$$

By using again Eq. (3.11), these equations can be further rewritten into relationships involving only R, G, and ϕ_1. We find

$$R(t_2, t_1, t_0) - \frac{G(t_2, t_1)^2}{R(t_3, t_2, t_1)} = R(t_3, t_1, t_0), \tag{A10}$$

$$\frac{G(t_3, t_1)G(t_2, t_1)}{G(t_3, t_2)} = R(t_3, t_1, t_0) - R(t_2, t_1, t_0), \tag{A11}$$

$$\frac{4G(t_3, t_2)G(t_3, t_1)}{G(t_3, t_2)} = \frac{1}{\phi_1(t_3, t_1)} - \frac{1}{\phi_1(t_3, t_2)}. \tag{A12}$$

This is the form of the addition theorem we used in order to simplify the expression of the correlation functions. We remark that in the case of the standard harmonic oscillator $(dh/dt = dg/dt = 0)$, Eqs. (A7)–(A9) are trivially satisfied in view of the addition theorems of hyperbolic functions.

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[3] Our definition of the entropy differs in sign from the conventional one used, e.g., in thermodynamics. Hence, a "classical" state of the system corresponds to a *minimum* of the entropy.

[4] For a recent review, see e.g., M. Jacob, in *Proceedings of the Third Johns Hopkins Workshop on Current Problems in High-Energy Particle Theory*, edited by R. Casalbuoni, G. Domokos, and S. Kövesi-Domokos

(Johns Hopkins University, Baltimore, 1979).
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[7]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 7.
[8]B. Pomorišac, Johns Hopkins University Technical Report No. COO 3285-38, 1980 (unpublished). As shown in this work, the requirements of finiteness and stability of the classical solution also impose some limitation on the allowed values of the parameters of the model.
[9]Some details of the numerical evaluation of these distributions together with an investigation of questions of local stability for various values of the parameters are described in Ref. 8. "Local stability" is always understood in the sense of Liapunoff. We wish to thank Professor P. Hartmann (Department of Mathematics, Johns Hopkins University) for several enlightening discussions concerning this point.
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